Excited Hadrons and Quark-Hadron Duality

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Research with **Pere Masjuan** and **Wojciech Broniowski**

To count or not to count

- The best way to look for missing states is to count them

$$1, 2, 3 \dots$$

- Thermodynamics is a way of counting
- The partition function of QCD counts

$$Z_{\rm QCD} = \sum_n e^{-E_n/T} \qquad H_{\rm QCD}\psi_n = E_n \psi_n$$

- Spectrum of QCD $\rightarrow$ Thermodynamics
- Colour singlet states (hadrons + ...???)
- Do we see quark-gluon substructure BELOW the "phase transition" ?
- Completeness relation in Hilbert space $\mathcal{H}_{\rm QCQ}$

$$\mathbf{1} = \sum_n |\Psi_n\rangle\langle\Psi_n| \approx \underbrace{\sum_n |\bar q q; n\rangle\langle \bar q q; n|}_{\rm mesons} + \underbrace{\sum_n |qqq; n\rangle\langle qqq; n|}_{\rm baryons} + \underbrace{\sum_n |\bar q q g; n\rangle\langle \bar q q g; n|}_{\rm hybrids} + \dots$$

- Given H, is there a sum rule involving ALL resonances ?.
- Quark-Hadron duality implies conditions on the spectrum

Approximation schemes in QCD

- Heavy quarks (quarkonium physics) $m_q \gg \Lambda_{\rm QCD}$
- Large N_c (meson dominance physics)
- Operator Product Expansion (condensates $\langle G^2 \rangle$, $\langle \bar q q \rangle$) $Q \gg \Lambda_{\rm QCD}$
- Chiral Perturbation Theory (real pions) $m_\pi \gg \Lambda_{\rm QCD}$

Quark-Hadron Duality: Physical observables can be computed in quark-gluon or hadronic language

- Two point functions (Spectrum)
- Three point functions (Form factors, Decays)
- Four point functions (Hadron-Hadron scattering, Structure functions, GPD's)

Operator Product Expansion

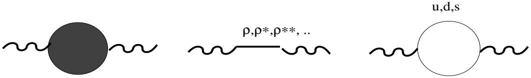
- Vector-Vector and Axial-Axial correlators

$$
\begin{aligned}
\Pi_V^{\mu a,\nu b}(q) &= i\int d^4x e^{-iq\cdot x}\langle 0|T\left\{J_V^{\mu a}(x)J_V^{\nu b}(0)\right\}|0\rangle \\
&= \Pi_V^T(q^2)\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)\delta^{ab}, \\
\Pi_A^{\mu a,\nu b}(q) &= i\int d^4x e^{-iq\cdot x}\langle 0|T\left\{J_A^{\mu a}(x)J_A^{\nu b}(0)\right\}|0\rangle \\
&= \Pi_A^T(q^2)\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)\delta^{ab} + \Pi_A^L(q^2)\,q^\mu q^\nu \delta^{ab},
\end{aligned}
$$

with $J_{V,A}^{\mu a} = \bar\psi i\gamma^\mu\{1,\gamma_5\}\frac{\tau^a}{2}\psi$ QCD currents. At large Q

$$
\begin{aligned}
\Pi_{V+A}^T(Q^2) &= \frac{1}{4\pi^2}\left\{-\frac{N_c}{3}\log\frac{Q^2}{\mu^2} + \frac{\pi}{3}\frac{\langle\alpha_S G^2\rangle}{Q^4} + \frac{256\pi^3}{81}\frac{\alpha_S\langle\bar q q\rangle^2}{Q^6}\right\} + \dots, \\
\Pi_{V-A}^T(Q^2) &= -\frac{32\pi}{9}\frac{\alpha_S\langle\bar q q\rangle^2}{Q^6} + \dots, \qquad (1)
\end{aligned}
$$

Narrow Meson resonances



- Saturating with narrow resonances

$$\Pi_V^T(Q^2) = \sum_{n=0}^{\infty} \frac{F_{V,n}^2}{M_{V,n}^2 + Q^2} + c.t.,$$

$$\Pi_A^T(Q^2) = \frac{f_\pi^2}{Q^2} + \sum_{n=0}^{\infty} \frac{F_{A,n}^2}{M_{A,n}^2 + Q^2} + c.t.,$$

- Spectral function

$$\mathrm{Im}\Pi_V^T(s) \to \sum_{n=0}^{\infty} F_{V,n}^2 \delta(M_{V,n}^2 - s) \sim \lim_{n\to\infty} \frac{F_{V,n}^2}{dn/dM_{V,n}^2}$$

$$\mathrm{Im}\Pi_V^T(s) \to \frac{N_c}{24\pi^2} \mathrm{Im}\log(-s) = \frac{N_c}{24\pi}$$

The radial Regge spectrum for light quarks

- Two scalar quarks in the CM frame

$$M = 2\sqrt{p^2 + m^2} + \sigma r \to 2m + \frac{p^2}{m} + \sigma r + \dots \tag{2}$$

- Bohr-Sommerfeld quantization $L = 0$ and $m = 0$

$$2\int_0^{M/\sigma} p_r dr = 2\pi(n + \alpha) \tag{3}$$

- Radial Regge spectrum for large n Anisovich, Anisovich, '02

$$M_n^2 = \mu^2 n + M_0^2 \qquad \mu^2 = 1.25(15)\mathrm{GeV}^2 \tag{4}$$

Regge phase-space

- For large meson masses, the level density is given by

$$\begin{aligned}\rho(M^2) &= \sum_n \delta(M^2 - M_n^2) \to \int dn \delta(M^2 - M_n^2) \\ &= \frac{1}{dM_n^2/dn}\Big|_{M_n^2=M^2} = \frac{dn}{dM^2} = \frac{1}{2\pi\sigma}\end{aligned} \tag{5}$$

- So, using the WKB approximation we get

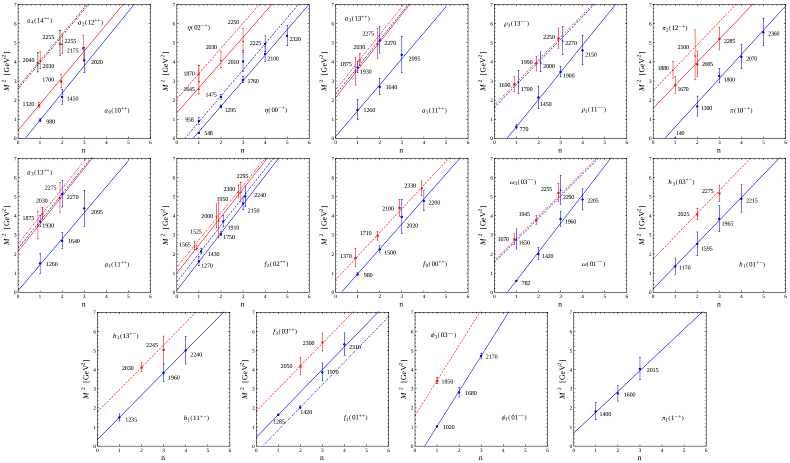
$$\frac{dn}{dM^2} = \frac{1}{\pi}\int_0^a \frac{dr}{p_r} = \frac{1}{2\pi\sigma} \tag{6}$$

- Finite quark mass corrections

$$\frac{dn}{dM^2} = \frac{1}{2\pi\sigma}\sqrt{1 - \frac{4m^2}{M^2}}\theta(M^2 - 4m^2) \tag{7}$$

- Two body phase space factor = Absortive part of two point correlators for FREE PARTICLES.

Empirical Regge Spectrum and Widths



- The **half-width rule**: $\Delta M_R = \Gamma_R/2$ or $\Delta M_R^2 = M_R \Gamma_R$

$$M^2 = 1.38(4)n + 1.12(4)J - 1.25(4)\,.$$

Large N_c scaling rules

$N_c \to \infty$ with αN_c fixed (t'Hoot, Witten)

- Hadronic spectrum (baryons and mesons are stable)

$$m_{\pi,\rho,\omega,\sigma} \sim N_c^0 \quad \Gamma_{\sigma,\rho,\omega} \sim 1/N_c \quad m_{N,\Delta} \sim N_c \quad \Gamma_\Delta \sim 1/N_c$$

- Couplings

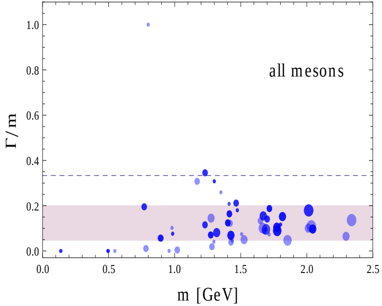
$$g_{MMM} \sim 1/\sqrt{N_c} \quad g_{MMMM} \sim 1/N_c \quad g_{BMM} \sim \sqrt{N_c}$$

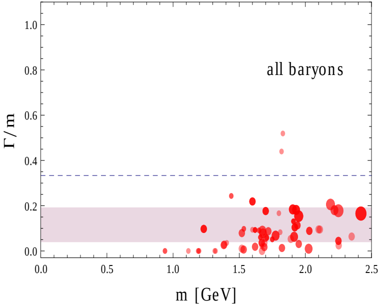
- Scattering

$$T(\pi\pi \to \pi\pi) \sim 1/N_c \, , \, T(\pi N \to \pi N) \sim N_c^0 \, , \, T(NN \to NN) \sim N_c$$

The hadron resonance spectrum

- Most states listed by the PDG are resonances (unstable with strong interactions)
- Many are classified according to the quark-model
- Meson and baryon spectrum resemble large N_c QCD properties





$$\frac{\Gamma}{m} \equiv \sum_{J,\alpha} (2J+1) \frac{\Gamma_{J,\alpha}}{m_{J,\alpha}} = 0.12(8) \qquad = \mathcal{O}(N_c{}^{-1})$$

Origin of resonances in QCD

- Hadrons are bound states of quarks
- Two point correlator at large distances: exponential fall-off (bound state in quenched QCD)

$$\langle 0| \left\{ J_V^{\mu a}(0,t) J_V^{\nu b}(0) \right\} |0\rangle \to Z e^{-m_\rho t}$$

- Unquenched result yields a mass-shift and deviations from the exponential fall-off
- What value should one quote ?
- If two methods give different values, how large could a discrepancy be admissible ?

Uncertainties in resonance masses

- A resonance is a pole of a scattering amplitude in the second Riemann sheet

$$T_s^{\mathrm{II}} \to \frac{g_R^2}{s - s_R} \qquad s_R = M_R^2 - i\Gamma_R M_R$$

- Complex energies cannot be measured nor simulated on the lattice
- This is not the usual definition of the PDG values
- Other definitions: speed plot, time-delay, Breit-Wigner etc.
- Lehman representation of the resonance two-point function

$$D(s) = \int_0^\infty \frac{\rho(\mu^2)}{\mu^2 - s + i0^+} = \frac{1}{s - M^2 - i\Gamma\sqrt{s}}$$

- Probabilistic interpretation of the line shape

$$P(\mu) = Z\rho(\mu)$$

Mass shift and width

- Canonical Meson propagator and self-energy

$$D(s) = \frac{1}{s - M^2 - \Sigma(s)} \to \frac{1}{s - M_R^2 + \mathrm{i} M_R \Gamma_R}$$

- Dispersion relation

$$\Sigma(s) = \mathrm{c.t.} + \frac{1}{\pi} \int ds' \frac{\mathrm{Im}\Sigma(s')}{s' - s}$$

- Absortive part

$$\mathrm{Im}\Sigma(s) = \Gamma(s) \sim s^{l+1/2}$$

- We need three subtractions $\to$ (no predictive power). Natural cut-offs of 1-2 GeV predict $\Delta m_\rho \sim 70 \sim \Gamma_\rho / 2$

Finite width effects in the space like region

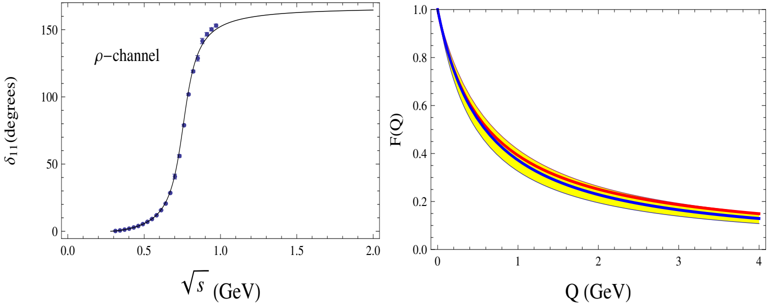
- Watson's final state theorem

$$\frac{F_V(s+i0^+)}{F_V(s-i0^+)} = \frac{T_{11}(s-i0^+)}{T_{11}(s+i0^+)} \equiv e^{2i\delta_{11}(s)} \qquad 4m_\pi^2 \le s \le 4m_K^2$$

- Omnes solution (not unique)

$$F_V(s) = \Omega(s) = \exp\left[\frac{s}{\pi}\int_{4m^2}^{\infty} \frac{\delta_J(s')}{s'(s-s')}\right]$$

- Sharp Resonance $\rightarrow$ VMD



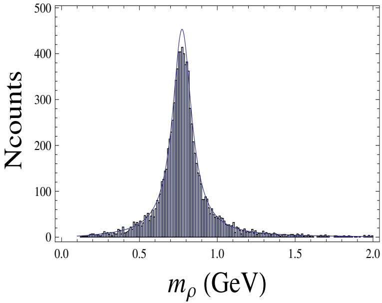
Random generation of a resonance

- Breit-Wigner distribution

$$D_{\rm BW}(\mu) = \frac{1}{\mu^2 - M^2 - i\Gamma\mu} \to P_{\rm BW}(\mu) = \frac{Z\Gamma\mu}{(\mu^2 - M^2)^2 + \Gamma^2\mu^2)} \qquad (8)$$

- Random distribution

$$\mu^2 = M_R^2 + M_R\Gamma_R \tan(\pi z/2) \qquad z \in U[0,1]$$



The **half-width rule**: $\Delta M_R = \Gamma_R/2$ or $\Delta M_R^2 = M_R\Gamma_R$

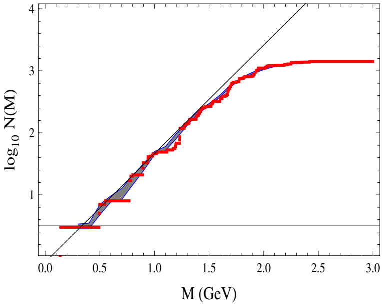
The physical spectrum and the half-width rule

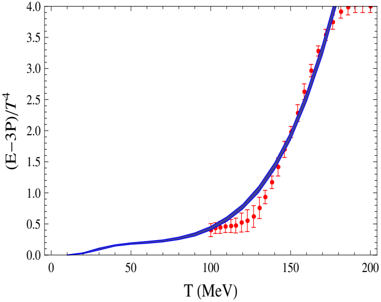
- The cummulative number

$$N(M) = \sum_i g_i \Theta(M - M_i)$$

- Trace anomaly and the resonance gas model from PDG

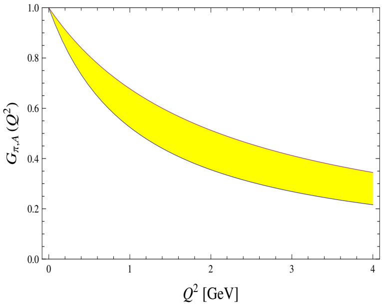
$$\Delta = \frac{E - 3P}{T^4} = \frac{1}{T^4} \sum_i g_i \int \frac{d^3k}{(2\pi)^3} \frac{(E_k - \vec{k} \cdot \nabla_k E_k)}{e^{E_k/T} \pm 1}$$





Pion Form Factors

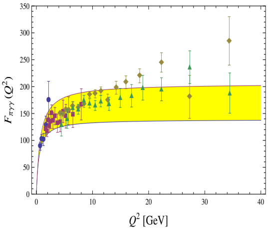
$$F_{em}(Q^2) = \frac{M_V^2}{M_V^2 + Q^2} \qquad G_A(Q^2) = \frac{M_A^2}{M_A^2 + Q^2}$$

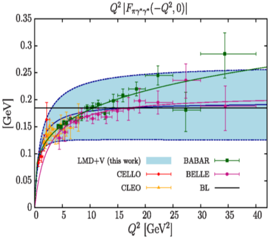


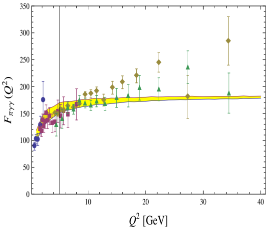
Pion Transition Form Factor

$$F_{\pi\gamma\gamma}(Q^2) = \frac{1}{4\pi^2 f_\pi} \sum_V \frac{c_V M_V^2}{M_V^2 + Q^2} \to \qquad \sum_V c_V = 1 \tag{9}$$

- 1 state $V = \rho$ (no pQCD constraint) Lattice: Gerardin et al. PRD94 (2016) 074507
- 2 states $V = \rho, \rho'$ (with pQCD constraint)







Nucleon Form Factors

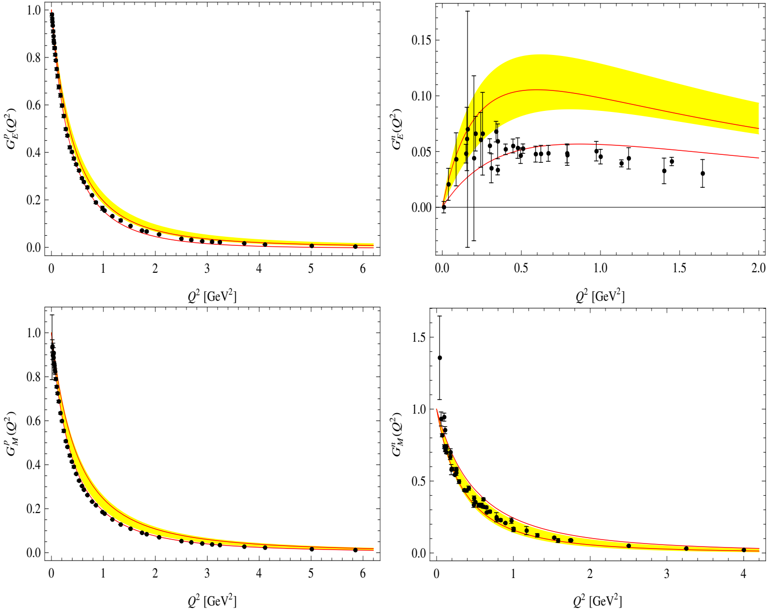
$$\langle N(p')|J_V^{\mu a}(0)|N(p)\rangle = \bar{u}(p')\frac{\tau^a}{2}\left[\gamma^\mu F_1^{I=1}(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M_N}F_2^{I=1}(q^2)\right]u(p)$$

$$t^{i+3}F_i(t) \to 0$$

$$F_1^{I=0}(t) = \sum_V \frac{g_{\omega NN}f_{\omega\gamma}}{m_\omega^2 - t} \qquad F_2^{I=0}(t) = \sum_V \frac{f_{\omega NN}f_{\omega\gamma}}{m_\omega - t}$$

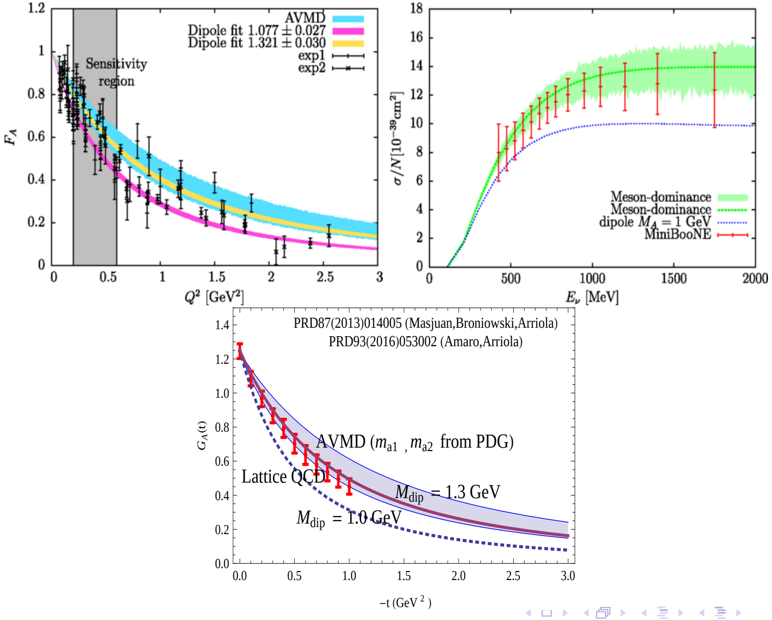
$$F_1^{I=1}(t) = \sum_V \frac{g_{\rho NN}f_{\rho\gamma}}{m_\rho - t} \qquad F_2^{I=1}(t) = \sum_V \frac{f_{\rho NN}f_{\rho\gamma}}{m_\rho - t}$$

SU(3) $\to g_{\omega NN} = 3g_{\rho NN} \sim 9$ Violations of 30%

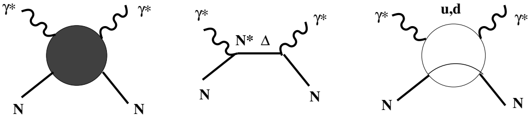


$G_E^p(Q^2)$
Q^2 [GeV^2]
$G_E^n(Q^2)$
Q^2 [GeV^2]
$G_M^p(Q^2)$
Q^2 [GeV^2]
$G_M^n(Q^2)$
Q^2 [GeV^2]

Nucleon Axial Form Factor



Deep inelastic scattering



- Forward Compton scattering

$$W^{ab}_{\mu\nu}(p,q;s) = \frac{1}{4\pi}\int d^4x e^{iq\cdot\xi}\left\langle p,s\middle|[J^a_\mu(\xi),J^{b\dagger}_\nu(0)]\middle|p,s\right\rangle \tag{10}$$

$$= \left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)M_N W_1(\nu,Q^2)+\left(p_\mu - q_\mu\frac{p\cdot q}{q^2}\right)\left(p_\nu - q_\nu\frac{p\cdot q}{q^2}\right)\frac{1}{M_N}W_2(\nu,Q^2)$$

$$= \sum_R \langle N|J_\mu|R\rangle\langle R|J_\nu|N\rangle\delta((p+q)^2-M_R^2) \tag{11}$$

- Bjorken limit:

$$Q^2\to\infty \qquad \text{with} \qquad x = Q^2/2p\cdot q \quad \text{fixed}\,.$$

- Scaling and Spin 1/2 (parton model)

$$M_N W_1(x,Q^2)\ \to, F_1(x) \quad \text{and} \quad \frac{p\cdot q}{M_N}W_2(x,Q^2)\ \to F_2(x) \qquad F_2(x)=2xF_1(x)$$

Narrow resonance approximation

- Invant mass

$$W^2 = (p+q)^2 = M_N^2 + Q^2 \left(\frac{1}{x} - 1 \right)$$

- Bjorken limit

$$W_2 = \sum_R \left[G_{N \to R}(q^2/M_R^2) \right]^2 \delta(W^2 - M_R^2) \to \int d\mu^2 \sum_i \left[G_{N \to i}(-Q^2/\mu^2) \right]^2 \frac{dn}{d\mu^2} \quad (12)$$

$$F_2(x) = \sum_i \left[G_{N \to i}(x/(1-x)) \right]^2 \frac{dn}{2\mu^2} \quad (13)$$

- Scaling limit

$$M_R^2 = \mu^2 n + M^2 \qquad G_{N \to R}(q) \equiv F(q^2/M_R^2)$$

- Drell-Yang relation

$$G_{N \to R}(-Q^2) \to \frac{1}{Q^n} \qquad F_2(x) \to (1-x)^{n-1}$$

The radial Regge spectrum for quark-diquark states

- Quark-diquark in the CM frame

$$M = \sqrt{p^2 + m_D^2} + p + \sigma r \qquad (14)$$

- Bohr-Sommerfeld quantization $L = 0$

$$2\int_0^{M/\sigma} p_r dr = 2\pi(n+\alpha) \to \frac{dn}{M^2} = \frac{1}{2\pi\sigma}(1 - \frac{m_D^2}{M^2}) \qquad (15)$$

- Radial Regge spectrum for large n

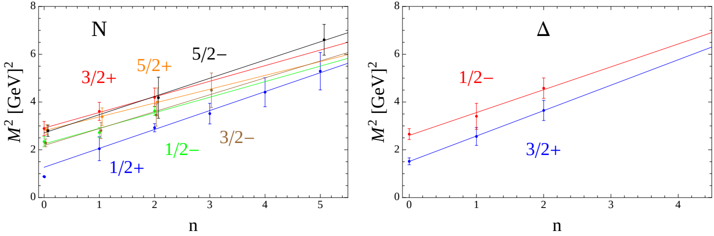
$$M_n^2 = \mu^2 n + M_0^2 \qquad (16)$$

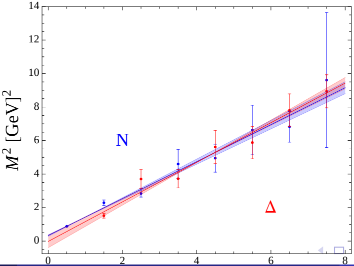
- Quark-Hadron duality in DIS requires Quark-Diquark dynamics !
- Problem of missing resonances →Quark model with qqq states predicts more states than observed

Regge Fits

Radial Regge trajectory parameterized as:

$$M_{n,J}^2 = a + \mu^2 n + \beta J \tag{17}$$





Regge Fits

N, J^P	μ [GeV]	χ^2/DOF	Δ, J^P	μ [GeV]	χ^2/DOF
$1/2+$	0.890(18)	0.02			
$3/2+$	0.808(24)	0.01	$3/2+$	1.032(6)	0.00
$5/2+$	0.760(13)	0.02			
$1/2-$	0.807(93)	0.13	$1/2-$	0.978(57)	0.06
$3/2-$	0.841(48)	0.07			
$5/2-$	0.873(16)	0.02			
Weighted Av.	0.824(8)			1.031(6)	
Combined fit	0.910(65)	0.13		1.004(181)	0.03

N, J^P	μ [GeV]	χ^2/DOF	Δ, J^P	μ [GeV]	χ^2/DOF
$1/2+$	0.854(57)	0.04			
$3/2+$	0.808(24)	0.01	$3/2+$	1.032(6)	0.00
$5/2+$	0.760(13)	0.02			
$1/2-$	0.807(93)	0.13	$1/2-$	0.978(57)	0.06
$3/2-$	0.762(25)	0.01			
$5/2-$	0.873(16)	0.02			
Weighted Av.	0.801(9)			1.031(6)	
Combined fit	0.887(46)	0.16		1.004(181)	0.03

	$\beta^2[GeV]^2$
N	1.050(24)
Δ	1.087(41)
Mean	1.059(21)

- Klempt and Forkel

$$M^2_{n,L} = 4\lambda^2(n + L + 3/2) \to \lambda = 0.510(5) \qquad \chi^2/DOF = 0.26 \tag{19}$$

- Santopinto *et al.* (Quark-Diquark relativistic dynamics)

$$M_{n,L} = a + bL + cn \tag{20}$$

$$\to \quad a = 1.53(12)\mathrm{GeV}^2 \quad b = 1.16(34)\mathrm{GeV}^2 \quad c = 1.11(13)\mathrm{GeV}^2 \tag{21}$$

$$\to \quad \chi^2/DOF = 1.36 \tag{22}$$

Conclusions

- Large N_C behaviour is a unique fingerprint of QCD
- Minimal number of resonances leads to parameter reduction
- Errors based on the half-width rule provide a guess on the uncertainties $\langle \Gamma/M \rangle = 0.12(8)$
- Quark-Hadron Duality requires infinitely resonance states
- Radial Regge formulas for hadrons (mesons and baryons)

$$M_n^2 \sim an + bJ + c$$

- Effective two-body dynamics (Quark-diquark for baryons)

References

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Chiral Perturbation Theory

- The chiral Lagrangian $u_\mu \sim \partial_\mu \phi + \dots$

$$
\begin{aligned}
\mathcal{L}_2^{\chi PT} &= \frac{f^2}{4} \langle\, u_\mu u^\mu + \chi_+ \,\rangle \,, \\
\mathcal{L}_4^{\chi PT} &= L_1 \langle\, u_\mu u^\mu \,\rangle^2 \,+\, L_2 \langle\, u_\mu u^\nu \,\rangle \langle\, u^\mu u_\nu \,\rangle \,+\, L_3 \langle\, u_\mu u^\mu u_\nu u^\nu \,\rangle \,+\, L_4 \langle\, u_\mu u^\mu \,\rangle \langle\, \chi_+ \,\rangle \\
&+ L_5 \langle\, u_\mu u^\mu \chi_+ \,\rangle \,+\, L_6 \langle\, \chi_+ \,\rangle^2 \,+\, L_7 \langle\, \chi_- \,\rangle^2 \,+\, L_8/2 \,\langle\, \chi_+^2 + \chi_-^2 \,\rangle \\
&- i L_9 \langle\, f_+^{\mu\nu} u_\mu u_\nu \,\rangle \,+\, L_{10}/4 \,\langle\, f_{+\mu\nu} f_+^{\mu\nu} - f_{-\mu\nu} f_-^{\mu\nu} \,\rangle \\
&+ H_1/2 \,\langle\, f_{+\mu\nu} f_+^{\mu\nu} + f_{-\mu\nu} f_-^{\mu\nu} \,\rangle \,+\, H_2/4 \,\langle\, \chi_+^2 - \chi_-^2 \,\rangle \,,
\end{aligned}
$$

- L_i low energy constants (encode unknown physics)
- Incorporates spontaneous chiral breaking
- Violates Unitarity

Resonance Chiral Theory

- Resonance Lagrangian in the Single Resonance Approximation (SRA)

$$\mathcal{L}_R = \mathcal{L}_V + \mathcal{L}_A + \mathcal{L}_S + \mathcal{L}_P \,,$$

$$\mathcal{L}_V = \frac{F_V}{2\sqrt{2}} \langle V_{\mu\nu} f_+^{\mu\nu} \rangle + \frac{i\, G_V}{2\sqrt{2}} \langle V_{\mu\nu} [u^\mu, u^\nu] \rangle \,,$$

$$\mathcal{L}_A = \frac{F_A}{2\sqrt{2}} \langle A_{\mu\nu} f_-^{\mu\nu} \rangle \,,$$

$$\mathcal{L}_S = c_d \langle S u_\mu u^\mu \rangle + c_m \langle S \chi_+ \rangle \,,$$

$$\mathcal{L}_P = i\, d_m \langle P \chi_- \rangle \quad .$$

- Decay rates $R \to 2\pi$

$$\Gamma_S = \frac{3 c_d^2 m_S^3}{16\pi f^4} \qquad \Gamma_V = \frac{G_V^2 m_V^3}{48\pi f^4}$$

Low energy limit

Integrate out resonances as heavy fields

$$L_1 = \frac{G_V^2}{8M_V^2}\,, \qquad L_2 = \frac{G_V^2}{4M_V^2}\,, \qquad L_3 = -\frac{3G_V^2}{4M_V^2} + \frac{c_d^2}{2M_S^2}\,,$$

$$L_4 = 0\,, \qquad L_5 = \frac{c_d c_m}{M_S^2}\,, \qquad L_6 = 0\,,$$

$$L_7 = 0\,, \qquad L_8 = \frac{c_m^2}{2M_S^2} - \frac{d_m^2}{2M_P^2}\,, \qquad L_9 = \frac{F_V G_V}{2M_V^2}\,,$$

$$L_{10} = -\frac{F_V^2}{4M_V^2} + \frac{F_A^2}{4M_A^2}\,, \qquad H_1 = -\frac{F_V^2}{8M_V^2} - \frac{F_A^2}{8M_A^2}\,, \qquad H_2 = \frac{c_m}{M_S^2} + \frac{d_m^2}{M_P^2}\,.$$

Short distance constraints

High energy (QCD) conditions on low and intermediate energy models

- Pion Form Factor

$$F^{v}_{\pi\pi}(q^2) = 1 + \frac{F_V G_V}{f^2} \frac{q^2}{m_V^2 - q^2} = \mathcal{O}(q^{-2}) \to F_V G_V = f_\pi^2$$

- Axial Form Factor

$$F^{a}_{\pi\gamma}(q^2) = \frac{F_A^2}{m_A^2 - q^2} + \frac{2F_V G_V - F_V^2}{m_V^2} = \mathcal{O}(q^{-2}) \to 2F_V G_V = F_V^2$$

- Weinberg sum rules (VV - AA correlators)

$$\Pi_{V-A}(q^2) = \frac{f_\pi^2}{q^2} + \frac{F_V^2}{m_V^2 - q^2} - \frac{F_A^2}{m_A^2 - q^2} = \mathcal{O}(q^{-4})$$

$$F_V^2 - F_A^2 = f_\pi^2 , \qquad m_V^2 F_V^2 - m_A^2 F_A^2 = 0 .$$

- Scalar Form Factor

$$F^s_{K\pi}(q^2) = 1 + \frac{4c_m}{f_\pi^2}\left[c_d + (c_m - c_d)\frac{m_K^2 - m_\pi^2}{m_S^2}\right]\frac{q^2}{m_S^2 - q^2} = \mathcal{O}(q^{-4})$$

$$4c_d c_m = f_\pi^2 \,, \qquad c_m - c_d = 0 \,.$$

- SS-PP correlator

$$\Pi_{S-P}(q^2) = 16B_0^2\left(\frac{c_m^2}{m_S^2 - q^2} - \frac{d_m^2}{m_P^2 - q^2} + \frac{f_\pi^2}{8q^2}\right) = \mathcal{O}(q^{-4}) \,.$$

$$8\left(c_m^2 - d_m^2\right) = f_\pi^2 \,, \qquad c_m^2 m_S^2 - d_m^2 m_P^2 \simeq 0 \,.$$

$$F_V = 2G_V = \sqrt{2}F_A = \sqrt{2}F \,, \qquad m_A = \sqrt{2}m_V \,,$$
$$c_m = c_d = \sqrt{2}d_m = \frac{f_\pi}{2} \,, \qquad m_P \simeq \sqrt{2}m_S \,,$$

- $\pi\pi$ scattering

$$A^{\rm SRA}(s,t,u) = \frac{m_\pi^2 - s}{f^2} + \frac{G_V^2}{f^4}\left\{ \frac{t(s-u)}{t-m_V^2} + \frac{u(s-t)}{u-m_V^2} \right\}$$
$$+\frac{2}{3f^4}\frac{\left[c_d(s-2m_\pi^2)+2m_\pi^2\ c_m\right]^2}{s-m_{S_8}^2}$$
$$+\frac{4}{f^4}\frac{\left[\bar{c}_d(s-2m_\pi^2)+2m_\pi^2\ \bar{c}_m\right]^2}{s-m_{S_1}^2} + \frac{8d_m^2}{f^4}\frac{m_\pi^4}{m_{P_8}^2-m_\pi^2}$$

- Forward amplitudes in t-channel (Froissart bound + Regge)

$$\begin{aligned} T^{\rm SRA}_{I_t=1}(\nu,0) &= \frac{2c_d^2 - f_\pi^2 + G_V^2}{f_\pi^4}\,\nu + \mathcal{O}(\nu^{-1}) \to f_\pi^2 = 2c_d^2 + G_V^2 \\ T^{\rm SRA}_{I_t=2}(\nu,0) &= 2\frac{c_d^2 m_S^2 - G_V^2 m_V^2}{f_\pi^4} + \mathcal{O}(\nu^{-2}) \to 2c_d^2 m_S^2 = G_V^2 m_V^2 \\ \to m_S &= m_V \end{aligned}$$

- Pion Transition form factor

$$
\begin{aligned}
F_{\pi\gamma\gamma^*}(q^2) &= \frac{1}{4\pi^2 f_\pi}\frac{m_V^2}{m_V^2-q^2} \to -\frac{6 f_\pi}{N_c q^2} \\
m_V^2 &= \frac{24\pi^2 f_\pi^2}{N_c}
\end{aligned}
$$

EVERYTHING IS DETERMINED FROM f_π !!

-
- The Low Energy Constants fulfilling all these constraints

$$
2L_1^{\rm SRA} = L_2^{\rm SRA} = -\frac{1}{2}L_3^{\rm SRA} = \frac{1}{2}L_5^{\rm SRA} = \frac{4}{3}L_8^{\rm SRA} = \frac{f^2}{8m_V^2} = \frac{N_c}{192\pi^2}
$$

- Can we estimate the error in a simple manner ?