Relativistic fluid dynamics with spin

Wojciech Florkowski

Institute of Nuclear Physics, Krakow and Jan Kochanowski University, Kielce, Poland

based on recent work with **B. Friman, A. Jaiswal, and E. Speranza**, arXiv:1705.00587 (nucl-th)

Excited QCD, Sintra, Portugal
May 7 – 13, 2017

- **Non-central heavy-ion collisions create fireballs with large global angular momenta** which may generate a spin polarization of the hot and dense matter in a way similar to the Einstein-de Haas and Barnett effects
- **Much effort has recently been invested in studies of polarization and spin dynamics of particles produced in high-energy nuclear collisions**, both from the experimental and theoretical point of view

L. Adamczyk et al. (**STAR**), (2017), arXiv:1701.06657 (nucl-ex), to appear in **Nature**
Global Λ hyperon polarization in nuclear collisions: evidence for the most vortical fluid

www.sciencenews.org/article/smashing-gold-ions-creates-most-swirly-fluid-ever

Our starting point is the **phase-space distribution functions for spin-1/2 particles** and antiparticles in local equilibrium. In order to incorporate the spin degrees of freedom, they have been **generalized from scalar functions to two by two spin density matrices** for each value of the space-time position x and momentum p, **F. Becattini et al., Annals Phys. 338 (2013) 32**

$$f^+_{rs}(x,p) = \frac{1}{2m}\bar{u}_r(p)X^+ u_s(p), \quad f^-_{rs}(x,p) = -\frac{1}{2m}\bar{v}_s(p)X^- v_r(p)$$

Following the notation used by F. Becattini et al., we introduce the matrices

$$X^\pm = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu\right] M^\pm$$

where

$$M^\pm = \exp\left[\pm\frac{1}{2}\omega_{\mu\nu}(x)\hat{\Sigma}^{\mu\nu}\right]$$

Here we use the notation $\beta^\mu = u^\mu/T$ and $\xi = \mu/T$, with the temperature T, chemical potential μ and four velocity u^μ. The latter is normalized to $u^2 = 1$. Moreover, $\omega_{\mu\nu}$ is the spin tensor, while $\hat{\Sigma}^{\mu\nu}$ is the spin operator expressed in terms of the Dirac gamma matrices, $\hat{\Sigma}^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$.

Spin/polarization tensor

$$\omega_{\mu\nu} \equiv k_\mu u_\nu - k_\nu u_\mu + \epsilon_{\mu\nu\beta\gamma} u^\beta \omega^\gamma.$$

We can assume that both k_μ and ω_μ are orthogonal to u^μ, i.e., $k \cdot u = \omega \cdot u = 0$,

$$k_\mu = \omega_{\mu\nu} u^\nu, \quad \omega_\mu = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta}\, \omega^{\nu\alpha} u^\beta.$$

It is convenient to introduce the dual spin tensor

$$\tilde{\omega}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \omega^{\alpha\beta} = \omega_\mu u_\nu - \omega_\nu u_\mu + \epsilon^{\mu\nu\alpha\beta} k_\alpha u_\beta. \tag{1}$$

One finds $\frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu} = k \cdot k - \omega \cdot \omega$ and $\frac{1}{2}\tilde{\omega}_{\mu\nu}\omega^{\mu\nu} = 2k \cdot \omega$. Using the constraint

$$k \cdot \omega = 0$$

we find the compact form

$$M^\pm = \cosh(\zeta) \pm \frac{\sinh(\zeta)}{2\zeta}\, \omega_{\mu\nu} \hat{\Sigma}^{\mu\nu}, \tag{2}$$

where

$$\zeta \equiv \frac{1}{2} \sqrt{k \cdot k - \omega \cdot \omega}. \tag{3}$$

We now assume also that $k \cdot k - \omega \cdot \omega \geq 0$, which implies that ζ is real.

The **charge current** (S. de Groot, W. van Leeuwen, and C. van Weert)

$$N^\mu = \int \frac{d^3p}{2(2\pi)^3 E_p} p^\mu \left[\mathrm{tr}(\mathrm{X}^+) - \mathrm{tr}(\mathrm{X}^-)\right] = n u^\mu$$

where 'tr' denotes the trace over spinor indices and n is the charge density

$$n = 4\cosh(\zeta)\sinh(\xi)\, n_{(0)}(T) = 2\cosh(\zeta)\left(e^{\xi} - e^{-\xi}\right) n_{(0)}(T)$$

Here $n_{(0)}(T) = \langle (u \cdot p) \rangle_0$ is the number density of spin 0, neutral Boltzmann particles, obtained using the thermal average

$$\langle \cdots \rangle_0 \equiv \int \frac{d^3p}{(2\pi)^3 E_p} (\cdots)\, e^{-\beta \cdot p},$$

where $E_p = \sqrt{m^2 + \mathbf{p}^2}$.

The **energy-momentum tensor** for a perfect fluid then has the form

$$T^{\mu\nu} = \int \frac{d^3p}{2(2\pi)^3 E_p} p^\mu p^\nu \left[\mathrm{tr}(\mathrm{X}^+) + \mathrm{tr}(\mathrm{X}^-) \right] = (\varepsilon + P) u^\mu u^\nu - P g^{\mu\nu},$$

where the energy density and pressure are given by

$$\varepsilon = 4 \cosh(\zeta) \cosh(\xi)\, \varepsilon_{(0)}(T)$$

and

$$P = 4 \cosh(\zeta) \cosh(\xi)\, P_{(0)}(T),$$

respectively. In analogy to the density $n_{(0)}(T)$, we define the auxiliary quantities $\varepsilon_{(0)}(T) = \langle (u \cdot p)^2 \rangle_0$ and $P_{(0)}(T) = -(1/3)\langle \left[p \cdot p - (u \cdot p)^2 \right] \rangle_0$.

Entropy current

The **entropy current** is given by an obvious generalization of the Boltzmann expression

$$S^\mu = -\int \frac{d^3p}{2(2\pi)^3 E_p}\, p^\mu \left(\mathrm{tr}\left[\mathrm{X}^+ (\ln \mathrm{X}^+ - 1)\right] + \mathrm{tr}\left[\mathrm{X}^- (\ln \mathrm{X}^- - 1)\right] \right)$$

This leads to the following entropy density

$$s = u_\mu S^\mu = \frac{\varepsilon + P - \mu\, n - \Omega w}{T},$$

where Ω is defined through the relation $\zeta = \Omega/T$ and

$$w = 4 \sinh(\zeta) \cosh(\xi)\, n_{(0)}.$$

This suggests that Ω should be used as a thermodynamic variable of the grand canonical potential, in addition to T and μ. Taking the pressure P to be a function of T, μ and Ω, we find

$$s = \left.\frac{\partial P}{\partial T}\right|_{\mu,\Omega}, \quad n = \left.\frac{\partial P}{\partial \mu}\right|_{T,\Omega}, \quad w = \left.\frac{\partial P}{\partial \Omega}\right|_{T,\mu}.$$

The conservation of energy and momentum requires that

$$\partial_\mu T^{\mu\nu} = 0.$$

This equation can be split into two parts, one longitudinal and the other transverse with respect to u^μ:

$$\begin{aligned} \partial_\mu[(\varepsilon + P)u^\mu] &= u^\mu \partial_\mu P \equiv \frac{dP}{d\tau}, \\ (\varepsilon + P)\frac{du^\mu}{d\tau} &= (g^{\mu\alpha} - u^\mu u^\alpha)\partial_\alpha P. \end{aligned} \tag{4}$$

Evaluating the derivative on the left-hand side of the first equation we find

$$T\,\partial_\mu(su^\mu) + \mu\,\partial_\mu(nu^\mu) + \Omega\,\partial_\mu(wu^\mu) = 0. \tag{5}$$

The middle term vanishes due to charge conservation,

$$\partial_\mu(nu^\mu) = 0. \tag{6}$$

Thus, in order to have entropy conserved in our system (for the perfect-fluid description we are aiming at), we demand that

$$\partial_\mu(wu^\mu) = 0. \tag{7}$$

Consequently, we self-consistently arrive at the equation for conservation of entropy, $\partial_\mu(su^\mu) = 0$.

In the absence of a net spin polarization, i.e., for $\zeta = 0$, we find the standard expression for the net charge density $n = 4\sinh(\xi)\, n_{(0)}$.

On the other hand, one may consider two linear combinations of the form $\partial_\mu \left[(n \pm w) u^\mu\right] = 0$. Then, we find $n \pm w = 4\sinh[(\mu \pm \Omega)/T]\, n_{(0)}$, which indicates that thermodynamic quantities corresponding to charge and spin of the particles couple.

In fact, Ω can be interpreted as a chemical potential related with spin. Interestingly, from a thermodynamic point of view, a system of particles with spin 1/2 can be seen as a two component mixture of scalar particles with chemical potentials $\mu \pm \Omega$.

The scheme defined so far, can be regarded as a minimal extension of the standard perfect-fluid hydrodynamics of charged particles, where all dynamic equations follow from the conservation laws. Equations derived above form a closed system of equations, which facilitates the study of spin dynamics. We may first solve these equations and subsequently use this solution as the **dynamic background for the spin dynamics**.

Spin dynamics

Since we use a symmetric form of the energy-momentum tensor $T^{\mu\nu}$, the spin tensor $S^{\lambda,\mu\nu}$ satisfies the conservation law,

$$\partial_\lambda S^{\lambda,\mu\nu} = 0.$$

For $S^{\lambda,\mu\nu}$ we use the form taken from the textbook by deGroot,

$$S^{\lambda,\mu\nu} = \int \frac{d^3p}{2(2\pi)^3 E_p}\, p^\lambda \, \mathrm{tr}\left[(X^+ - X^-)\hat{\Sigma}^{\mu\nu}\right] = \frac{w u^\lambda}{4\zeta}\omega^{\mu\nu}$$

Using the conservation law for the spin density and introducing the rescaled spin tensor $\bar{\omega}^{\mu\nu} = \omega^{\mu\nu}/(2\zeta)$, we obtain

$$u^\lambda \partial_\lambda \, \bar{\omega}^{\mu\nu} = \frac{d\bar{\omega}^{\mu\nu}}{d\tau} = 0,$$

with the normalization condition $\bar{\omega}_{\mu\nu}\,\bar{\omega}^{\mu\nu} = 2$.

The tensor $\bar{\omega}_{\mu\nu}$ can be decomposed in the similar way as the original spin tensor, with the two rescaled four vectors $\bar{k}_\mu = k_\mu/(2\zeta)$ and $\bar{\omega}_\mu = \omega_\mu/(2\zeta)$, satisfying the constraints

$$\bar{k}\cdot u = 0, \quad \bar{\omega}\cdot u = 0, \quad \bar{k}\cdot\bar{\omega} = 0, \quad \bar{k}\cdot\bar{k} - \bar{\omega}\cdot\bar{\omega} = 1,$$

which leave only four independent components in $\bar{k}_\mu$ and $\bar{\omega}_\mu$.

The last condition above is fulfilled by employing the parameterization

$$\bar{k}_\mu = m_\mu \sinh(\psi), \quad \bar{\omega}_\mu = n_\mu \cosh(\psi).$$

The four-vectors m_μ and n_μ are space-like and normalized to -1,

$$m_\mu m^\nu = -1, \quad n_\mu n^\mu = -1.$$

We thus find two coupled equations

$$\begin{aligned}
&\frac{dm_\mu}{d\tau}\sinh(\psi) + m_\mu \cosh(\psi)\frac{d\psi}{d\tau} + m_\nu a^\nu \sinh(\psi) u_\mu + \epsilon_{\mu\nu\beta\gamma} u^\nu a^\beta n^\gamma \cosh(\psi) = 0,\\
&\frac{dn_\mu}{d\tau}\cosh(\psi) + n_\mu \sinh(\psi)\frac{d\psi}{d\tau} + n_\nu a^\nu \cosh(\psi) u_\mu + \epsilon_{\mu\nu\alpha\beta} u^\nu a^\beta m^\alpha \sinh(\psi) = 0,
\end{aligned} \tag{8}$$

where $a^\mu = du^\mu/d\tau$ is the acceleration of the fluid element and $\frac{d\psi}{d\tau} = \epsilon_{\mu\nu\beta\gamma} m^\mu u^\nu a^\beta n^\gamma$.

The hydrodynamic flow is defined by the four-vector u^μ with the components

$$u^0 = \gamma, \quad u^1 = -\gamma\,\tilde{\Omega}\,y, \quad u^2 = \gamma\,\tilde{\Omega}\,x, \quad u^3 = 0,$$

where $\tilde{\Omega}$ is a constant, $\gamma = 1/\sqrt{1-\tilde{\Omega}^2 r^2}$, and r denotes the distance from the center of the vortex in the transverse plane, $r^2 = x^2 + y^2$. Due to limiting light speed, the assumed flow profile may be realised only within a cylinder with the radius $R < 1/\tilde{\Omega}$. The total time (convective) derivative takes the form

$$\frac{d}{d\tau} = u^\mu \partial_\mu = -\gamma\tilde{\Omega}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right).$$

Equation (9) can be used to find the fluid acceleration

$$a^\mu = \frac{du^\mu}{d\tau} = -\gamma^2\tilde{\Omega}^2(0, x, y, 0).$$

As expected the spatial part of the four-acceleration points towards the centre of the vortex, as it describes the centripetal acceleration.

Spin/polarization tensor

It is easy to see that the equations of the hydrodynamic background are satisfied if T, μ and Ω are proportional to the Lorentz-γ factor

$$T = T_0\gamma, \quad \mu = \mu_0\gamma, \quad \Omega = \Omega_0\gamma,$$

with T_0, μ_0 and Ω_0 being constants. **One possibility is that the vortex represents an unpolarized fluid with $\omega_{\mu\nu} = 0$ and thus, with $\Omega_0 = 0$.**

Another possibility is that the particles in the fluid are polarized and $\Omega_0 \neq 0$. In the latter case we expect that the spin tensor has the structure

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\tilde{\Omega}/T_0 & 0 \\ 0 & \tilde{\Omega}/T_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the parameter T_0 has been introduced to keep $\omega_{\mu\nu}$ dimensionless. This form yields $k^\mu = \tilde{\Omega}^2(\gamma/T_0)\,(0,x,y,0)$ and $\omega^\mu = \tilde{\Omega}(\gamma/T_0)\,(0,0,0,1)$. As a consequence, we find $\zeta = \tilde{\Omega}/(2T_0)$, which, for consistency with the hydrodynamic background equations, implies

$$\tilde{\Omega} = 2\,\Omega_0.$$

Vortex solution 3

It follows that $\bar{k}^\mu = \gamma\tilde{\Omega}r\,(0, x/r, y/r, 0)$ and $\bar{\omega}^\mu = \gamma\,(0,0,0,1)$,
leading to $m^\mu = (0, x/r, y/r, 0)$, $n^\mu = (0,0,0,1)$,
$\cosh(\psi) = \gamma$, and $\sinh(\psi) = \gamma\tilde{\Omega}r$.
With all these quantities determined, it is rather straightforward to show that our spin-evolution equations are fulfilled.

We observe that $d\psi/d\tau = 0$, since the four-vectors m^μ and a^μ are parallel. We also note that the spin tensor agrees with the thermal vorticity, namely

$$\varpi_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu\right)$$

as emphasised in the works by Becattini and collaborators.

In this work **we have introduced a hydrodynamic framework, which includes the evolution of the spin density in a consistent fashion**. Equations that determine the dynamics of the system follow solely from conservation laws. Thus, they can be regarded as a minimal extension of the well established perfect-fluid picture.

Our framework can be used to determine the space-time dynamics of fluid variables, now including also the spin tensor, from initial conditions defined on an initial space-like hypersurface. This property makes them useful for practical applications in studies of polarization evolution in high-energy nuclear collisions and also in other physics systems exhibiting fluid-like, collective dynamics connected with non-trivial polarization phenomena. In particular, the possibility to study the dynamics of systems in local thermodynamic equilibrium represents an important advance compared to studies, where global equilibrium was assumed.