Fluctuations of the gravitational field generated by dark substructures

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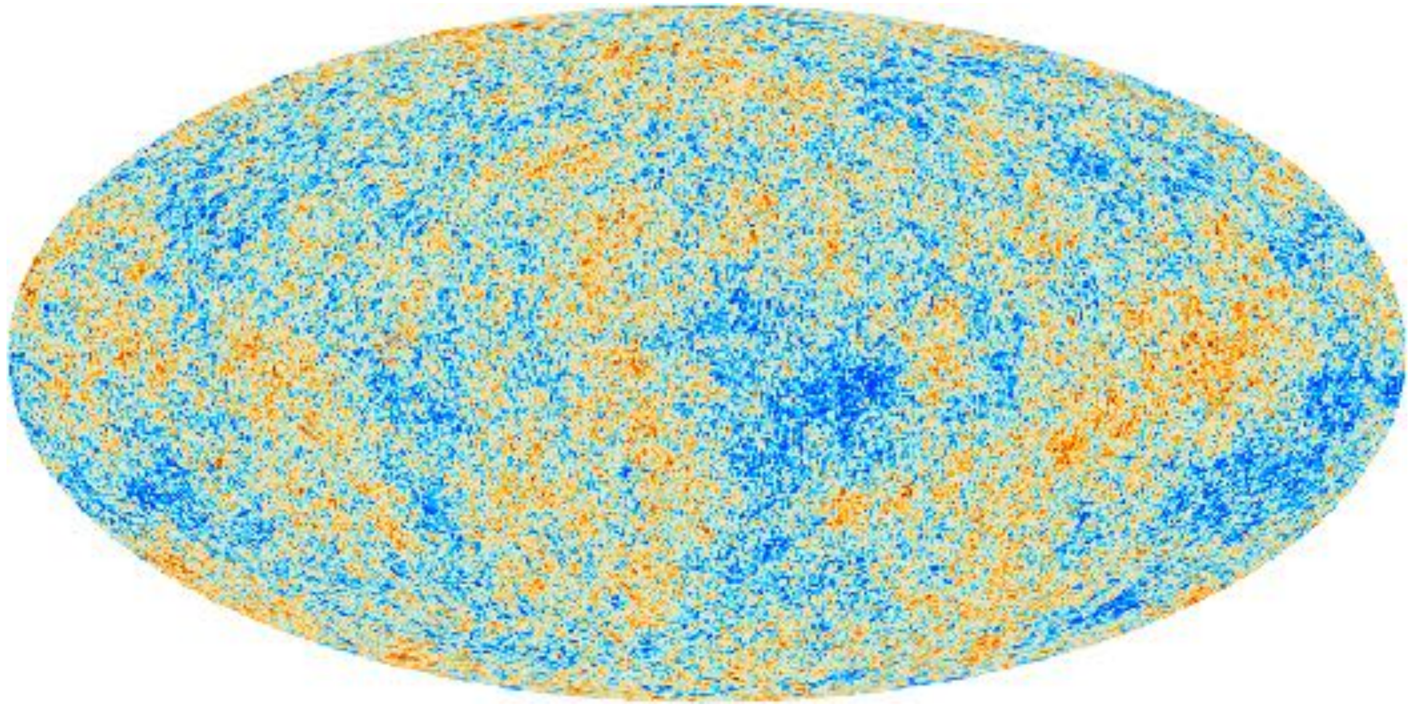
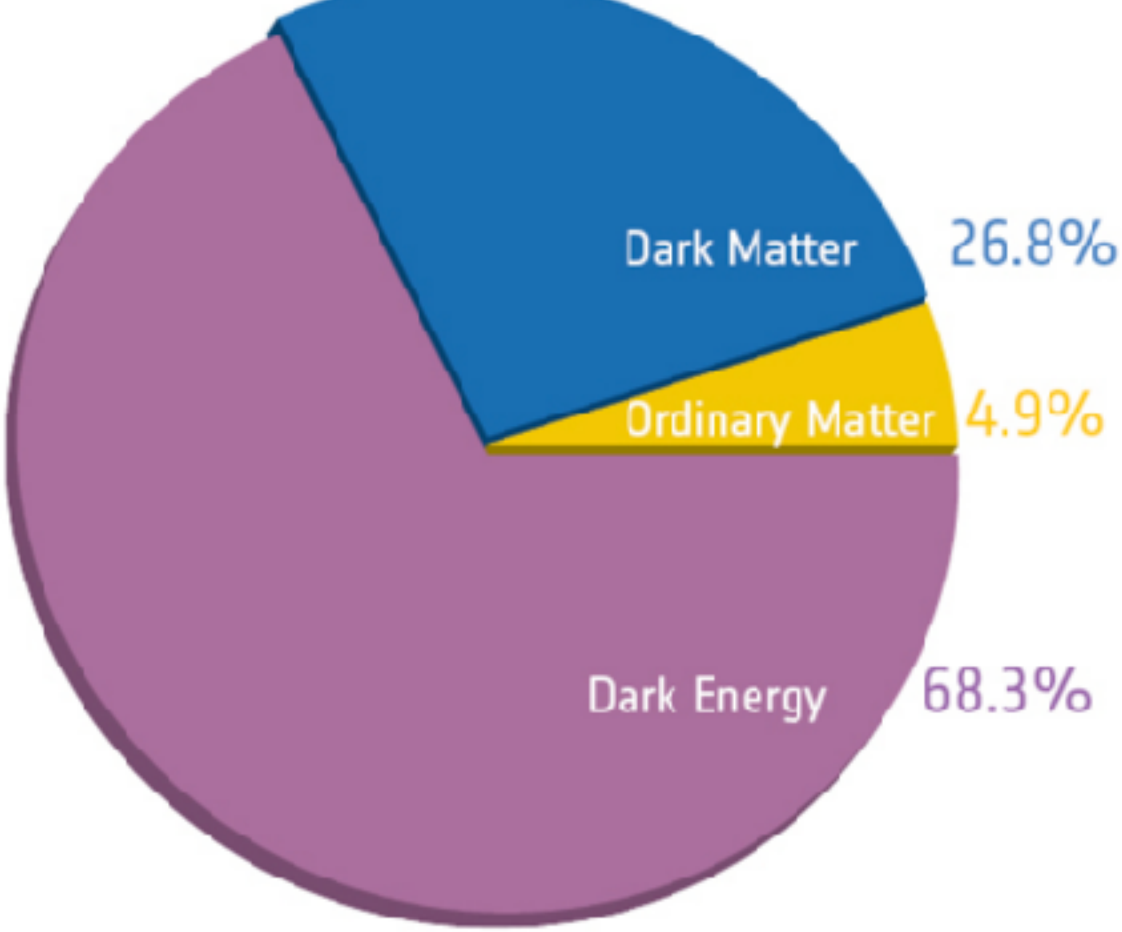
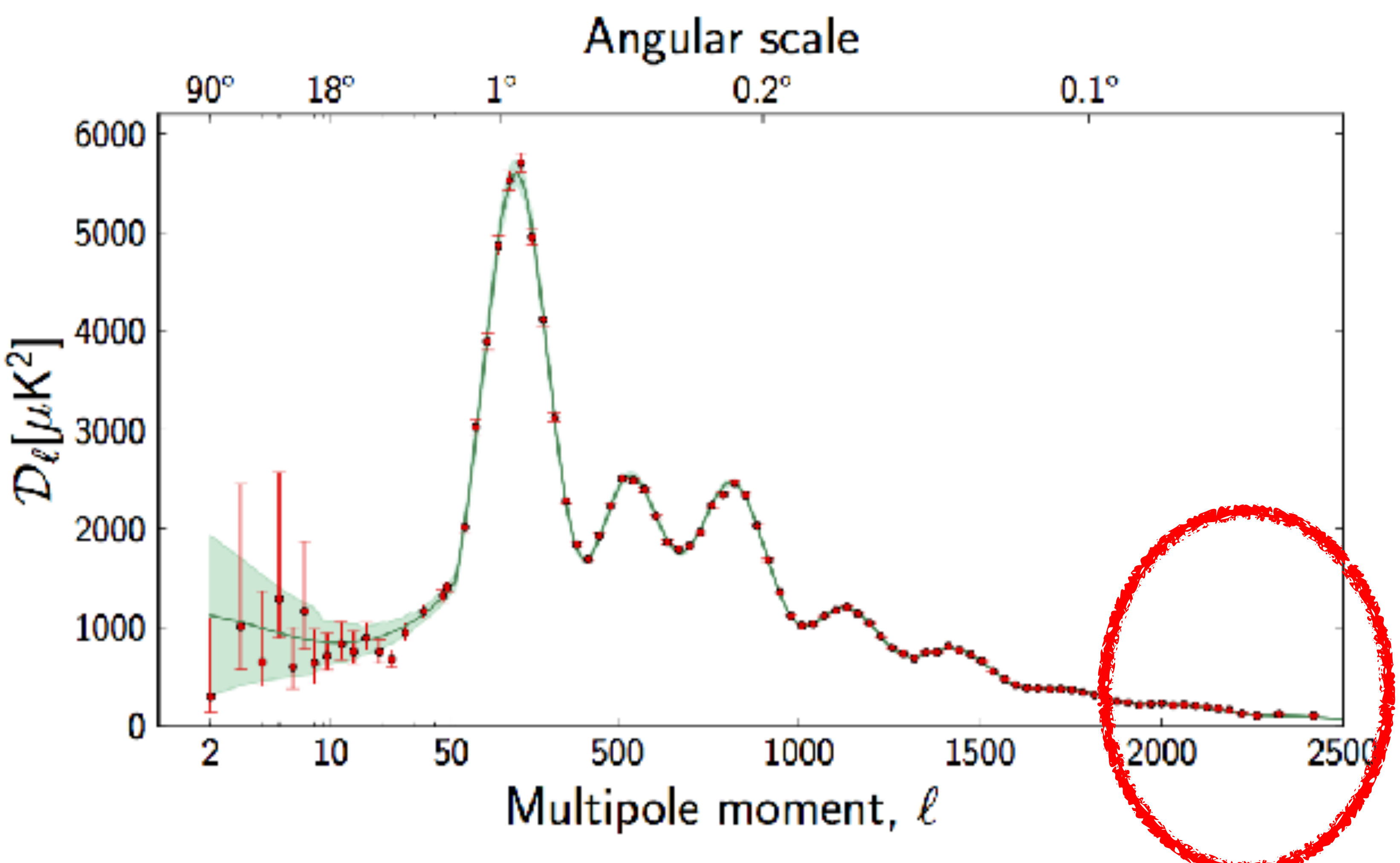
Tallinn
18 June 2018



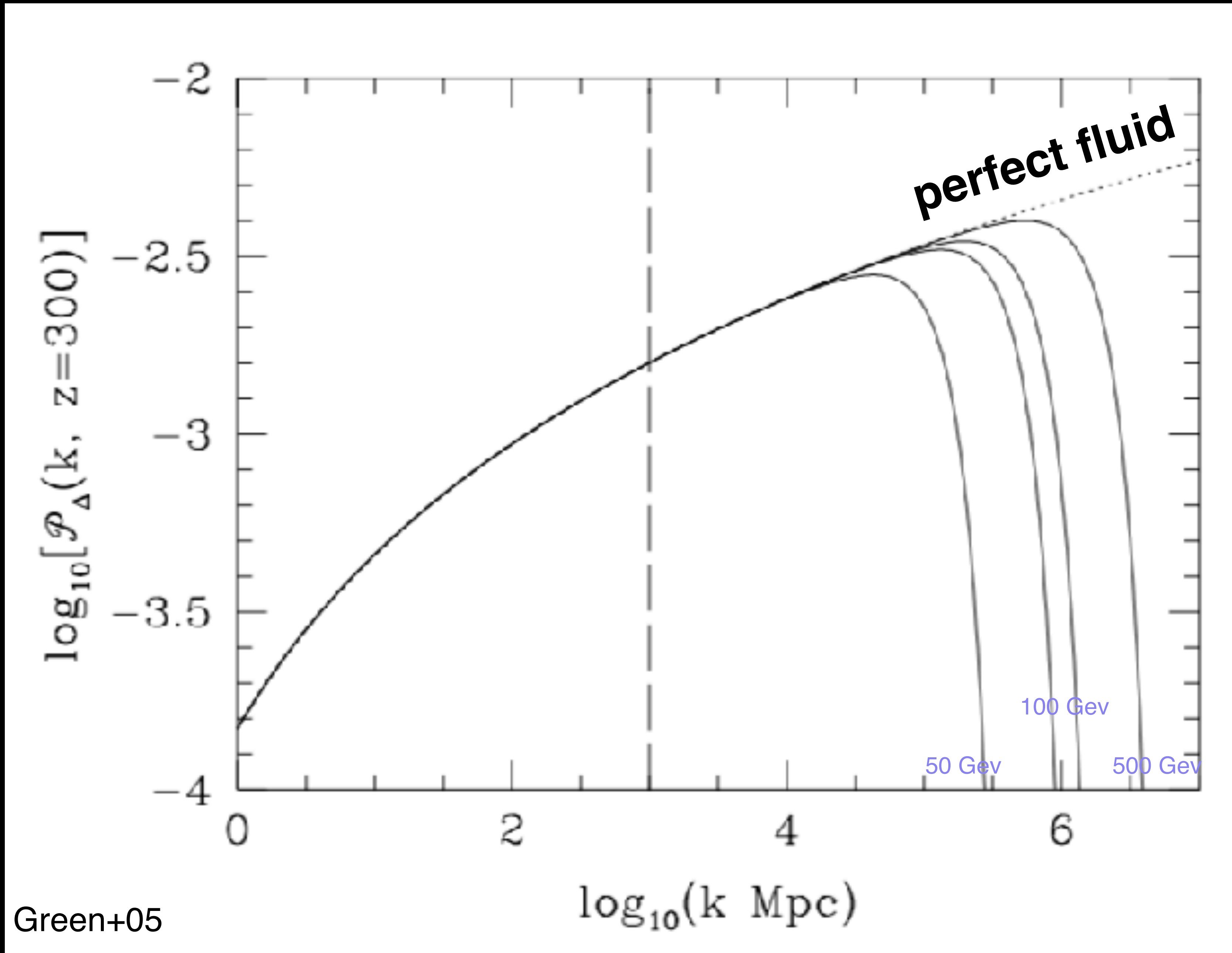




Classical physics

No deviation from DM = perfect fluid

interacts through gravity only



DM damping on small scales through quantum effects

Free streaming length:

$\lambda_{FS} \equiv$ *"average distance travelled by a DM particle before it falls in a potential well"*

HDM: λ_{FS} ~ 20 Mpc (30 ev / m_ν)

WDM: λ_{FS} ~ 100 kpc (1 kev / m_ν)

CDM: λ_{FS} ~ 3.7 pc (100 Gev / m_ν)$^{1/2}$

Spherical collapse model:

$$M(< \lambda_{FS}, z = 60) \gtrsim 10^{-6} M_\odot$$

DM clumps with *planet* size!

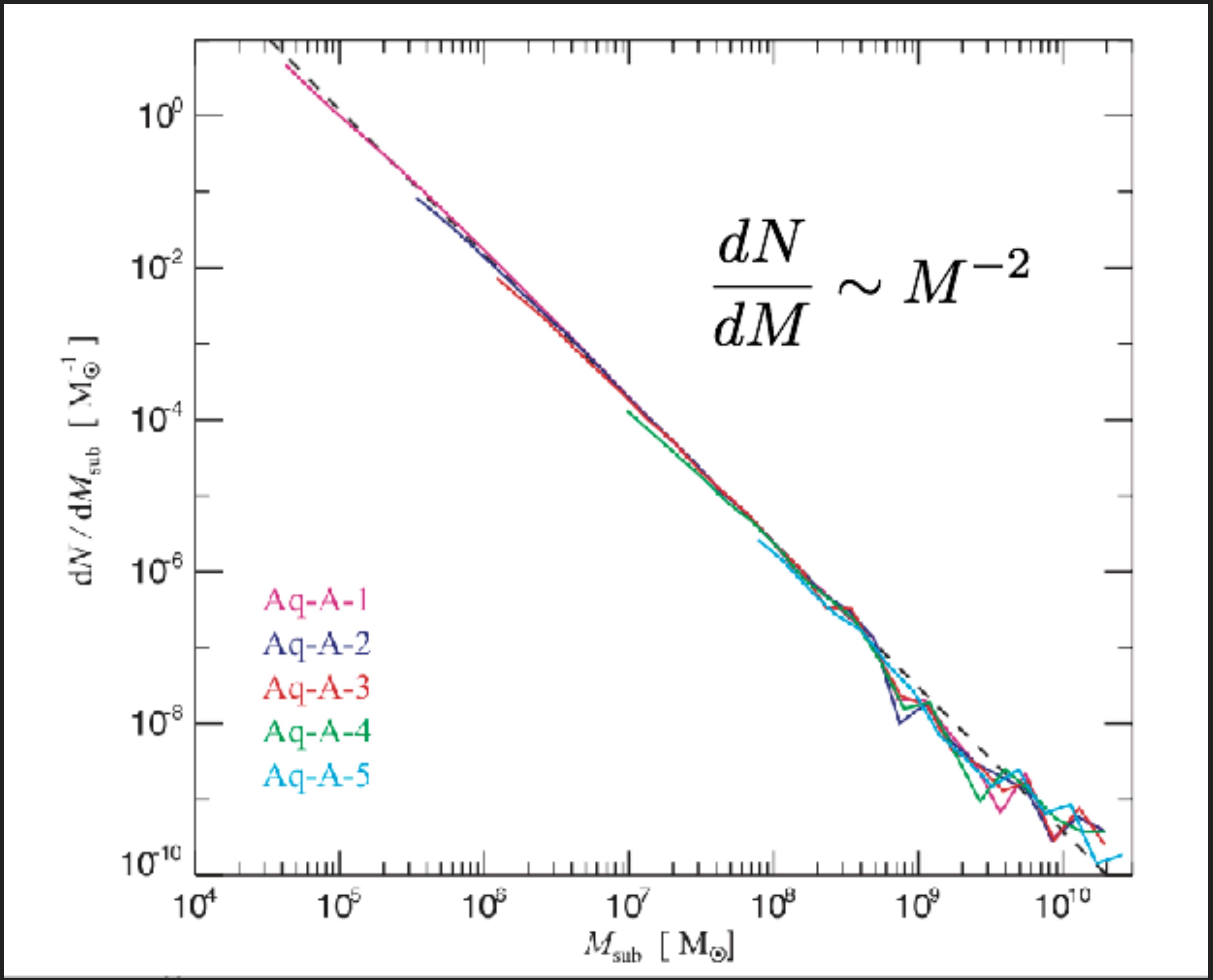
DM-ONLY N-BODY SIMULATIONS

Springel + 08

Aquarius simulation of **Milky Way** halo

Aq-A-1: N ~ 5x10^9 particles

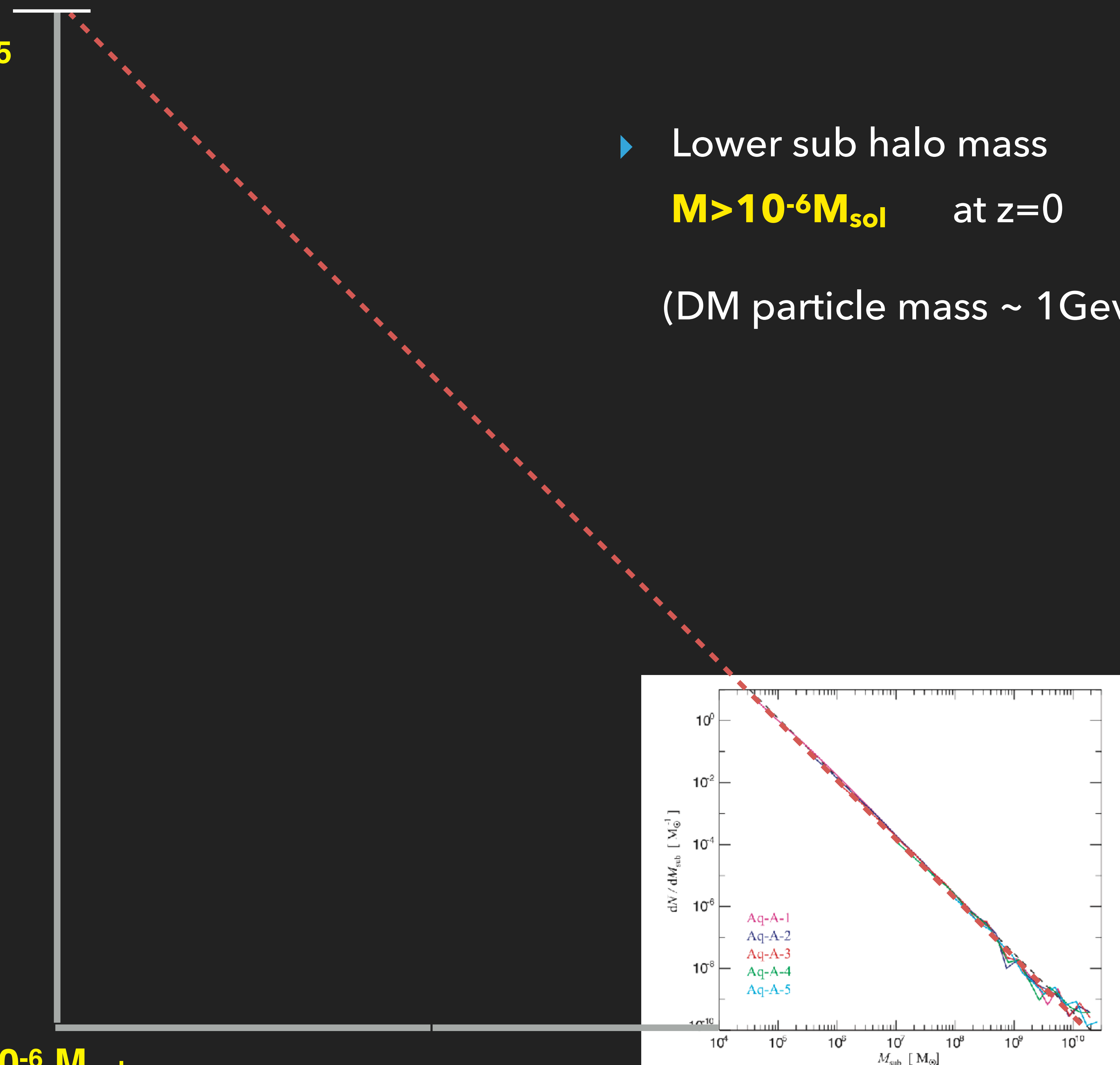
DM-ONLY N-BODY SIMULATIONS



- Cold Dark Matter (CDM) subhaloes follow a power-law mass function

EXTRAPOLATION DOWN TO FREE-STREAMING LENGTH

$N \sim 10^{15}$



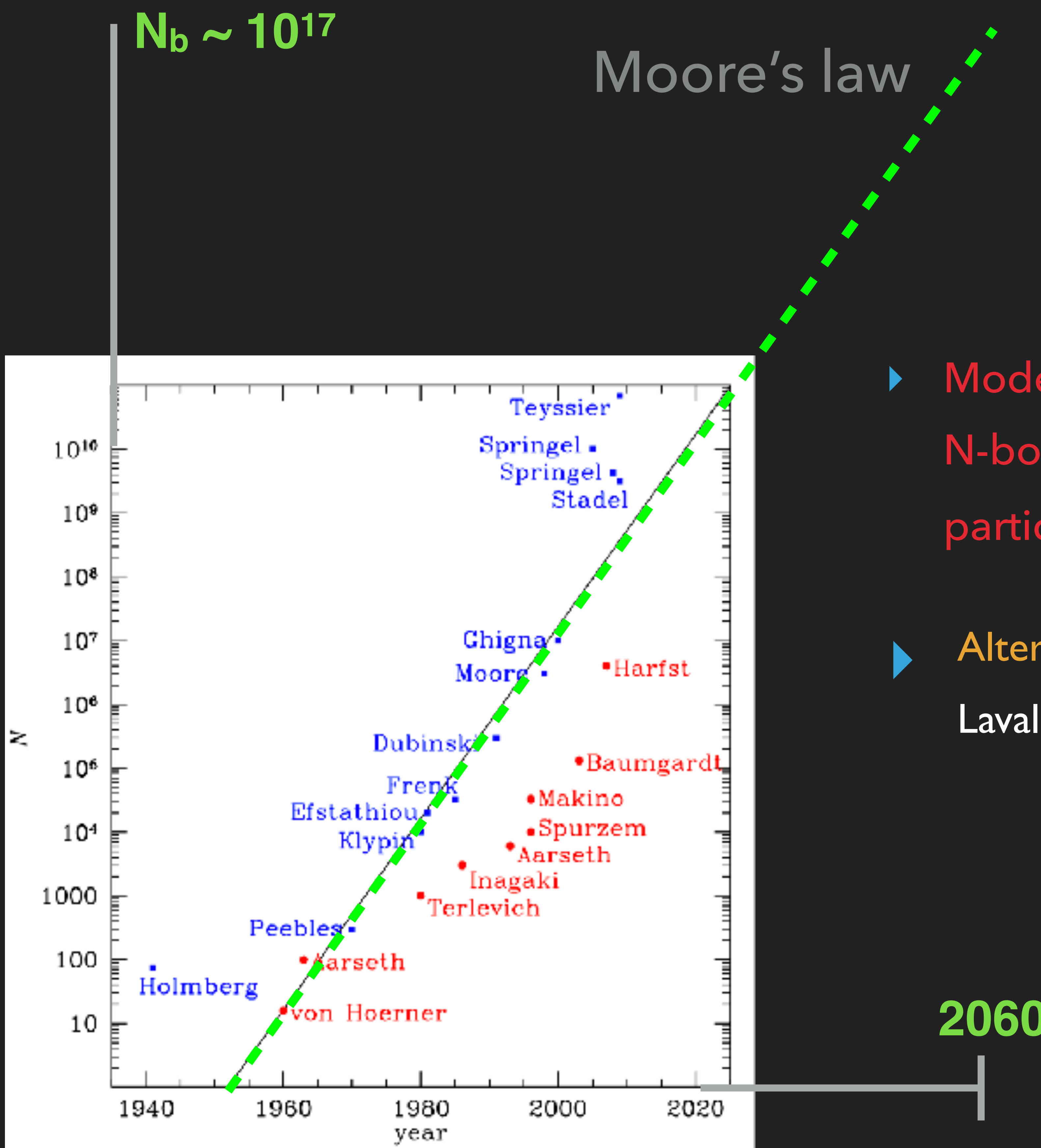
- Lower sub halo mass

$M > 10^{-6} M_{sol}$ at z=0

(DM particle mass ~ 1Gev)

WIMPS: **10^{-6} M_{sol}**

LIMITATIONS OF N-BODY METHODS



- Modelling ~10^{15} sub haloes with current N-body methods and at least >100 particles per sub halo requires **N_b ~10^{17}**
- Alternative: semi-analytic models (e.g. Stef & Lavalle 2017)

1. **Theory:** model effects of microhaloes on the dynamics of visible systems

2. **Sensitivity :** are dynamics useful to test perfect fluid model?

3. **Detectability:** what observations provide strongest constraints?

GRAVITATIONAL PERTURBATIONS ON VISIBLE OBJECTS

$$\frac{d\mathbf{r}^2}{dt^2} = -\nabla\Phi(\mathbf{r}, t) + \sum_{i=1}^{N} \mathbf{f}_i$$

stochastic equations of motion

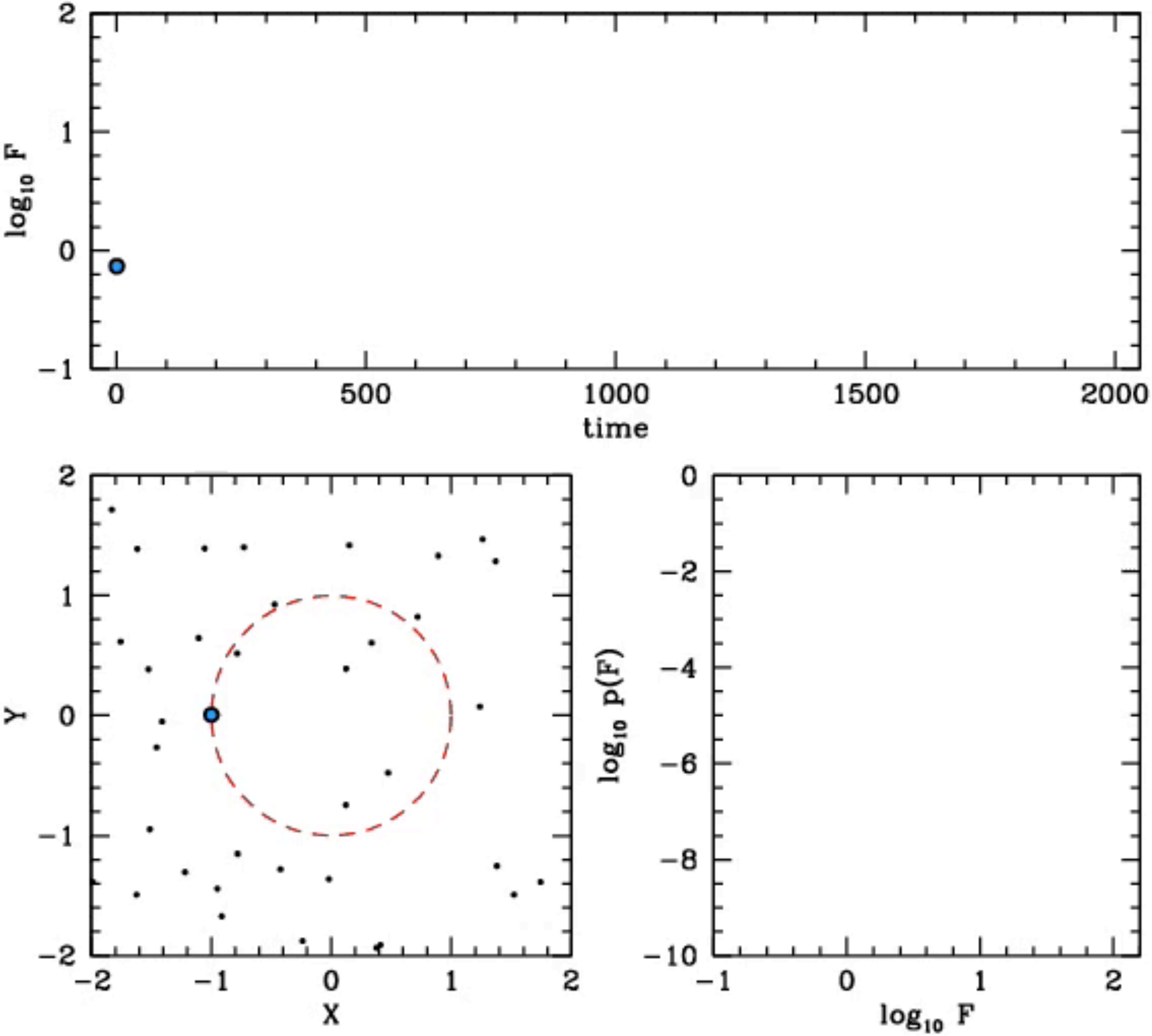
$$\mathbf{F} \equiv \sum_{i=1}^{N} \mathbf{f}_i$$

noise term induced by substructures

A large population of extended substructures generates a stochastic gravitational field **F** that is fully specified by the function

p(F) : probability to experience a combined force within F, F+dF

Direct calculation of $\mathbf{F} = \sum_{i=1}^{N} \mathbf{f}_i$



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Holtsmark (1919) method to describe motion of charged particles in an **homogeneous** plasma

$$p(\mathbf{F}) = \frac{1}{V}\int d^3r_1 \times \ldots \times \frac{1}{V}\int d^3r_N \delta\left(\mathbf{F} - \sum_i \mathbf{f}_i\right)$$

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FT

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$$\tilde{p}(\mathbf{k}) = \int d^3F e^{i\mathbf{k}\cdot\mathbf{F}} p(\mathbf{F}) = \exp[-\phi(\mathbf{k})]$$

where $\phi(\mathbf{k}) \equiv n \int_V d^3r \left(1 - e^{i\mathbf{k}\cdot\mathbf{f}}\right)$ and $n \equiv N/V$

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FT

$$\tilde{p}(\mathbf{k}) = \int d^3F\,e^{i\mathbf{k}\cdot\mathbf{F}}p(\mathbf{F}) = \exp[-\phi(\mathbf{k})]$$

FT^{-1}

$$p(\mathbf{F}) = \tfrac{1}{(2\pi)^3}\int d^3k\,\exp\big[-i\mathbf{k}\cdot\mathbf{F} - \phi(\mathbf{k})\big]$$

where $\phi(\mathbf{k}) \equiv n\int_V d^3r\left(1 - e^{i\mathbf{k}\cdot\mathbf{f}}\right)$ and $n \equiv N/V$

POINT-MASS PARTICLES: HOLTSMARK (1919) DISTRIBUTION

$$\mathbf{f} = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

$$\phi(\mathbf{k}) = \tfrac{4}{15}(2\pi GM)^{3/2} n k^{3/2} \equiv a k^{3/2}$$

isotropically oriented in Fourier space

$$p(\mathbf{F}) = \tfrac{1}{2\pi^2}\int_0^\infty dk\, k^2 \exp(-a k^{3/2}) \frac{\sin(kF)}{kF}$$

Holtsmark distribution (1919)

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Holtsmark distribution (1919)

- Weak-force limit

$$\lim_{F\to 0} p(\mathbf{F}) = \frac{1}{2\pi^2}\int_0^\infty k\, k^2 \exp(-ak^{3/2}) = \frac{1}{3\pi^2 a^2}.$$

Flat
larger number of distant substructures cancels with declining forces

-Strong-force limit

$$\lim_{F\to\infty} p(\mathbf{F}) \approx \frac{1}{2}(GM)^{3/2} n F^{-9/2}.$$

Power-law tail
due to contribution of single **nearest** particle

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Holtsmark distribution (1919)

** (finite) force average

$$\langle F \rangle = \int_0^\infty d^3F\, p(\mathbf{F}) F = 8.879\, GMn^{2/3}$$

** (divergent) amplitude of fluctuations

$$\langle F^2 \rangle = \int_0^\infty d^3F\, p(\mathbf{F}) F^2 \propto \int \frac{dF}{F^{1/2}} \to \infty$$

Fluctuations grow up to arbitrarily-large values as integration time-steps increases

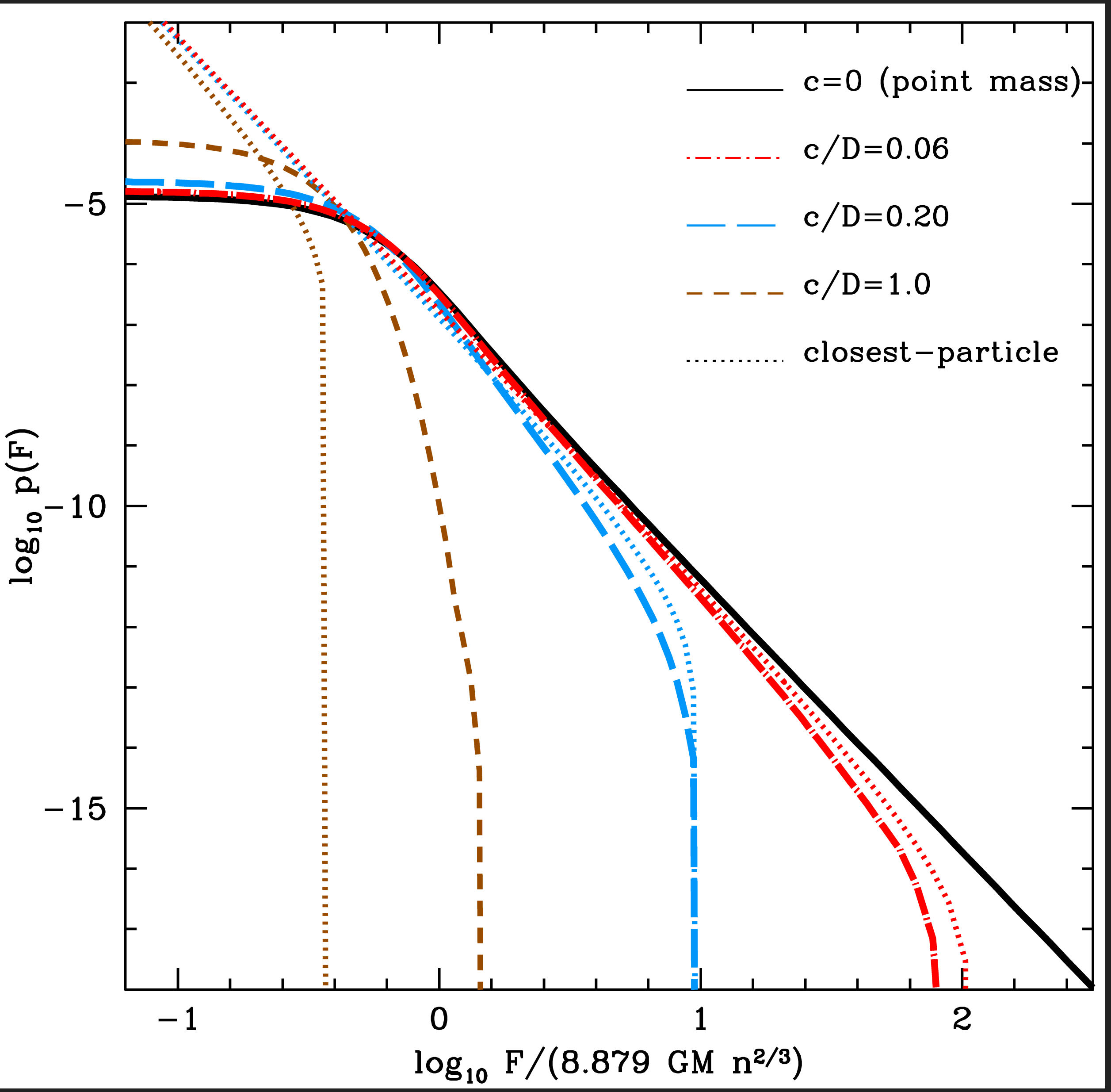
EXTENDED SUBSTRUCTURES

$$\mathbf{f} = -\frac{GM}{(r+c)^2}\hat{\mathbf{r}}$$

$$\phi(\mathbf{k}) = A(k)k^{3/2}$$

Hernquist (1990) sphere

$$\rho = \frac{M}{2\pi c^3}\frac{1}{(r/c)(1+r/c)^3}$$



p(F)
* truncated at F = f_0=GM/c^2
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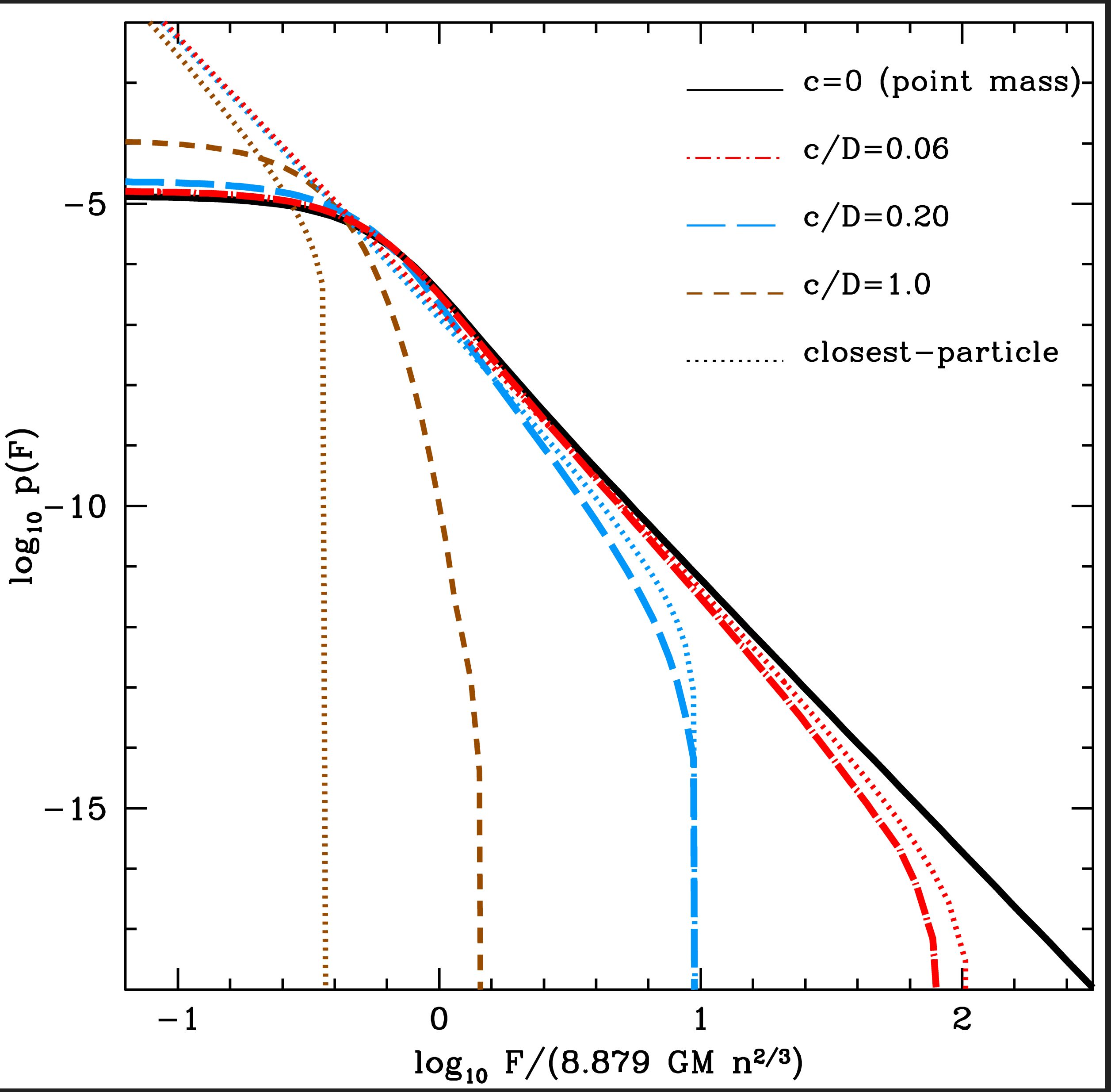
(non-divergent) amplitude of fluctuations

$$\langle F^2\rangle = \int_0^\infty d^3F p(\mathbf{F})F^2$$

$$= 2\pi(GM)^{3/2}n\int_0^{f_0}\frac{dF}{F^{1/2}}\left(1-\sqrt{F/f_0}\right)^2$$

$$= \frac{4\pi}{3}(GM)^{3/2}n\sqrt{f_0}$$

$$= \frac{4\pi}{3}\frac{(GM)^2 n}{c}$$



p(F)
* truncated at $F = f_0 = GM/c^2$
* point-mass as $c/D \longrightarrow 0$

the truncation of the large-force spectrum implies a finite variance

CDM SUBHALO POPULATIONS

Stats subhalo ensembles defined by

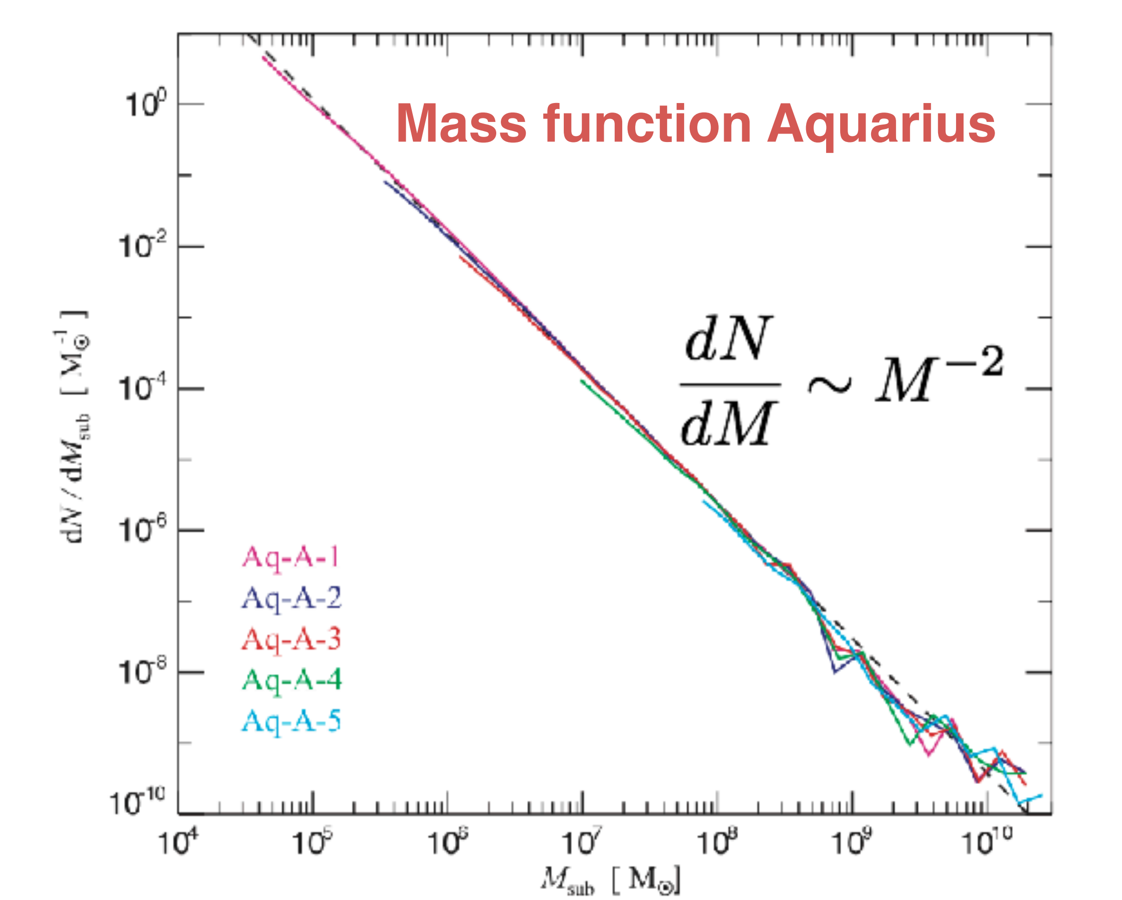
1) mass function

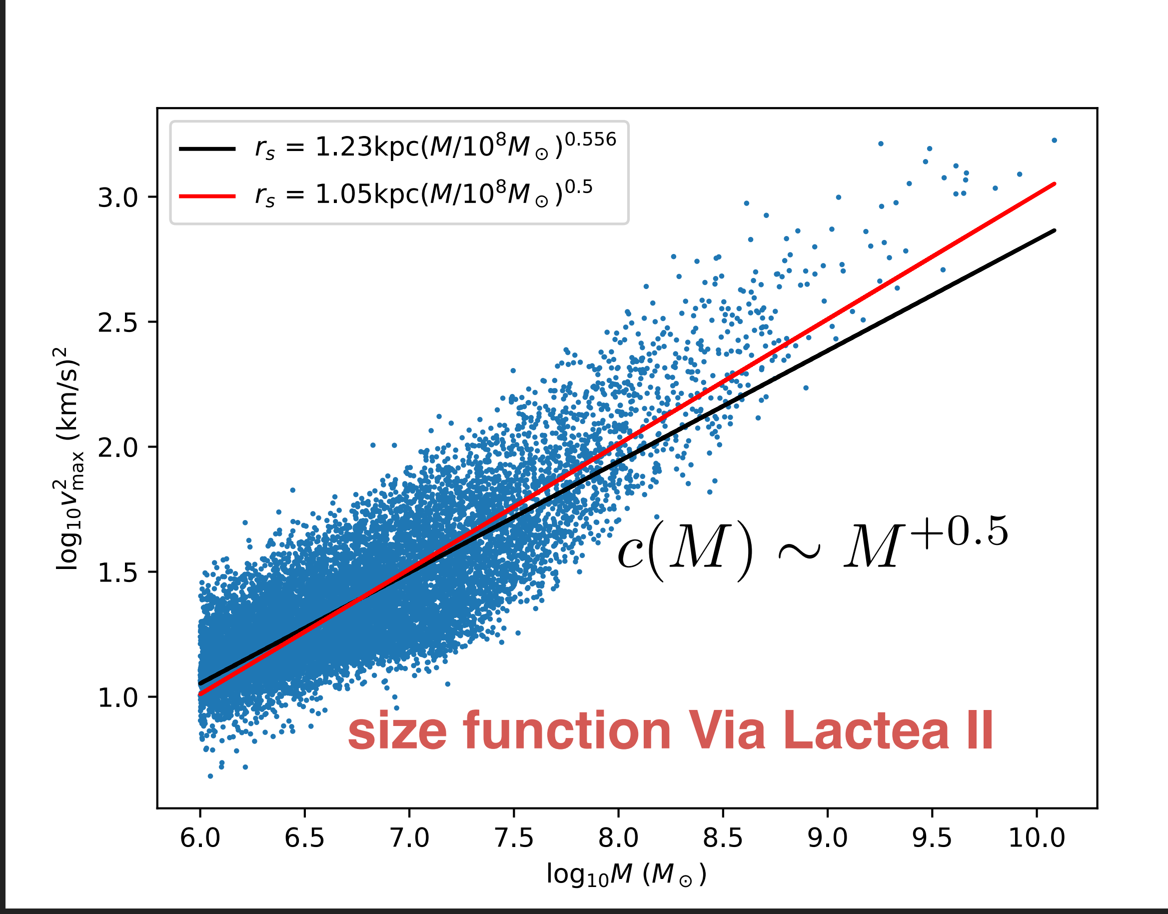
2) size function

$$n \rightarrow \int\int \frac{d^2 n}{dM\,dc} dM\,dc$$

$$\frac{d^2 n}{dM\,dc} = B_0 \left(\frac{M}{M_0}\right)^{\alpha} \delta\left[c - c_0 \left(\frac{M}{M_0}\right)^{\beta}\right]$$

double power-law





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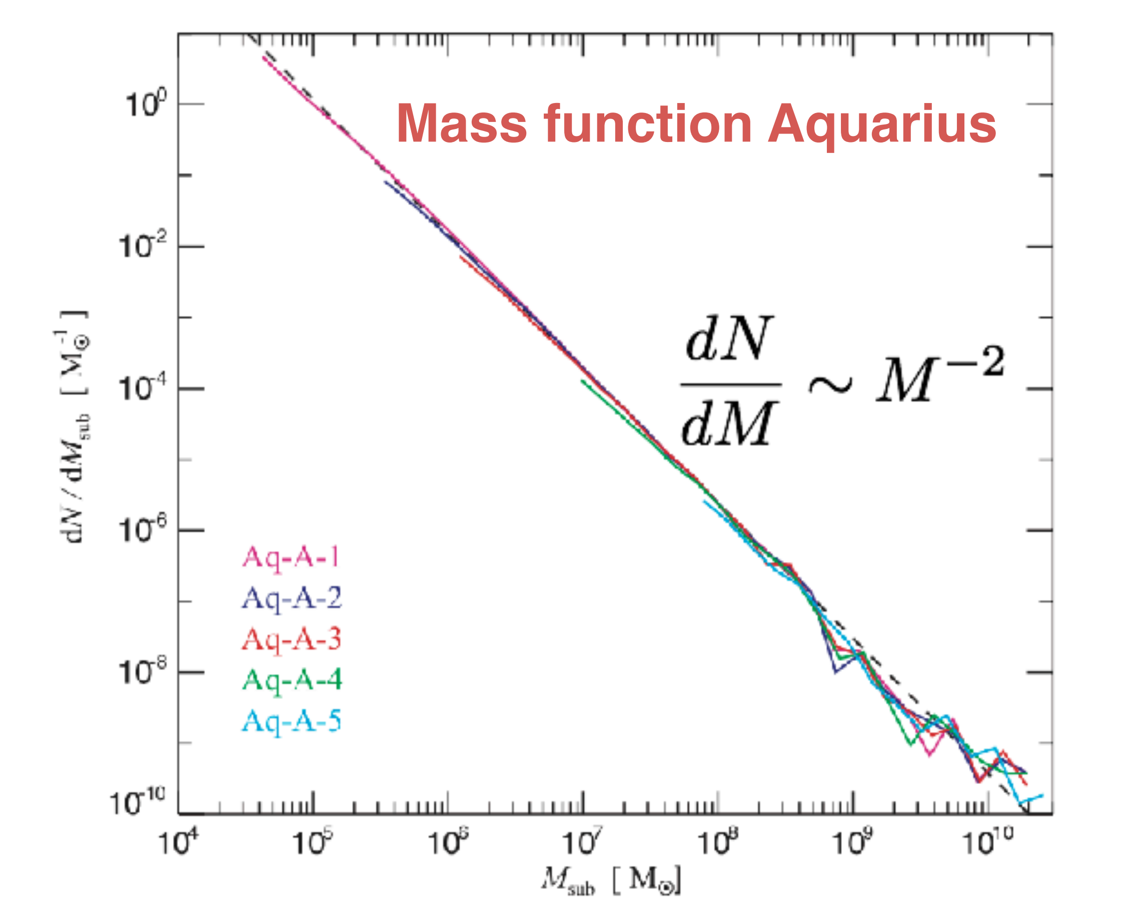
double power-law

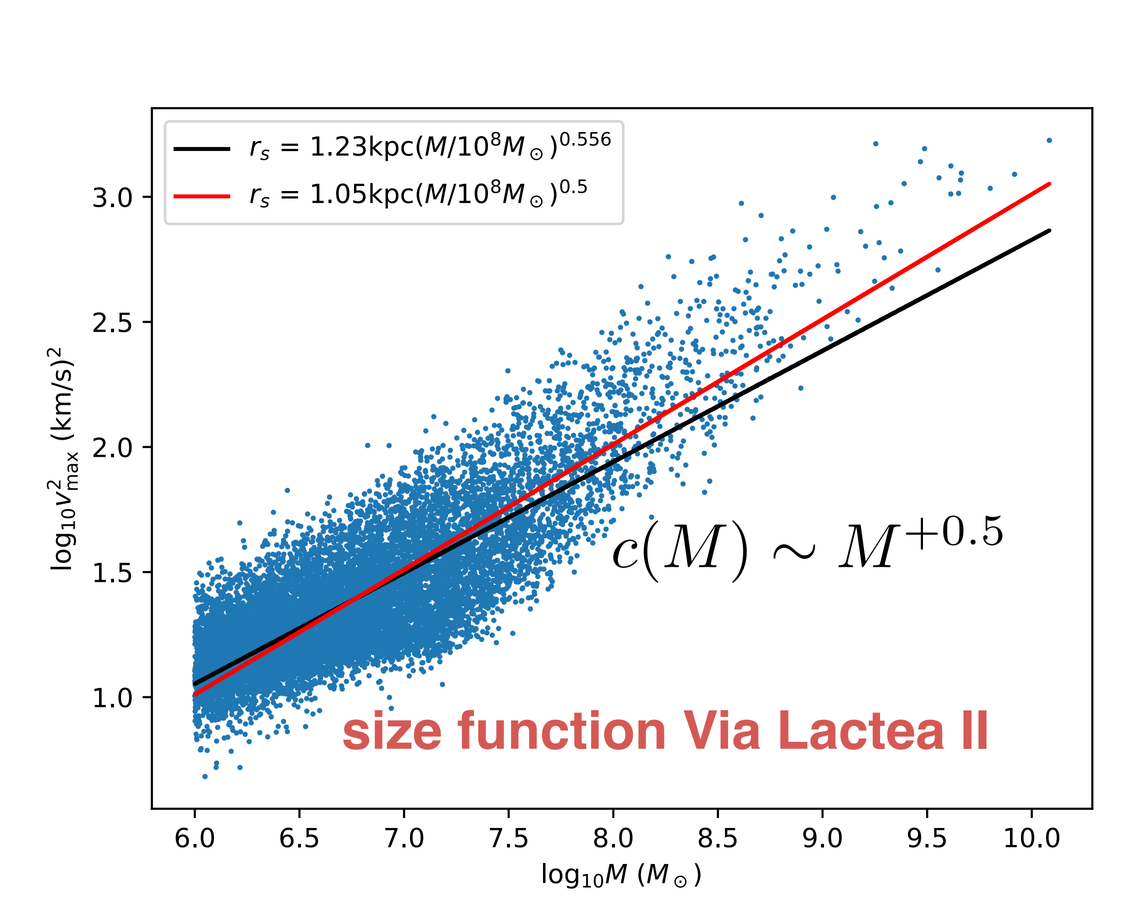
$$\langle F^2 \rangle = \int_{M_1}^{M_2} dM \int dc \frac{d^2 n}{dM\, dc} \frac{4\pi}{3} \frac{G^2 M^2}{c}$$

$$= \frac{4\pi}{3} M_0^{\beta-\alpha} \frac{G^2 B_0}{c_0} \times \frac{M_2^{3+\alpha-\beta} - M_1^{3+\alpha-\beta}}{3+\alpha-\beta}$$

CDM: $\boxed{3 + \alpha - \beta \simeq 0.5}$ → Fluctuations dominated by massive objects

$$\langle F^2 \rangle \simeq \frac{8\pi}{3} M_0^{\beta-\alpha} \frac{G^2 B_0}{c_0} M_2^{+0.5}$$

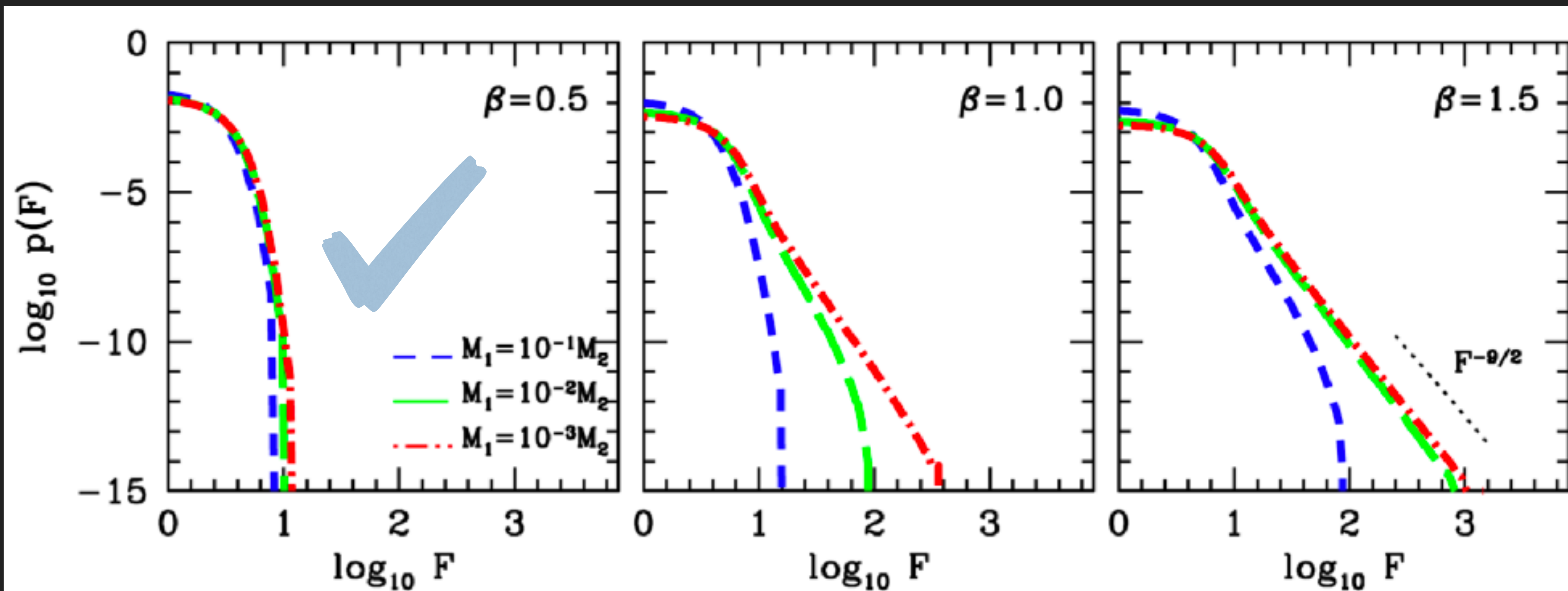




CDM SUBHALO POPULATIONS

$\alpha = -2$

steepening size function (β)



3 + α - β = + 0.5

3 + α - β = 0.0

3 + α - β = - 0.5

MACRO-STRUCTURE DOMINATED

MICRO-STRUCTURE DOMINATED

STOCHASTIC TIDAL FIELD

$$\frac{d^2\mathbf{R}'}{dt^2} = -\underbrace{\nabla\Phi_s(\mathbf{R}')}_{\text{self-gravity}} + \underbrace{\sum_{i=1}^{N} t_i}_{\text{fluctuating tidal tensor}} \cdot \mathbf{R}',$$

$$\sum_{i=1}^{N} t_i = \sum_{i=1}^{N} \frac{\partial f_i^k}{\partial x^j} = \frac{\partial}{\partial x^j} \sum_{i=1}^{N} f_i^k = \frac{\partial F^k}{\partial x^j}$$

combined tidal tensor (noise term)

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combined tidal tensor (noise term)

diagonalize:

$$\mathbf{f}_t \equiv t \cdot \mathbf{R}' \approx \lambda R$$

$$\mathbf{F}_t = \sum_{i=1}^{N} \lambda_i R = \mathbf{\Lambda} R$$

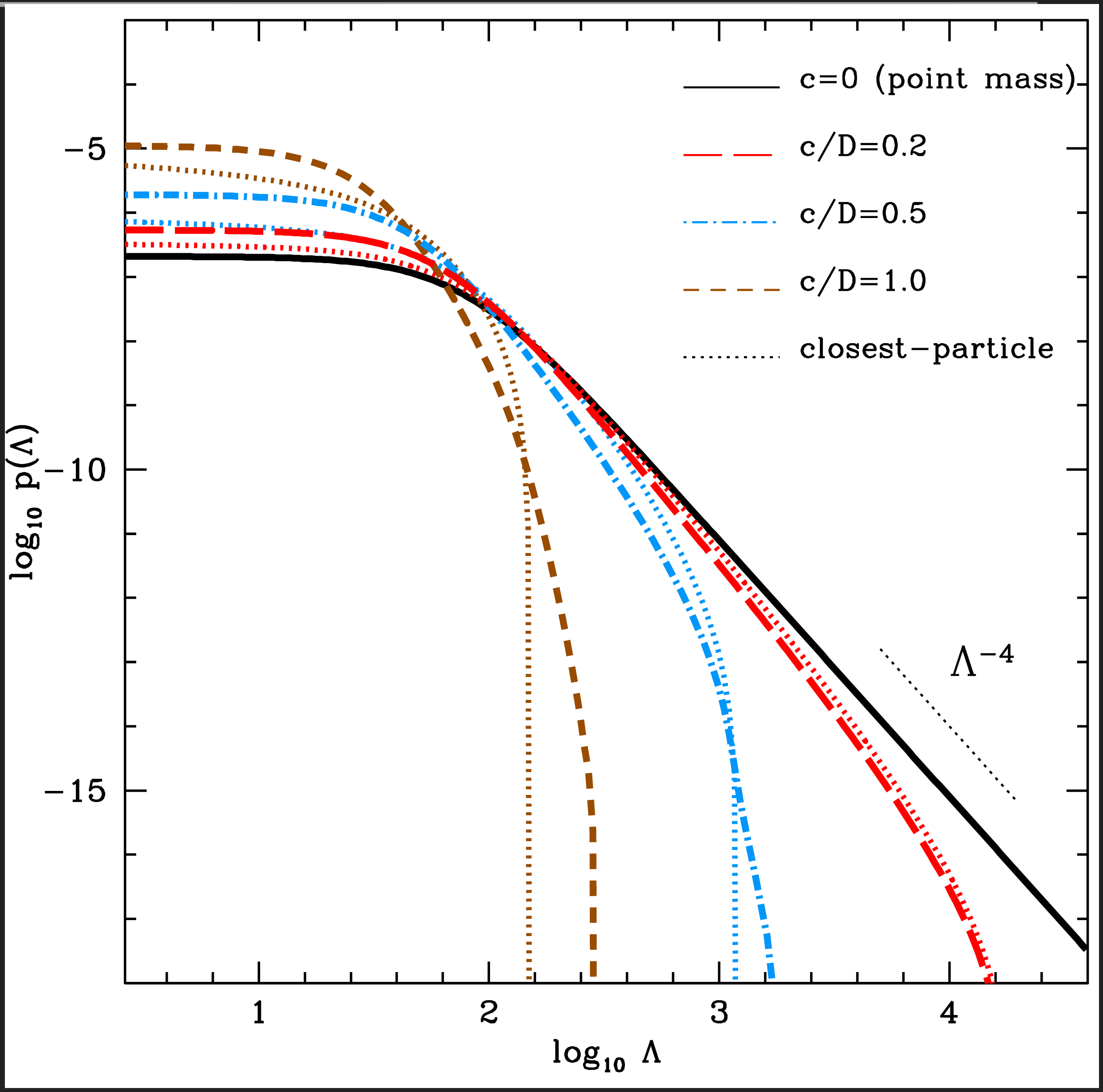
combined **tidal force** induced by a set of N-substructures

EXTENDED SUBSTRUCTURES : TIDES

$$\lambda = \frac{2GM}{(r+c)^3}\mathbf{u}$$

Hernquist (1990) sphere

$$\phi(\mathbf{k}) = n \int_V d^3r \left(1 - e^{i\mathbf{k}\cdot\lambda}\right)$$
$$= Q(k)k$$



p(Λ)

* truncated at $\Lambda = \lambda_0 = 2GM/c^3$
* point-mass as c/D—>0

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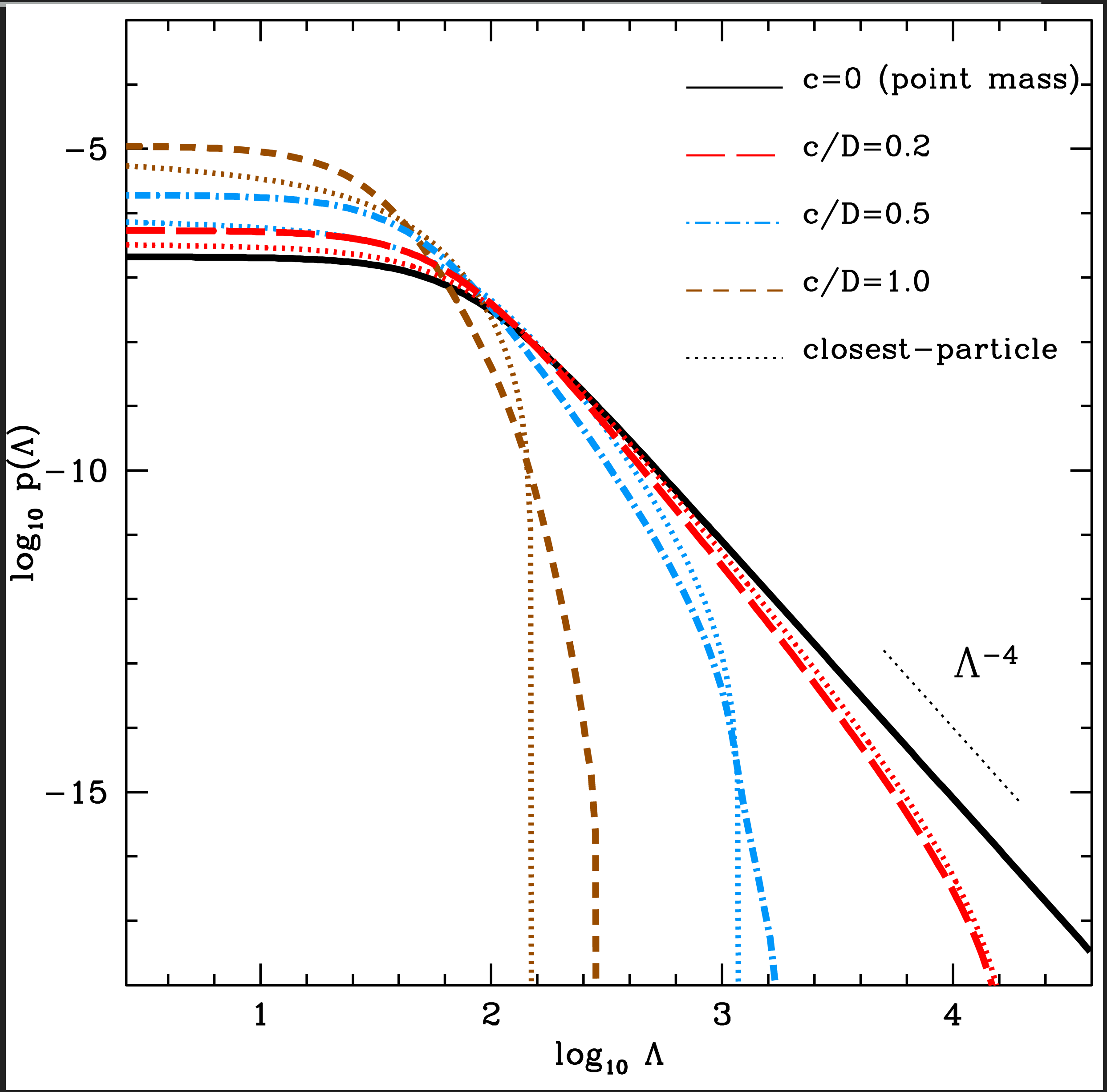

(non-divergent) amplitude of TIDAL fluctuations

$$\langle \Lambda^2 \rangle = \int_0^{\lambda_0} d^3\Lambda \, p(\Lambda) \Lambda^2$$

$$\approx \frac{8\pi}{15} \frac{(GM)^2 n}{c^3}$$

p(Λ)
* truncated at $\Lambda = \lambda_0 = 2GM/c^3$
* point-mass as c/D—>0

…. to be compared w/FORCE fluctuations

$$\langle F^2 \rangle = \frac{4\pi}{3} \frac{(GM)^2 n}{\boxed{c}}$$

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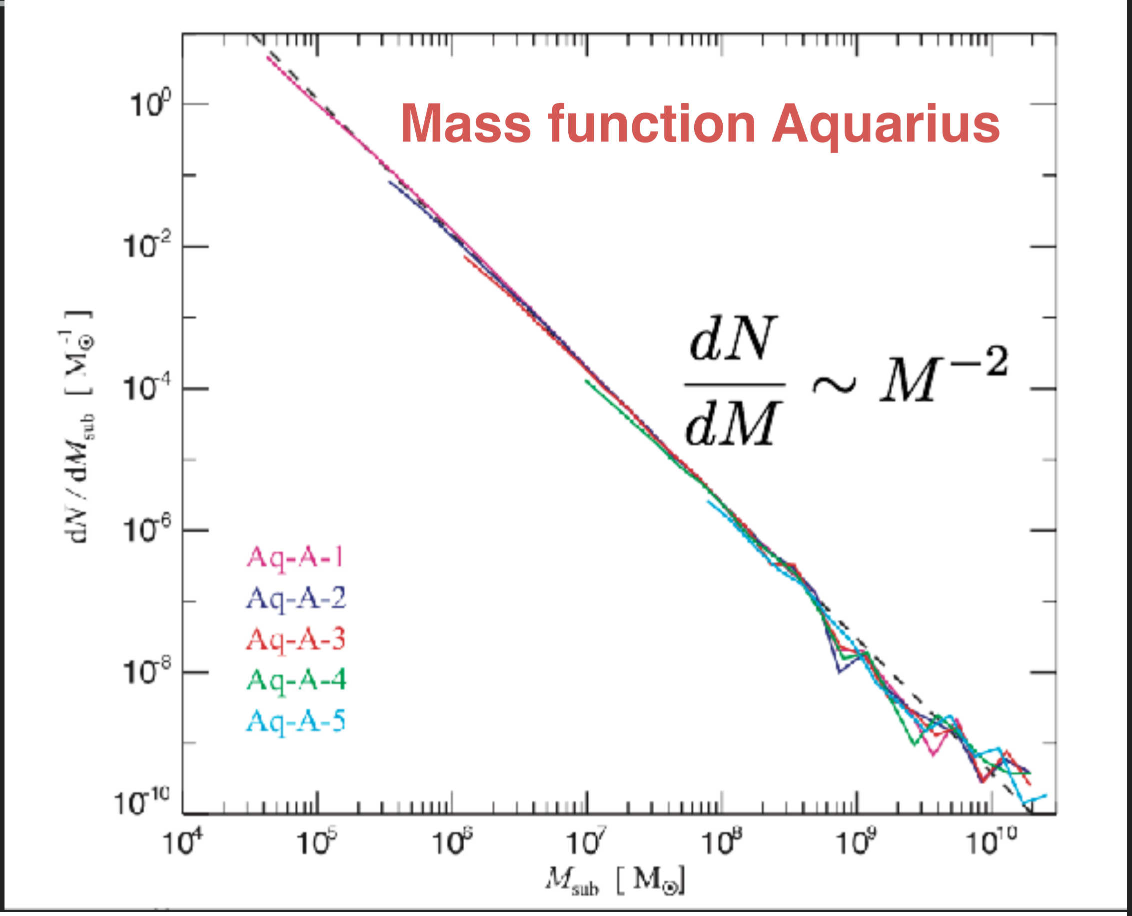
double power-law

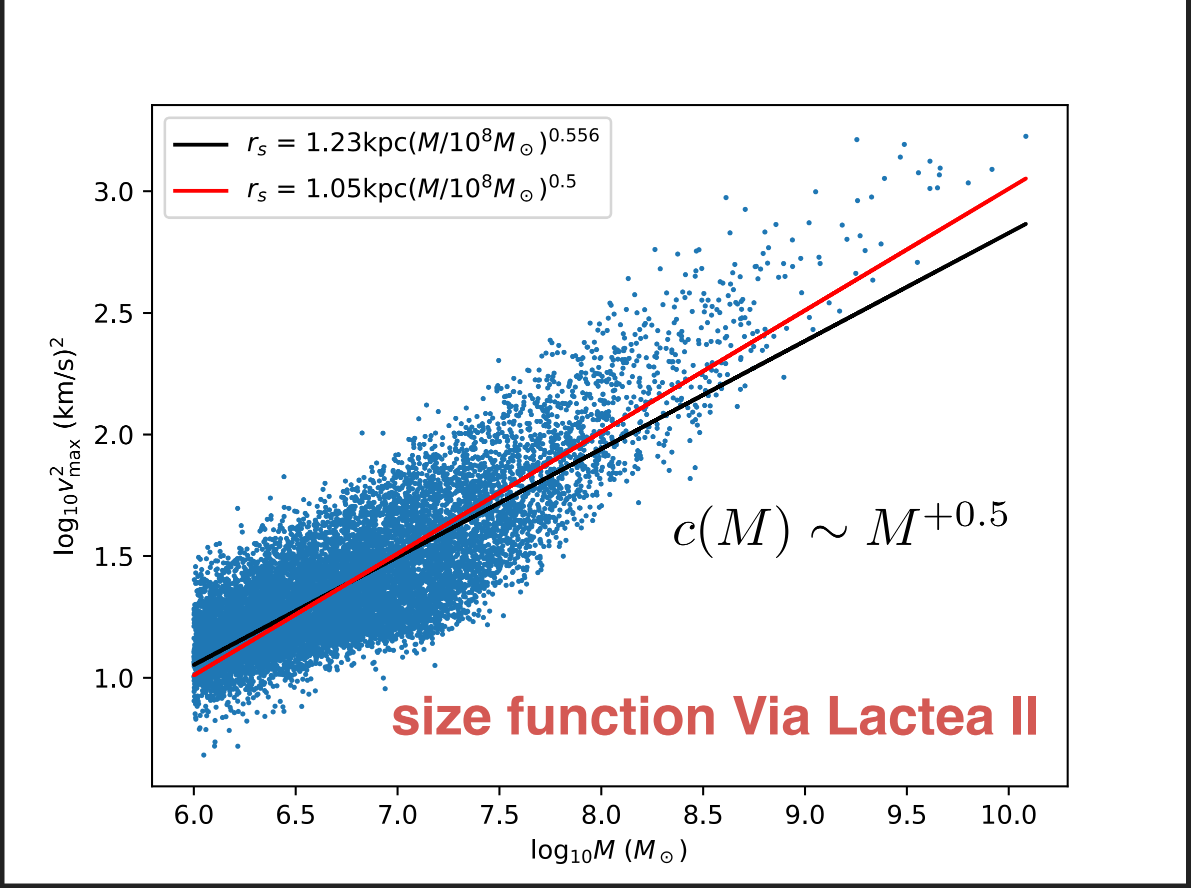
$$\langle \Lambda^2 \rangle = \frac{8\pi}{15} \frac{G^2 B_0}{M_0^{\alpha}} \int_{M_1}^{M_2} dM \, \frac{M^{\alpha+2}}{c(M)^3}$$

$$= \frac{8\pi}{15} M_0^{3\beta-\alpha} \frac{G^2 B_0}{c_0^3} \times \frac{M_2^{3+\alpha-3\beta} - M_1^{3+\alpha-3\beta}}{3+\alpha-3\beta}$$

CDM: $\boxed{3 + \alpha - 3\beta \simeq -0.5}$ ⟹ Fluctuations dominated by **smallest** objects

$$\langle \Lambda^2 \rangle \simeq \frac{16\pi}{15} M_0^{3\beta-\alpha} \frac{G^2 B_0}{c_0^3} \times M_1^{-1/2}$$





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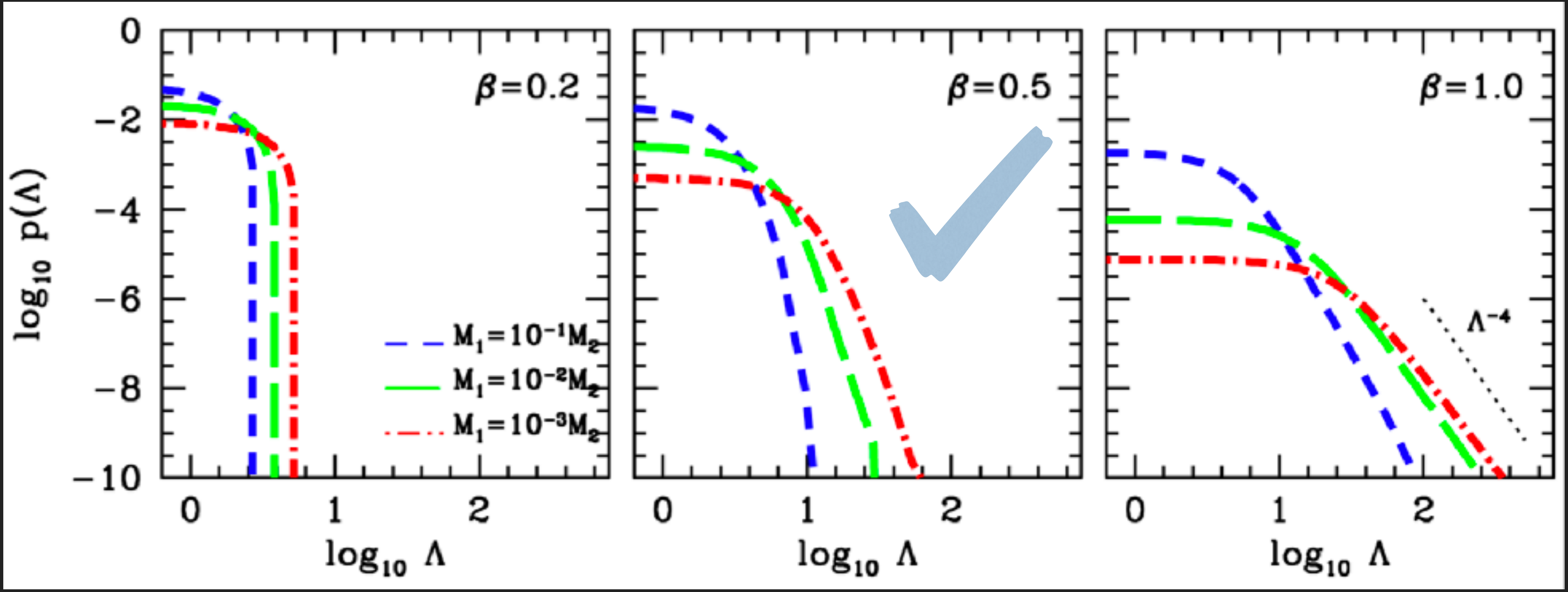


3 + α - 3β = + 0.4

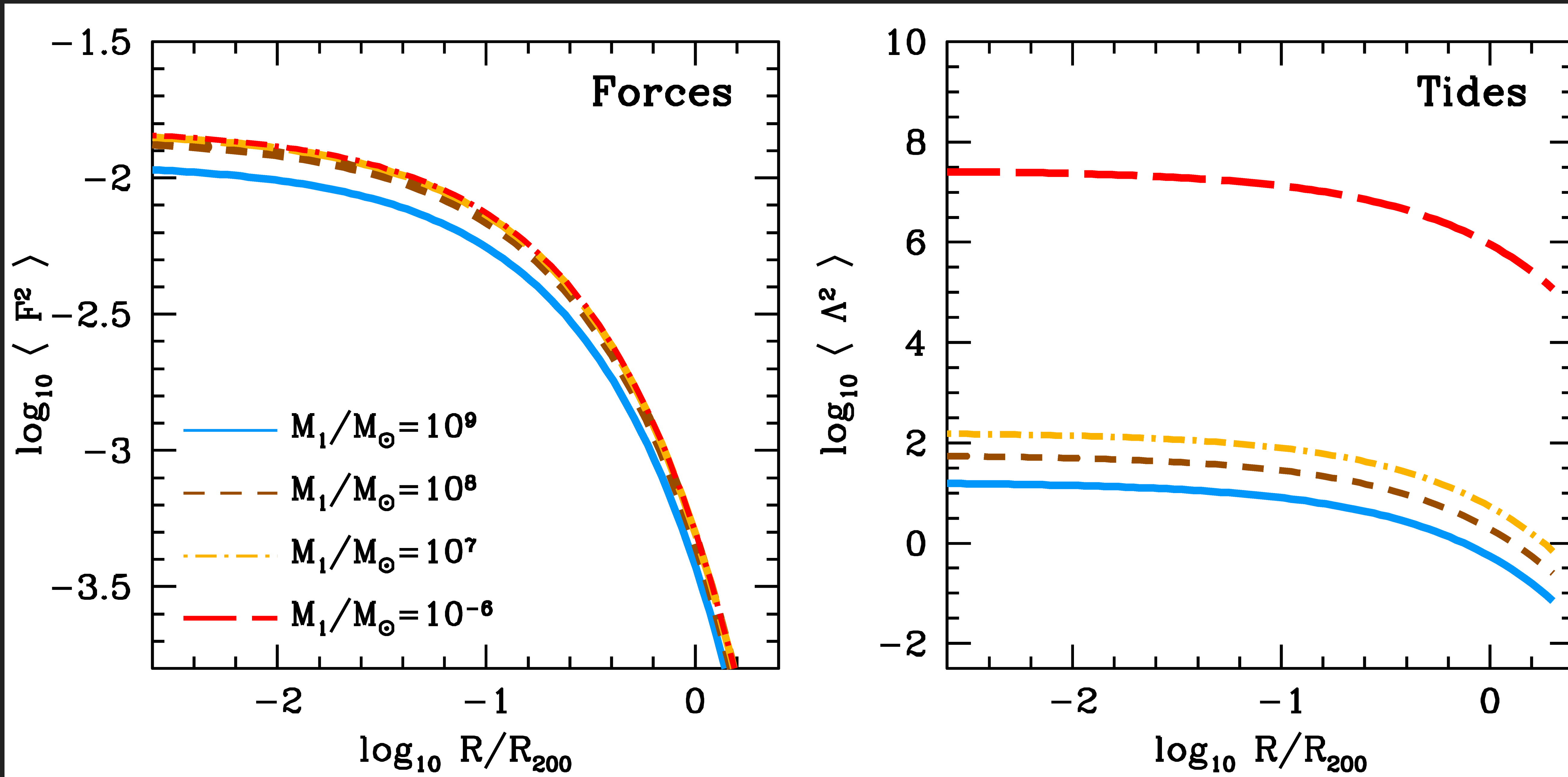
MACRO-STRUCTURE DOMINATED

3 + α - 3β = -0.5

MICRO-STRUCTURE DOMINATED

3 + α - 3β = -2

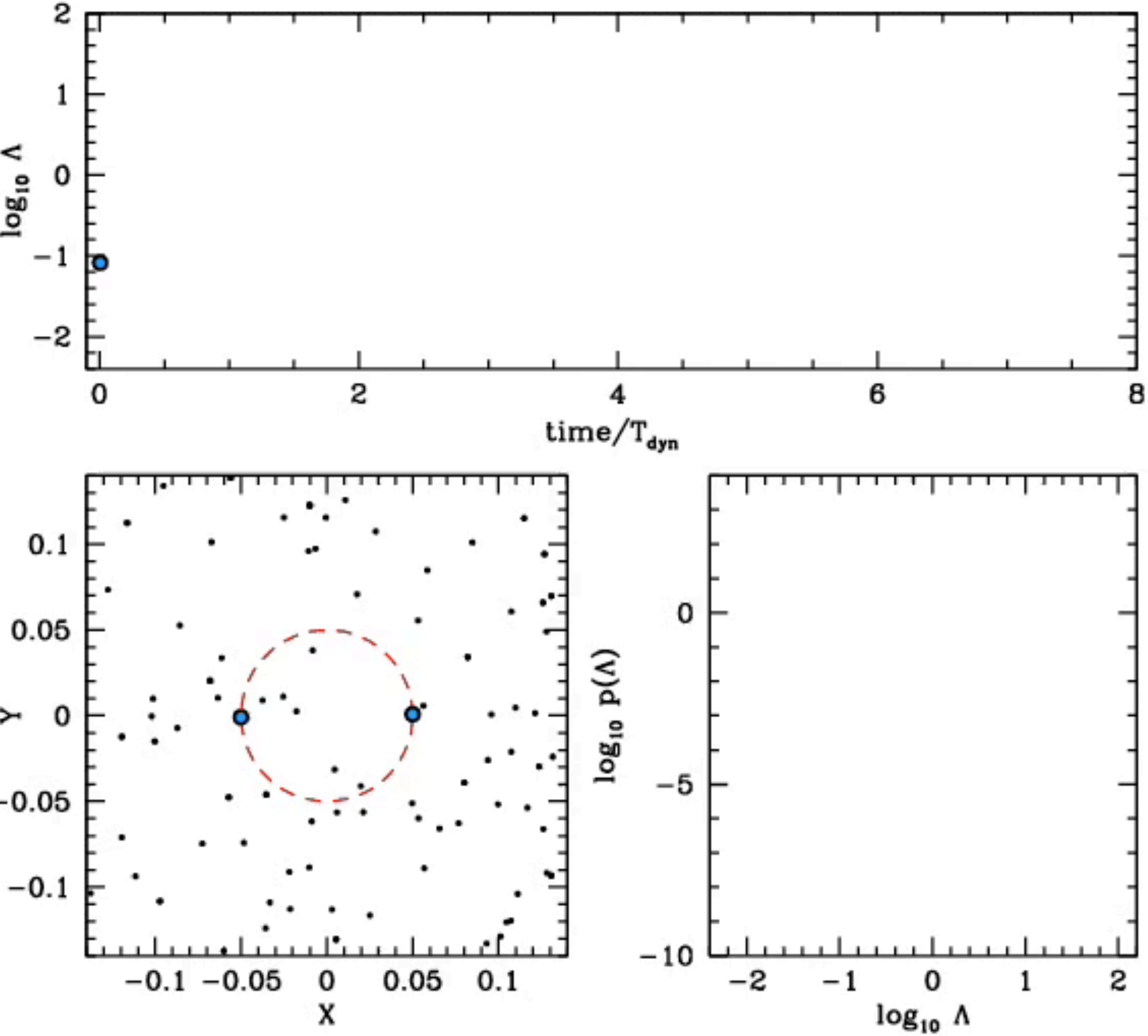
MICRO-STRUCTURE DOMINATED



- **N-body** simulations that do not resolve M_1 strongly **underestimate** tidal fluctuations
- **WDM - CDM** widely different tidal fields **(~6 orders of magnitude <Λ²>)**

How can we constrain p(Λ) ??

Direct calculation of $\mathbf{F}_t = \sum_{i=1}^{N} t_i \cdot \mathbf{R}'$



HEATING OF WEAKLY-BOUND OBJECTS: WIDE BINARIES

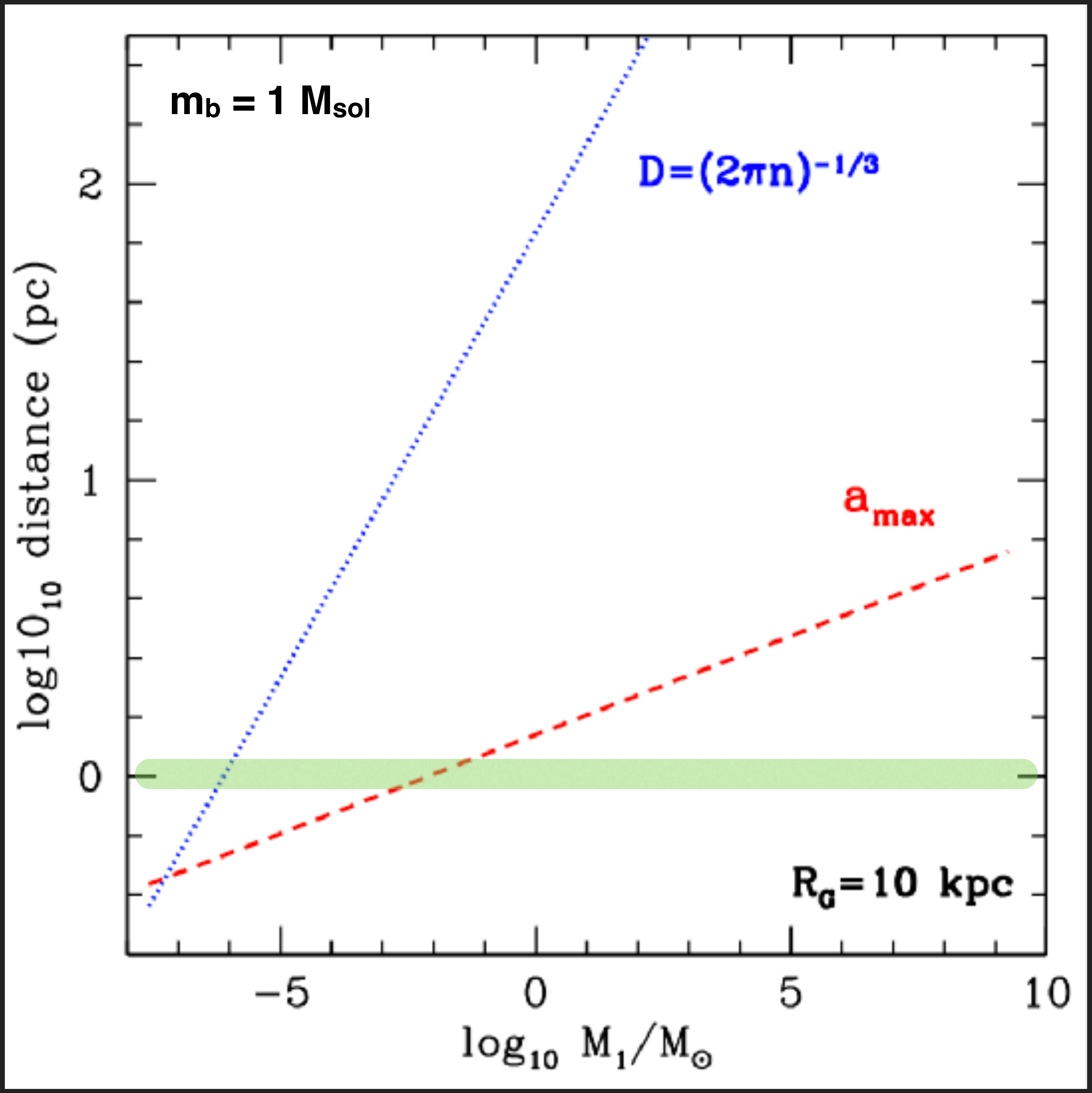
$$G m_b / R_{\rm max}^2 = \langle \Lambda^2 \rangle^{1/2} R_{\rm max}$$

(self-gravity = external force)

semi-major axis: $R_{\rm max} = 2 a_{\rm max}$

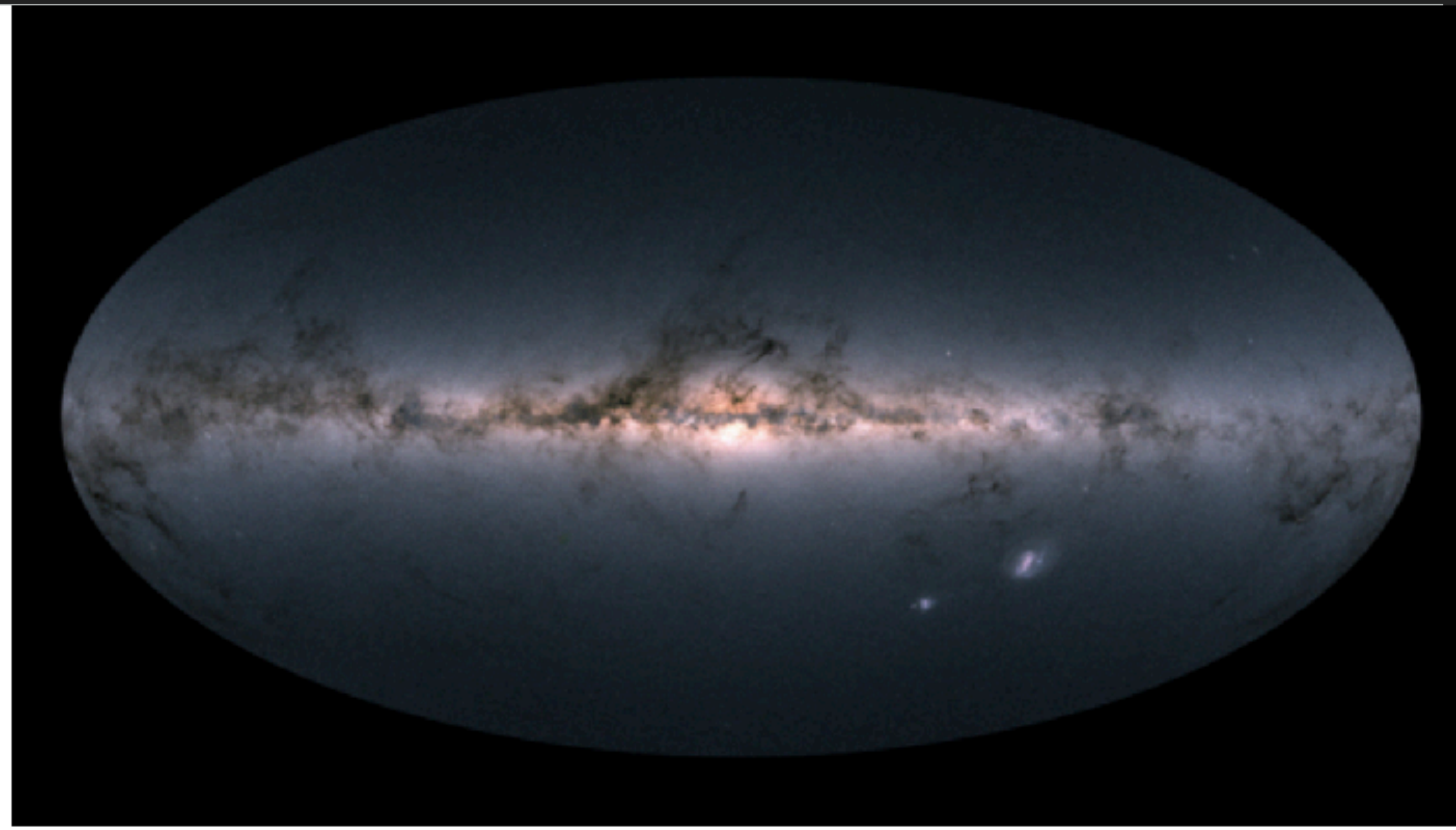
binaries with semi-major axis
a < 1pc
sensitive to subhaloes with
M<1 M_{sol}

Estimates for extrapolation of Aquarius halo



WIDE BINARIES IN GAIA

Oh+17 using TGAS

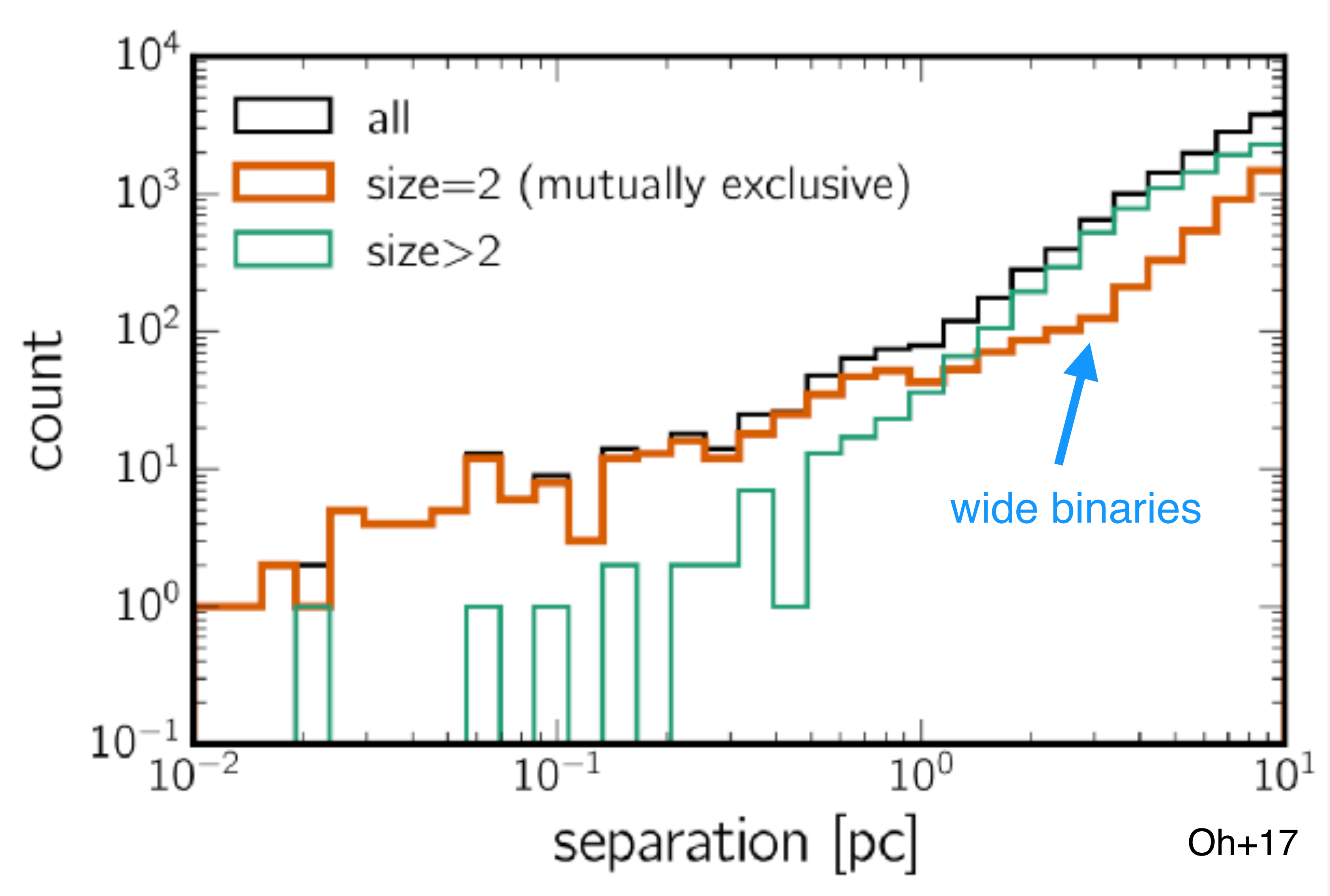


Gaia creates richest star map of our Galaxy – and beyond

25 April 2018 ESA's Gaia mission has produced the richest star catalogue to date, including high-precision measurements of nearly 1.7 billion stars and revealing previously unseen details of our home Galaxy. Read more

WIDE BINARIES IN GAIA DR1

comoving pairs of stars in TGAS (2e6 stars)



Oh+17 using TGAS

Full dynamical modelling of observations required

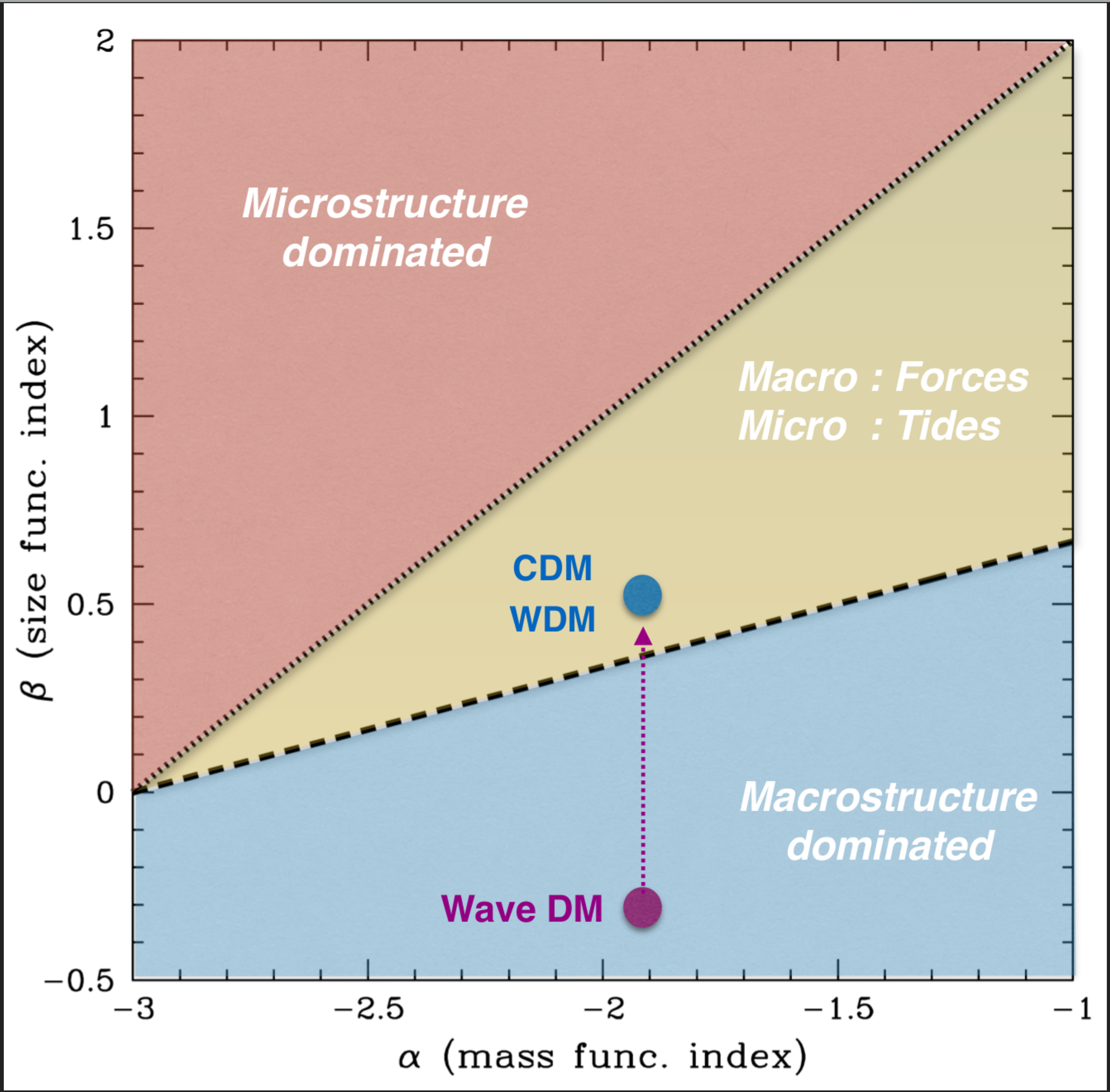
- stellar ages
- wide binary formation models: isolated / association
- orbits in smooth MW potential + statistical sampling p(Λ)
- add baryonic substructures (stars, GMCs)

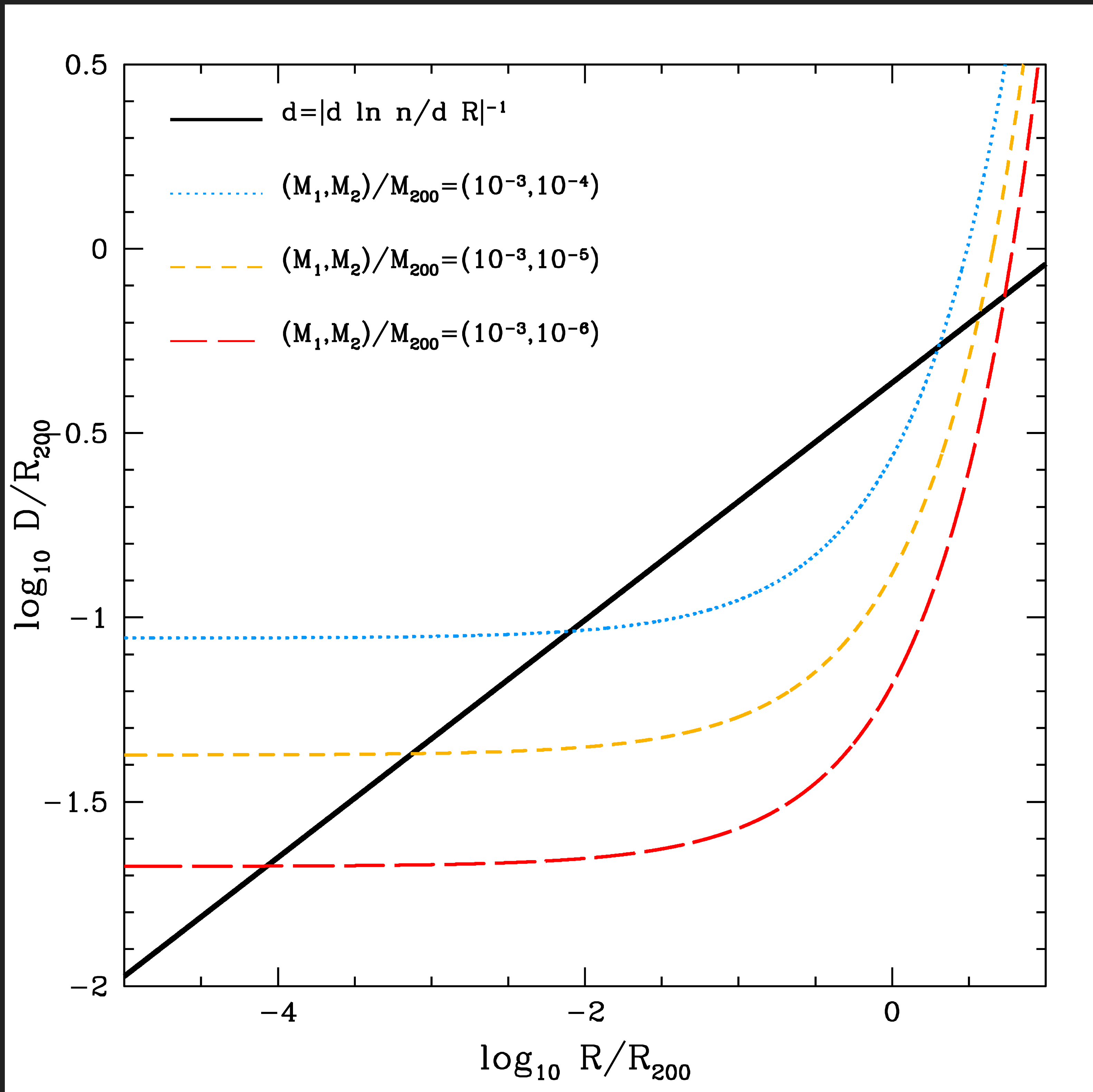
SUMMARY

- WIMP-CDM deviations from perfect fluid $M > M_{FS} \sim 10^{-6}\, M_{sol}$
- N-body simulations of structure formation $M > 10^4 - 10^6\, M_{sol}$ (10 - 12 orders of magnitude above free streaming length!)
- Combined subhalo forces dominated by largest satellites (orbits/tidal streams not sensitive to small substructures)
- Stochastic fluctuations of tidal field are dominated by smallest sub haloes ($M \sim M_{FS}$)
- Opens up the possibility to test CDM mass function by measuring tidal fluctuations
- Gaia will be provide key observations of weakly-bound systems in the Milky Way (e.g. wide binaries)

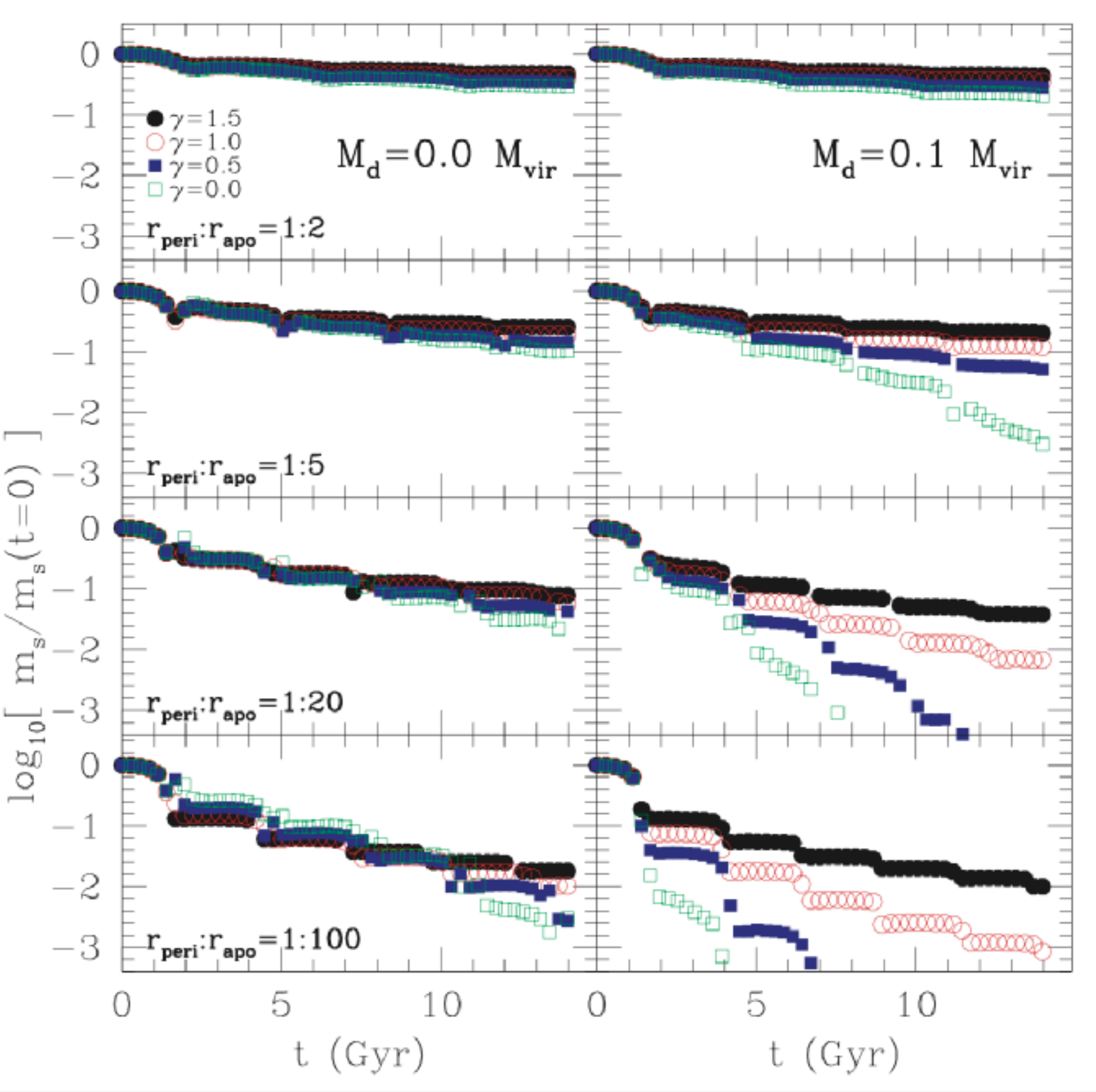
QUESTIONS

- Mass & size function down $M\sim10^{-6}M_{sol}$ at z=0? N-body methods?
- Time-evolution   d^2 **n(r,t)** / dMdc ?
- Disruption of micro haloes by tidal field of MW disc ?
- Baryonic substructures (GMCs, stars, etc)   Hydro + SF + feedback?


Microstructure dominated
Macro : Forces
Micro : Tides
CDM
WDM
Wave DM
Macrostructure dominated
2
1.5
1
0.5
0
−0.5
−3
−2.5
−2
−1.5
−1
β (size func. index)
α (mass func. index)



d=|d ln n/d R|$^{-1}$
$(M_1,M_2)/M_{200}=(10^{-3},10^{-4})$
$(M_1,M_2)/M_{200}=(10^{-3},10^{-5})$
$(M_1,M_2)/M_{200}=(10^{-3},10^{-6})$
0.5
0
−0.5
−1
−1.5
−2
−4
−2
0
$\log_{10}$ D/R_{200}
$\log_{10}$ R/R_{200}

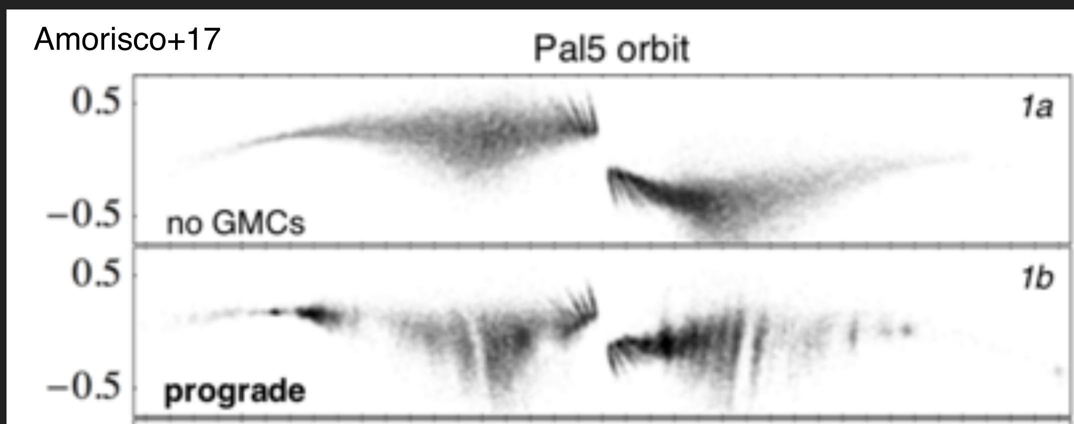


$\gamma=1.5$
$\gamma=1.0$
$\gamma=0.5$
$\gamma=0.0$
$M_d=0.0\ M_{vir}$
$M_d=0.1\ M_{vir}$
$r_{peri}:r_{apo}=1:2$
$r_{peri}:r_{apo}=1:5$
$r_{peri}:r_{apo}=1:20$
$r_{peri}:r_{apo}=1:100$
$\log_{10}[\ m_s/m_s(t=0)\ ]$
t (Gyr)
t (Gyr)
0
−1
−2
−3
0
5
10

DETECTION OF DARK SUB HALOES

Detection extremely challenging owing to the low-mass and lack of luminous matter

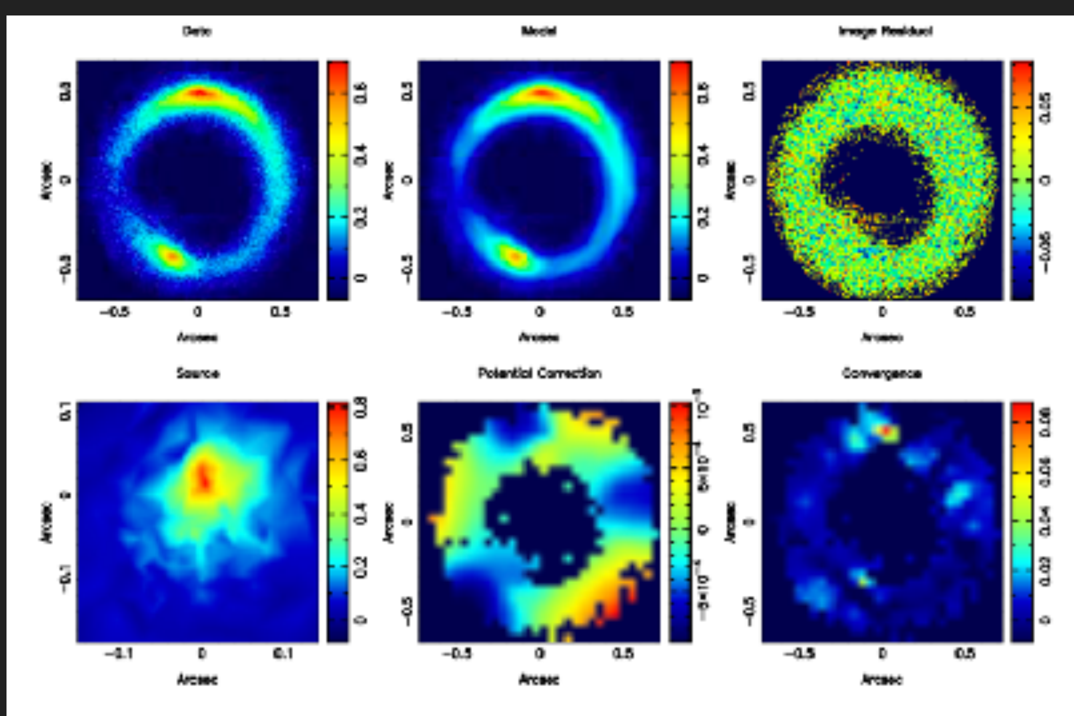
1) TIDAL STREAM HEATING



* **sensitive only to 'massive' substructures with $M>10^6\ M_{sol}$** (Ibata et al. 2002; Johnston et al. 2002; Yoon et al. 2011; Carlberg 2014; Erkal & Belokurov 2015; Ngan et al. 2016, Erkal et al. 2016; Bovy et al. 2017)

* **can be perturbed by GMCs with $M<10^7 M_{sol}$** (Amorisco+17)

2) LENSING



Vegetti+12

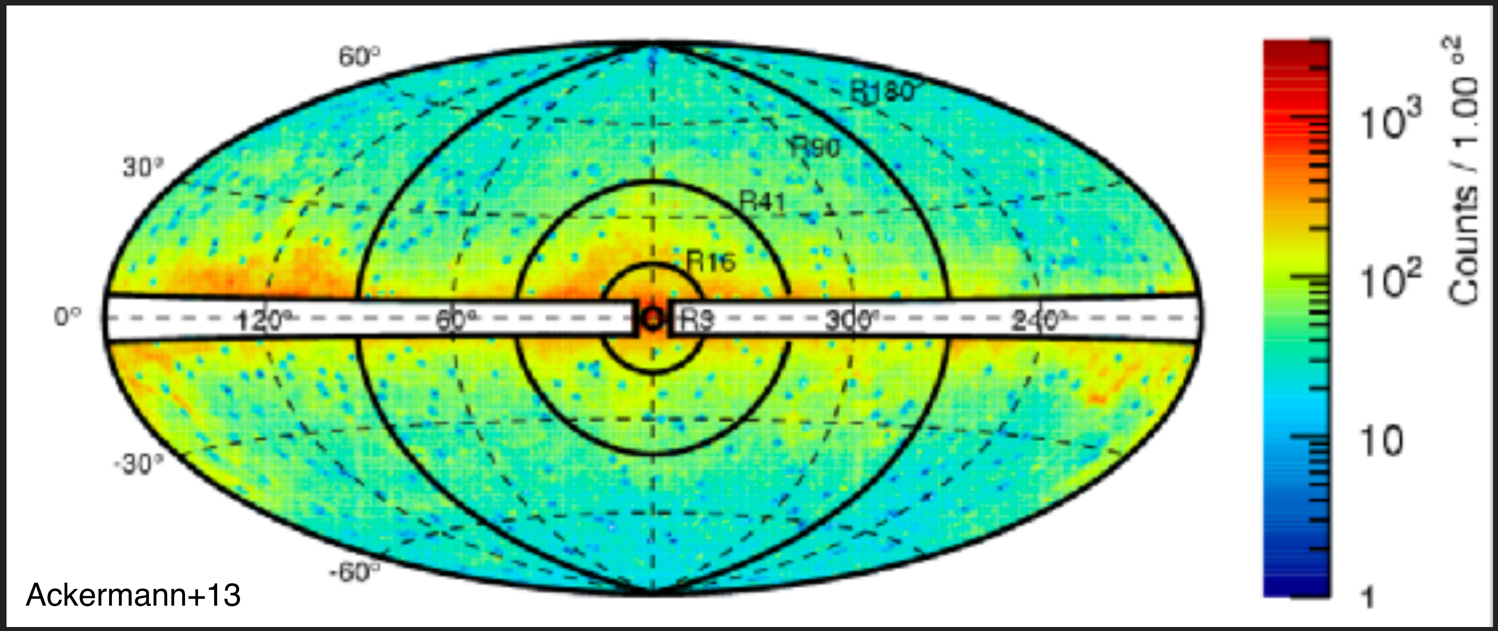
* **perturb. in Einstein rings around lensed galaxies are also expected to be dominated by relatively massive subhaloes with $M>10^7\ M_{sol}$** (Li et al. 2016)

* **number of lensed galaxies relatively small. Very high-resolution data needed** (Koopmans 2005; Vegetti & Koopmans 2009; Li et al. 2013; Vegetti et al. 2014)

DETECTION

Detection extremely challenging owing to the low-mass and lack of luminous matter

3) GAMMA-RAYS ANNIHILATION



* **Degenerate with DM-particle model**
(e.g.Diemand et al. 2005; Koushiappas 2009; Ackermann et al. 2014; Bringmann et al. 2014; Zechlin+17)

* **baryonic sources (e.g. BHs, neutron stars…)**

HOLTSMARK (1919) DISTRIBUTION

$$\mathbf{f} = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

$$\phi(\mathbf{k}) = \tfrac{4}{15}(2\pi GM)^{3/2} n k^{3/2} \equiv a k^{3/2}$$

isotropically oriented in Fourier space

$$p(\mathbf{F}) = \tfrac{1}{2\pi^2}\int_0^\infty dk\, k^2 \exp(-ak^{3/2})\,\tfrac{\sin(kF)}{kF}$$

Holtsmark distribution (1919)

- Weak-force limit

$$\lim_{F\to 0} p(\mathbf{F}) = \frac{1}{2\pi^2}\int_0^\infty \text{k}\, k^2 \exp(-ak^{3/2}) = \frac{1}{3\pi^2 a^2}.$$

Flat.
larger number of distant substructures cancels with declining forces

-Strong-force limit

$$\lim_{F\to\infty} p(\mathbf{F}) \approx \frac{1}{2}(GM)^{3/2} n F^{-9/2}.$$

Power-law tail
due to contribution of single nearest particle

probability to find **closest** particle between **r**, r+d**r**

$$p(\mathbf{r})d^3r \sim \exp\left(-\tfrac{4}{3}\pi r^3 n\right) 4\pi r^2 n dr$$

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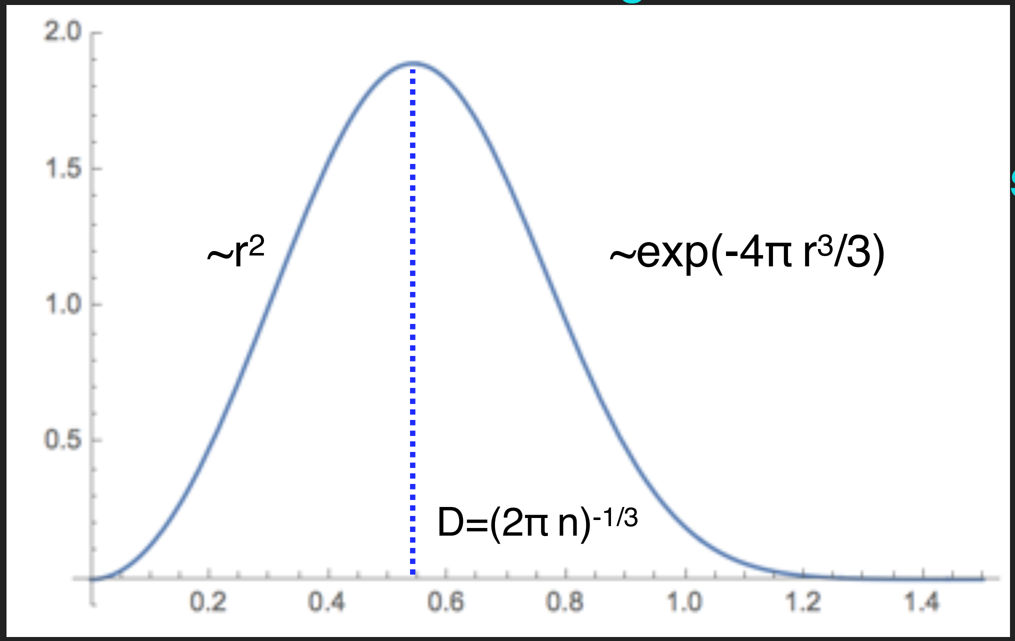

st

probability to find **closest** particle between **r**, r+dr

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due to contribution of single nearest particle

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$$p(\mathbf{r})d^3r \sim \exp\left(-\tfrac{4}{3}\pi r^3 n\right)4\pi r^2 n\,dr$$

$$|F| = \frac{GM}{r^2} \qquad dr = \tfrac{1}{2}(GM)^{1/2}F^{3/2}dF$$

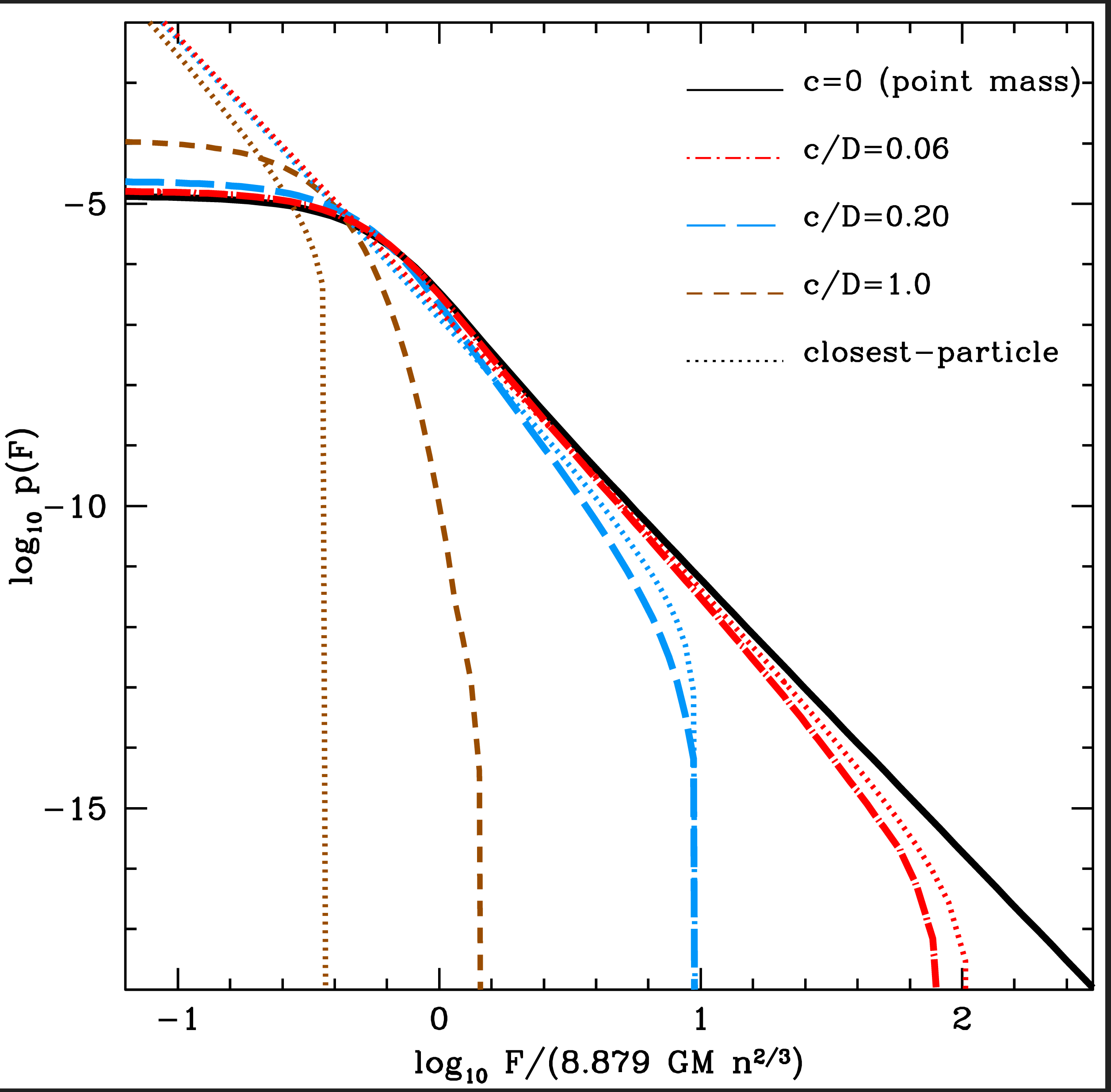
$$p(\mathbf{r})d^3r = p(\mathbf{F})d^3F \qquad p(\mathbf{F}) = n\frac{r^2 dr}{F^2 dF} = \tfrac{1}{2}(GM)^{3/2} n F^{-9/2}$$

EXTENDED SUBSTRUCTURES

$$\mathbf{f} = -\frac{GM}{(r+c)^2}\hat{\mathbf{r}} \quad \Rightarrow \quad \phi(\mathbf{k}) = A(k)k^{3/2}$$

Hernquist (1990) sphere

$$\rho = \frac{M}{2\pi c^3}\frac{1}{(r/c)(1+r/c)^3}$$



probability to find **closest** particle between **r**, r+dr

$$p(\mathbf{r})d^3r \sim \exp\left(-\tfrac{4}{3}\pi r^3 n\right)4\pi r^2 n dr$$

p(F)
* truncated at F = f_0=GM/c^2
* point-mass as c/D—>0

$$F = \frac{GM}{(r+c)^2} \quad \Rightarrow \quad dr = \tfrac{1}{2}(GM)^{1/2}F^{3/2}dF$$

$$p(\mathbf{r})d^3r = p(\mathbf{F})d^3F \quad \Rightarrow \quad p(\mathbf{F}) = \tfrac{1}{2}(GM)^{3/2}nF^{-9/2}\left(1-\sqrt{F/f_0}\right)^2 \ \text{for} \ F < f_0 = \frac{GM}{c^2}$$

SUBHALOES IN THE MILKY WAY

Aquarius simulations (Springel+08)

$$\frac{dn}{dM}(R,M) = B_0\left(\frac{M}{M_0}\right)^{\alpha} \exp\left\{-\frac{2}{\gamma}\left[\left(\frac{R}{R_{-2}}\right)^{\gamma} - 1\right]\right\}$$

$$(\gamma, R_{-2}, R_{200}) = (0.678, 199\,\mathrm{kpc}, 246\,\mathrm{kpc})$$

$$B_0 = 2.02 \times 10^{-13} M_\odot^{-1}\mathrm{kpc}^{-3}$$

$$M_0 = 2.52 \times 10^{7} M_\odot$$

$$M_{200} = 1.84 \times 10^{12} M_\odot$$

$$N = 4\pi \int_0^{R_{200}} R^2 dR \int_{M_1}^{M_{200}} dM \frac{dn}{dM}$$

$$\sim 10^{15} \quad \text{for} \quad M_1 = 10^{-6} M_\odot$$

Via Lactea simulations (Diemand+07)

$$c = c_0 \left(\frac{M}{M_0}\right)^{\beta}$$

$$(c_0, \beta) = (0.55\,\mathrm{kpc}, 0.5)$$

(bold) extrapolation of size function down to

$$M_1 = 10^{-6} M_\odot$$

$$c_1 \sim 0.4\,\mathrm{pc}$$

(remains unexplored !!)

A STATISTICAL METHOD (HOLTSMARK 1919)

$$p(\mathbf{F}) = \frac{1}{V}\int d^3r_1 \times \ldots \times \frac{1}{V}\int d^3r_N \delta\left(\mathbf{F} - \sum_i \mathbf{f}_i\right)$$

N>>1 substructures distributed **homogeneously** over a volume **V**

A STATISTICAL METHOD (HOLTSMARK 1919)

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Fourier transform

$$\tilde{p}(\mathbf{k}) = \int d^3F e^{i\mathbf{k}\cdot\mathbf{F}} p(\mathbf{F}) = \frac{1}{V}\int d^3r_1 \times \ldots \times \frac{1}{V}\int d^3r_N \int d^3F e^{i\mathbf{k}\cdot\mathbf{F}} \delta\left(\mathbf{F} - \sum_j \mathbf{f}_j\right)$$

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spatially uncorrelated = random locations

$$= \left[\tfrac{1}{V}\int d^3r e^{i\mathbf{k}\cdot\mathbf{f}}\right]^N$$

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N-power: $\left[1 - \frac{1}{V}\int_V d^3r\left(1 - e^{i\mathbf{k}\cdot\mathbf{f}}\right)\right]^N \approx \exp\left[-n\int_V d^3r\left(1 - e^{i\mathbf{k}\cdot\mathbf{f}}\right)\right]$ for N>>1

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Number density = **N / V**

A STATISTICAL METHOD (HOLTSMARK 1919)

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define: $\phi(\mathbf{k}) \equiv n\int_V d^3r\left(1 - e^{i\mathbf{k}\cdot\mathbf{f}}\right)$

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define: $\phi(\mathbf{k}) \equiv n\int_V d^3r\left(1 - e^{i\mathbf{k}\cdot\mathbf{f}}\right)$ ➡ $\tilde{p}(\mathbf{k}) = \exp[-\phi(\mathbf{k})]$

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define: $\phi(\mathbf{k}) \equiv n\int_V d^3r\left(1 - e^{i\mathbf{k}\cdot\mathbf{f}}\right)$ $\Rightarrow$ $\tilde{p}(\mathbf{k}) = \exp[-\phi(\mathbf{k})]$

$$p(\mathbf{F}) = \tfrac{1}{(2\pi)^3}\int d^3k \exp\left[-i\mathbf{k}\cdot\mathbf{F} - \phi(\mathbf{k})\right]$$

Inverse Fourier transform

POINT-MASS PARTICLES: HOLTSMARK (1919) DISTRIBUTION

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$$p(\mathbf{F}) = \tfrac{1}{2\pi^2}\int_0^\infty dk\, k^2 \exp(-a k^{3/2})\frac{\sin(kF)}{kF}$$

Holtsmark distribution (1919)

- Weak-force limit

$$\lim_{F\to 0} p(\mathbf{F}) = \frac{1}{2\pi^2}\int_0^\infty \mathrm{k}\, k^2 \exp(-ak^{3/2}) = \frac{1}{3\pi^2 a^2}.$$

-Strong-force limit

$$\lim_{F\to\infty} p(\mathbf{F}) \approx \frac{1}{2}(GM)^{3/2} n F^{-9/2}.$$