Analytical models for fringing fields of electrostatic deflectors

work done within the JEDI collaboration

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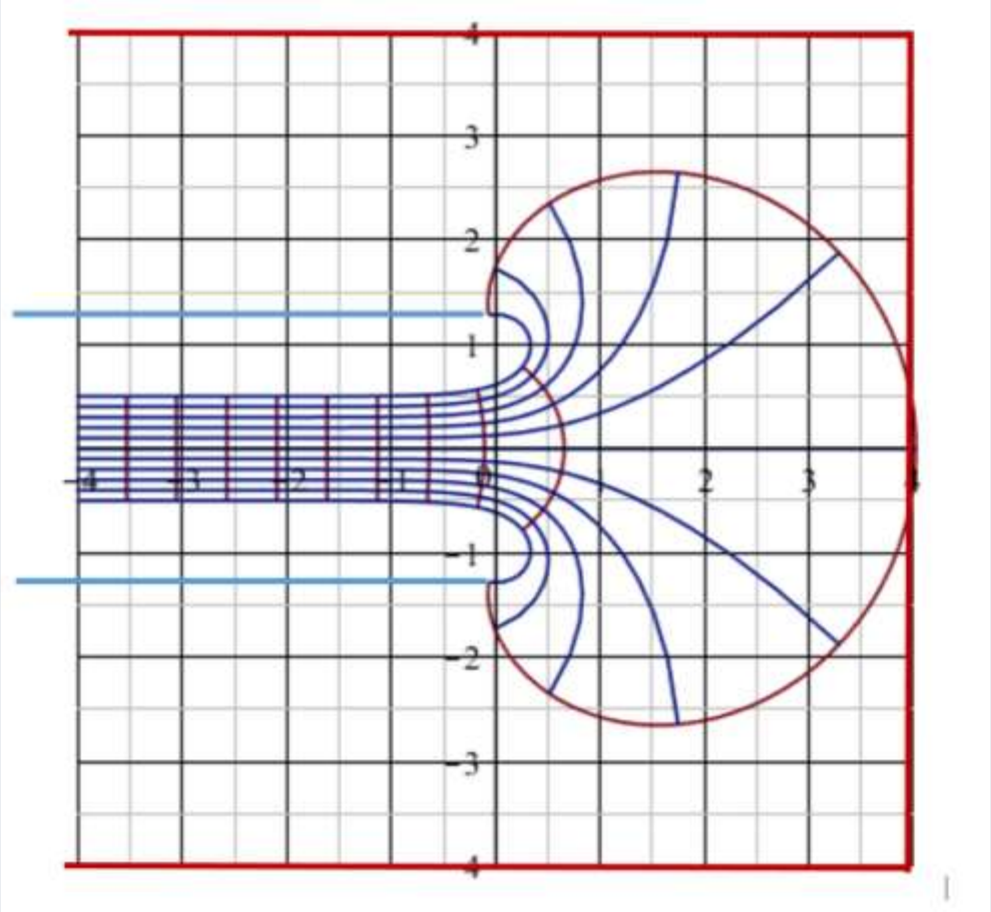
Overview

Main questions:

- What is the contribution of fringing fields to spin decoherence (ex: sextupolar component)?
- Why analytical models?
  - Easier and faster than finite elements for implementation in beam tracking code
  - Gives transfert functions (mapping) for trajectories and spin
  - Suited for long term behaviour and analytical model of (eg) one turn mapping and machine design (correction by sextuple for instance)
- What is presented
  - 2D realistic models based on conformal transforms for cylindrical deflectors
  - Trajectory of the reference particle (deviates from a circle)
  - Transfer functions (non-linear mapping) for trajectories and spin

Future: finalization of this preliminary work (under way) and implementation in Bmad (under way)

Question: how to get a realistic and accurate model for fields and fringing fields including boudary conditions ?



The motivation for analytical calculation is to get accurate models for code implementation. Finite elements are required to check/validate the models and 2D vs 3D

Reminder: Electrostatics in the complex plane

For $z = x + i \cdot y$ the **conformal transform** $Z = F(z) = X + i \cdot Y = \frac{1}{\pi} \cdot [1 + z + e^z]$

transforms the infinite planar capacity $[-\infty, +\infty] \times [-\pi, +\pi]$ to an half-infinite planar with gap equal to 2 and ending in X=0.

Potential: $V(Z) = V(z) = y$

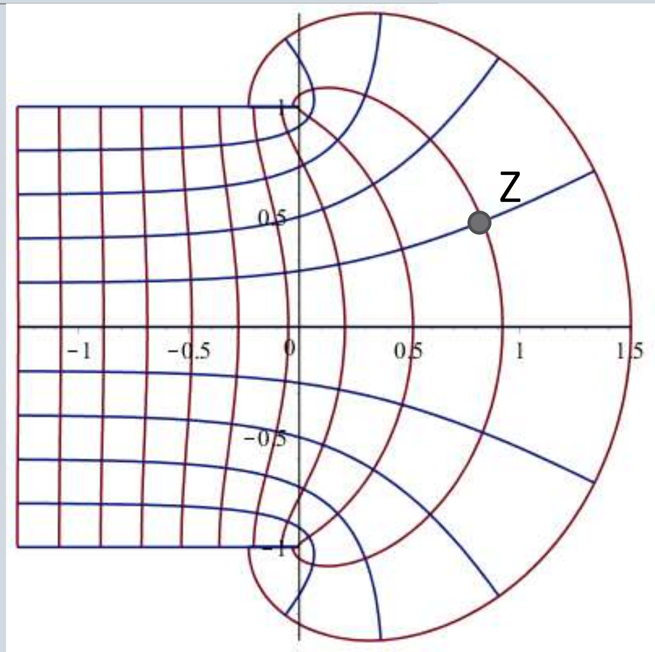
Reciprocal: $z = \pi Z - 1 - W(e^{\pi Z - 1})$ with W=Lambert function

More generally, any CT transforms a set of orthogonal lines to another one. The lines describe a potential/field lines set

$$\underline{V} = V_{scalar} + i \cdot \phi_E$$

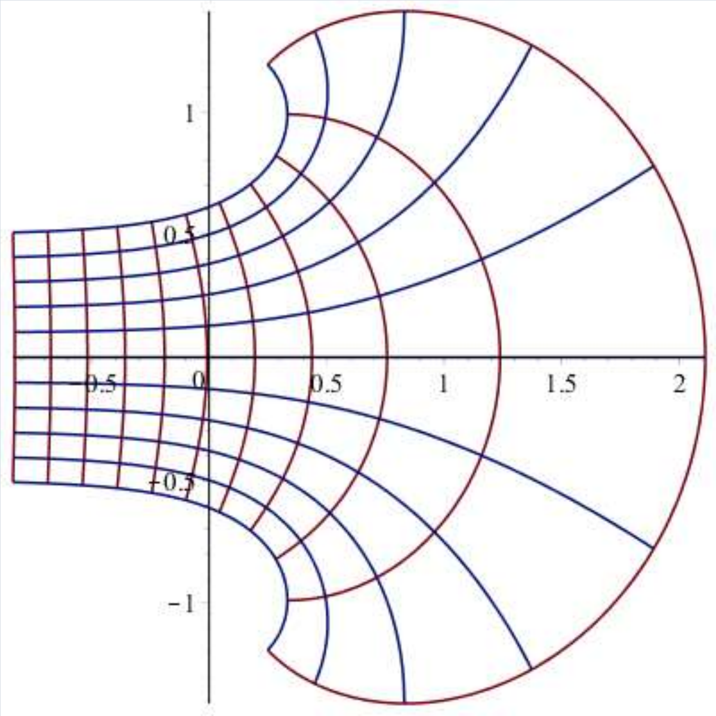
$$-\frac{d\underline{V}}{dz} = \underline{E} = E_x - i \cdot E_y$$

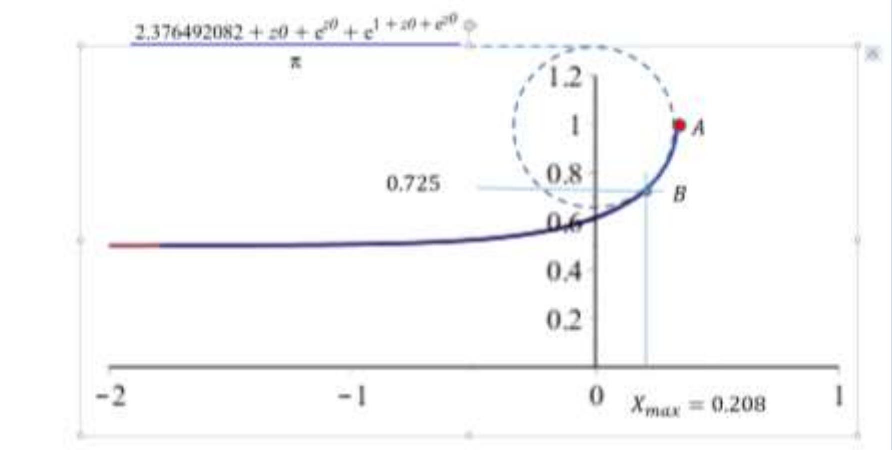
In our example: $\underline{V} = i \cdot z \rightarrow -\underline{E} = \frac{dz}{dZ} = \frac{1}{\frac{dZ}{dz}} = \frac{2\pi i}{1+e^z} = \frac{\pi i}{1+W[e^{\pi Z-1}]}$



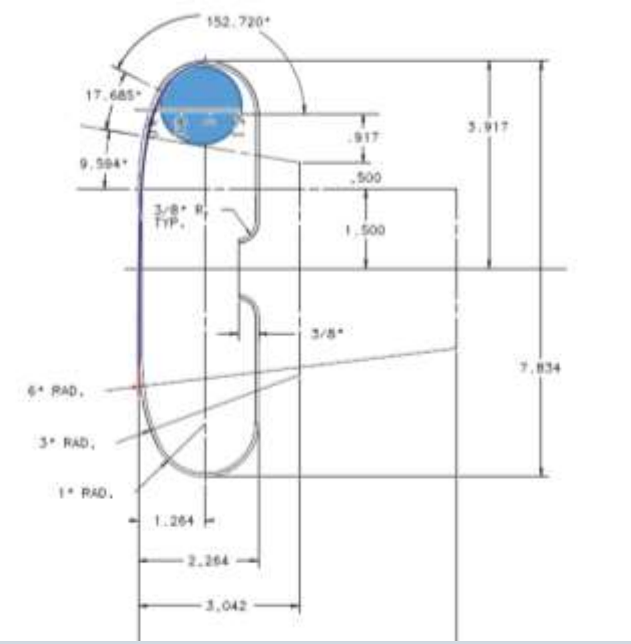
Step 1: A more realistic profile: a double transform

$$\left\{-\frac{\pi}{2} \leq Im(z) \leq \frac{\pi}{2}\right\} : T\colon\ z \mapsto Z_1 = [1 + z + e^{z}] \mapsto Z_2 = \frac{1}{\pi} \cdot [1.376 + Z_1 + e^{Z_1}]$$



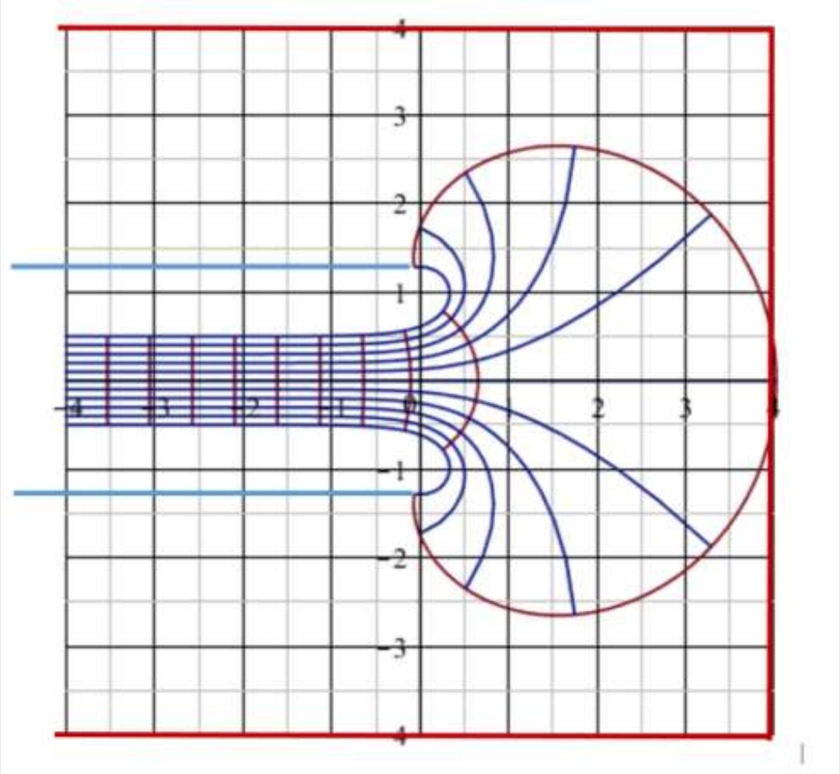


1.376 is only a matter of longitudinal origin

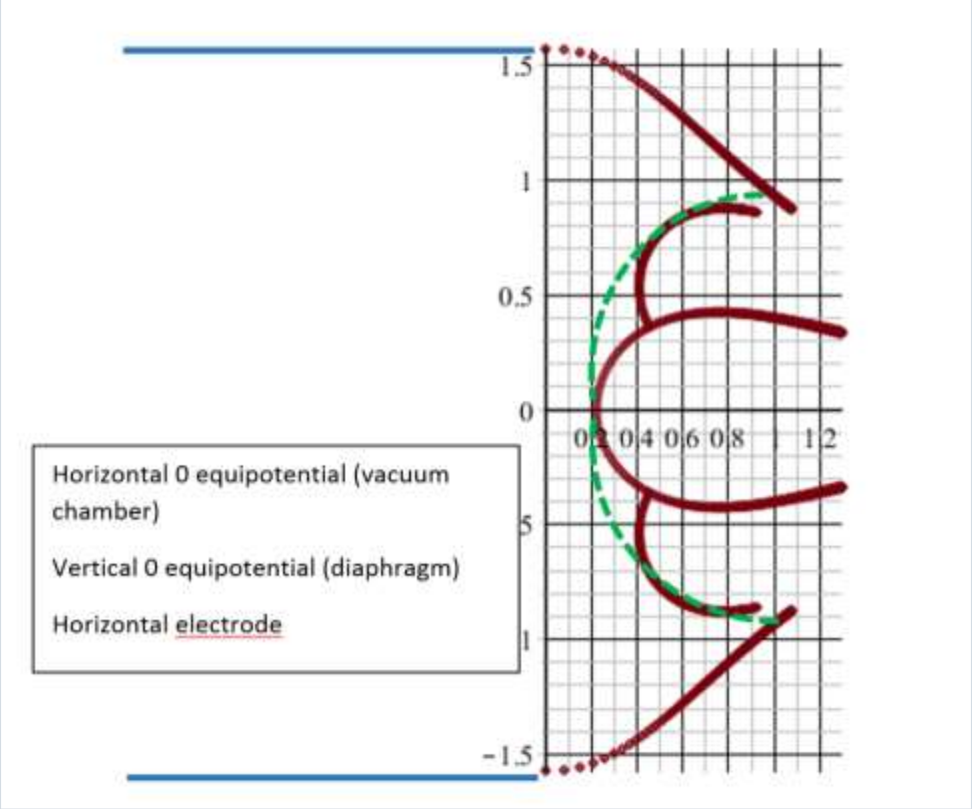


A way to do analytical calculation on a real –Aachen- profile

The (hard) reality: the real deflector



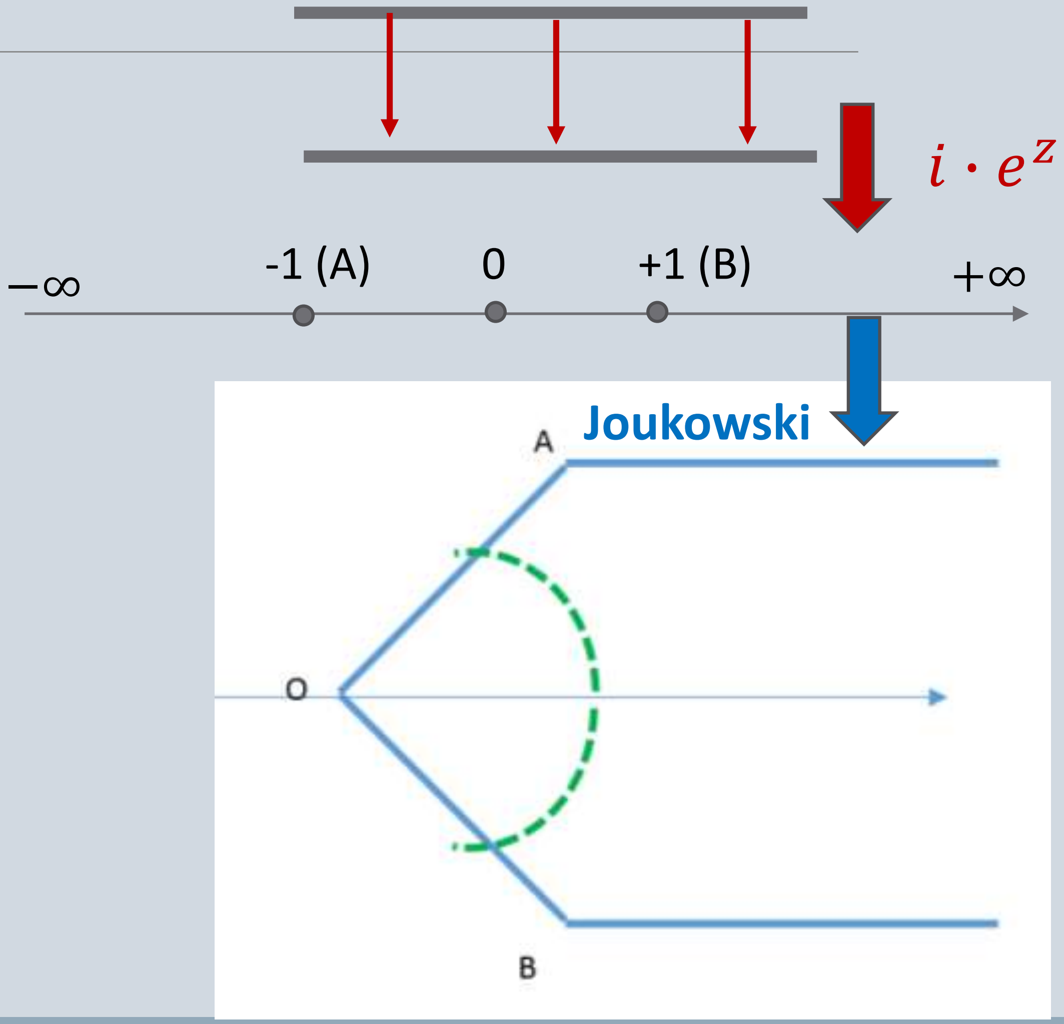
T^{-1}



Gap units

Instead of an infinite planar capacitor

Step 2: including the reality AND square boundary conditions



Questions:

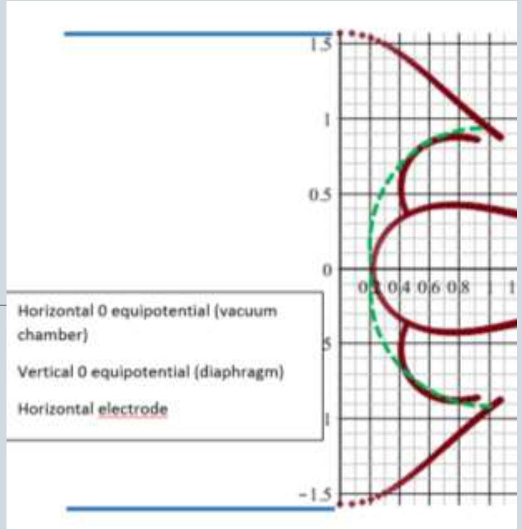
- How to get this picture from the infinite paralel deflector?
- What is the green line (average of the zero equipentials) ?

Anwers:

1- A new (sorry) conformal transform – Joukovski-

2- A boundary condition (green) easy to compute

1) The boundaries V=+-1



We use the Joukovski transform (*in fact for any angles*)

$$g_1(z) = \int_0^z \frac{dt}{t^{1/2}\cdot[t-1]^{1/4}\cdot[t+1]^{1/4}} = \frac{5}{2}\cdot e^{-\frac{i\cdot\pi}{5}}\cdot t^{2/5}\cdot H_{\frac{1}{5},\frac{1}{5},\frac{6}{5}}(t^2)$$

transforms the line (y=0) into the external contour

The inside (half-plane y>0) is the image of the $y \pm \frac{\pi}{2}$ (infinite parallel deflector) via

$$g_2(z) = i\cdot e^z \text{ (previous slide)}$$

Finaly: $G(z) = 0.2 + \frac{\pi}{2} - g_2 o g_1(z) = 0.2 + \frac{\pi}{2} - \frac{5}{2}\cdot e^{\frac{2}{5}z}\cdot H_{\frac{1}{5},\frac{1}{5},\frac{6}{5}}(e^{2z})$ is the solution

G is not so complicated

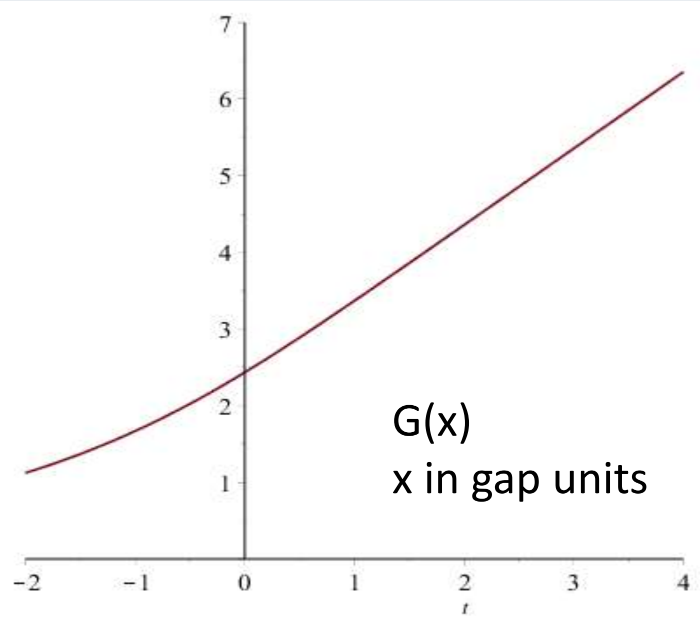
- The reciprocal of G must be calculated
- G is a very regular function
- G is universal (to be inverted one time only!)

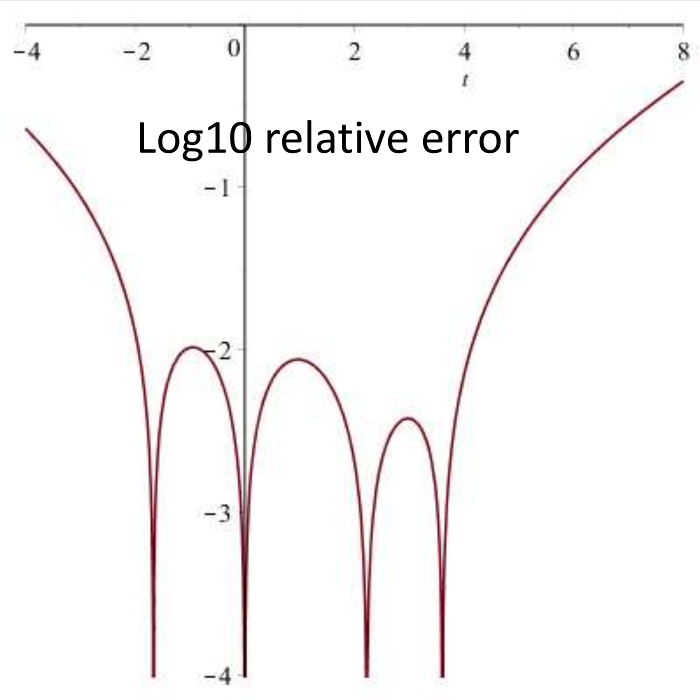
In gap units, at second order only:

$$G(x) \sim exp[0.89 + 0.3417x - 0.0259x^2] = exp(P(x))$$

Bottom picture: relative error on G(x) for x real (decimal logarithm)

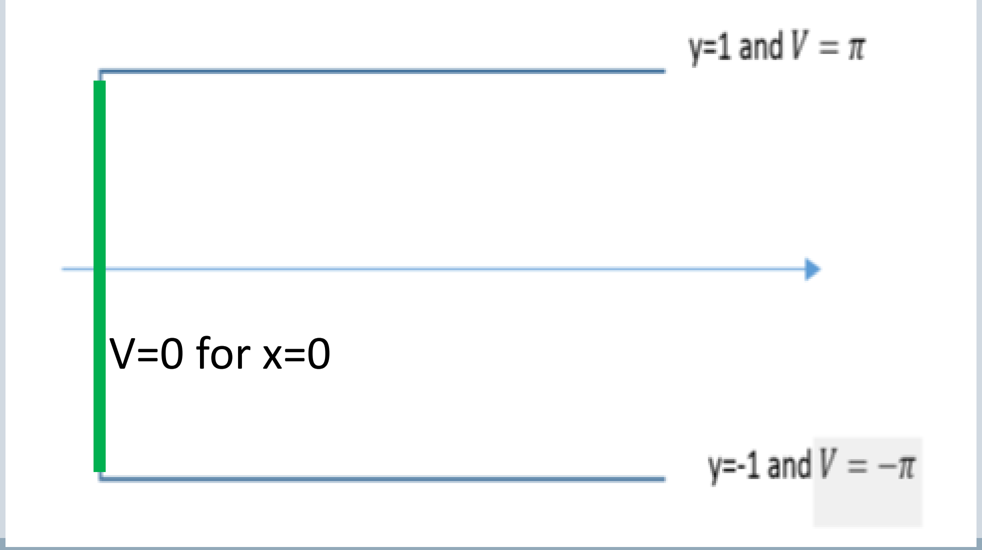
Used to check the coherence of the model, but has to be improved.The use of a second order polynomial was only to get a very easy reciprocal for G

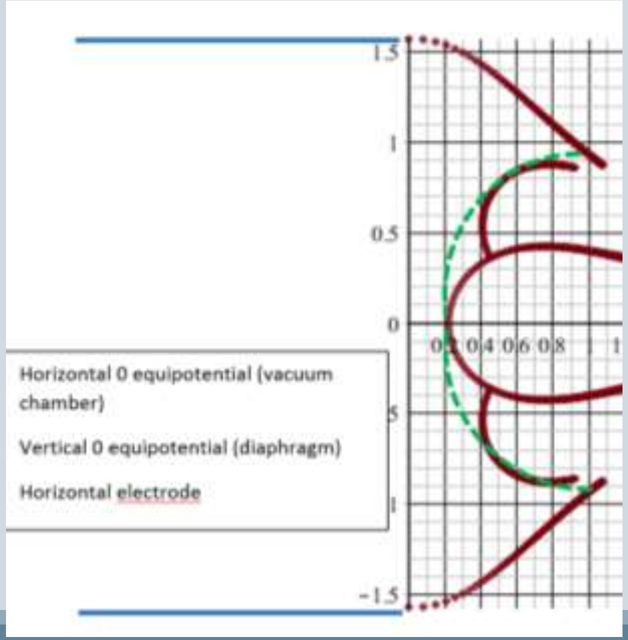


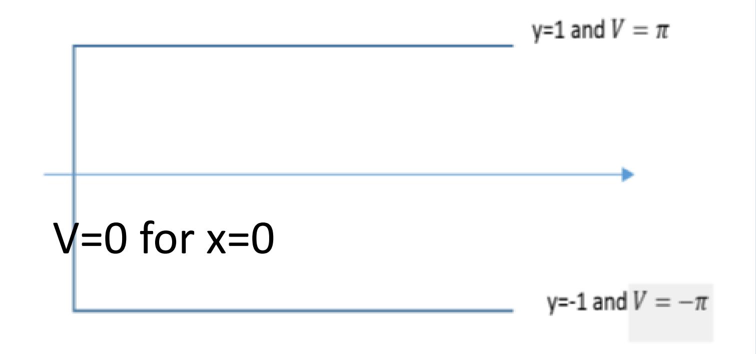


2) The boundary V=0

- The average zero-equipotential (for « square » boundary conditions), is close to a circle
- The circle is the image of a vertical segment
- The original configuration is not an infinite parallel capacitor but a semi-infinite capacitor with a vertical segment with V=0







Last ingredient: the potential in a half box

A well known result : $V(x,y) = \int_{-\infty}^{+\infty} \frac{\mathrm{sh}(\omega y)\cdot\sin(\omega x)}{\mathrm{sh}(\omega)\cdot\omega} \cdot d\omega$ but unuseful

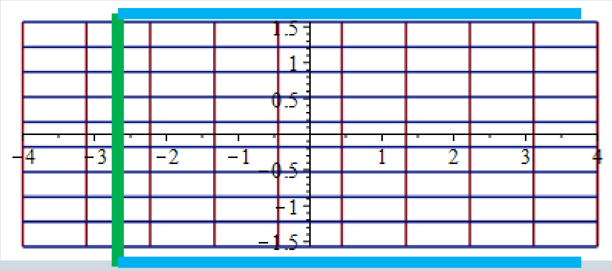
We found the following nice results (a little bit tricky but useful)

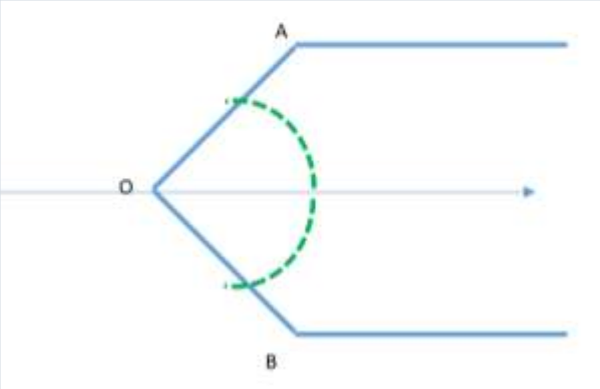
$$V(x,y) = \int_{-\infty}^{+\infty} \frac{sh(\omega y) \cdot \sin(\omega x)}{\omega \cdot sh(\omega)} \cdot d\omega = arctan\left[\frac{sh(\pi x) \cdot \sin(\pi y)}{1 + ch(\pi x) \cdot \cos(\pi y)}\right]$$

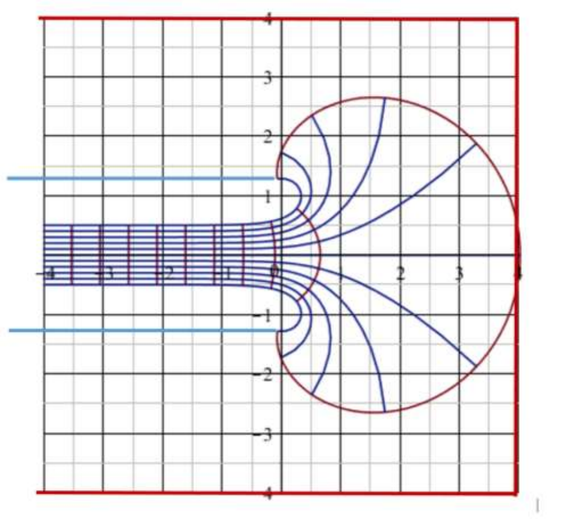
$$\underline{E}(z) = i \cdot \pi \cdot th\left[\frac{\pi \cdot z}{2}\right] \rightarrow \underline{V}(z) = i \cdot ln\left[th^2\left(\frac{\pi \cdot z}{2}\right) - 1\right] + \pi \cdot sign(Im(z)) \quad \text{for } Re(z) \geq 0$$

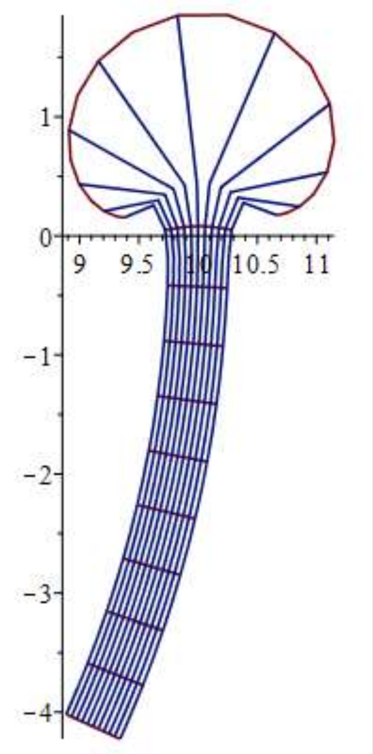
$$\underline{V}(z) = i \cdot ln\left[th^2\left(\frac{\pi \cdot z}{2}\right) - 1\right] + \pi \cdot sign(Im(z)) \quad \text{for } Re(z) \geq 0$$

A lot of transformations









Here
ρ_0=10 and G=1 for illustrati

$$G(z) = 0.2 + \frac{\pi}{2} - \frac{5}{2} \cdot e^{\frac{2}{5}z} \cdot H_{\frac{1}{5},\frac{1}{5},\frac{6}{5}}(e^{2z}) \mapsto Z_1 = \left[1 + G(z) + e^{G(z)}\right] \mapsto Z_2 = \frac{1}{\pi} \cdot [1.376 + Z_1 + e^{Z_1}] \mapsto Z_3 = \rho_0 \cdot e^{iG \cdot Z_3/\rho_0}$$

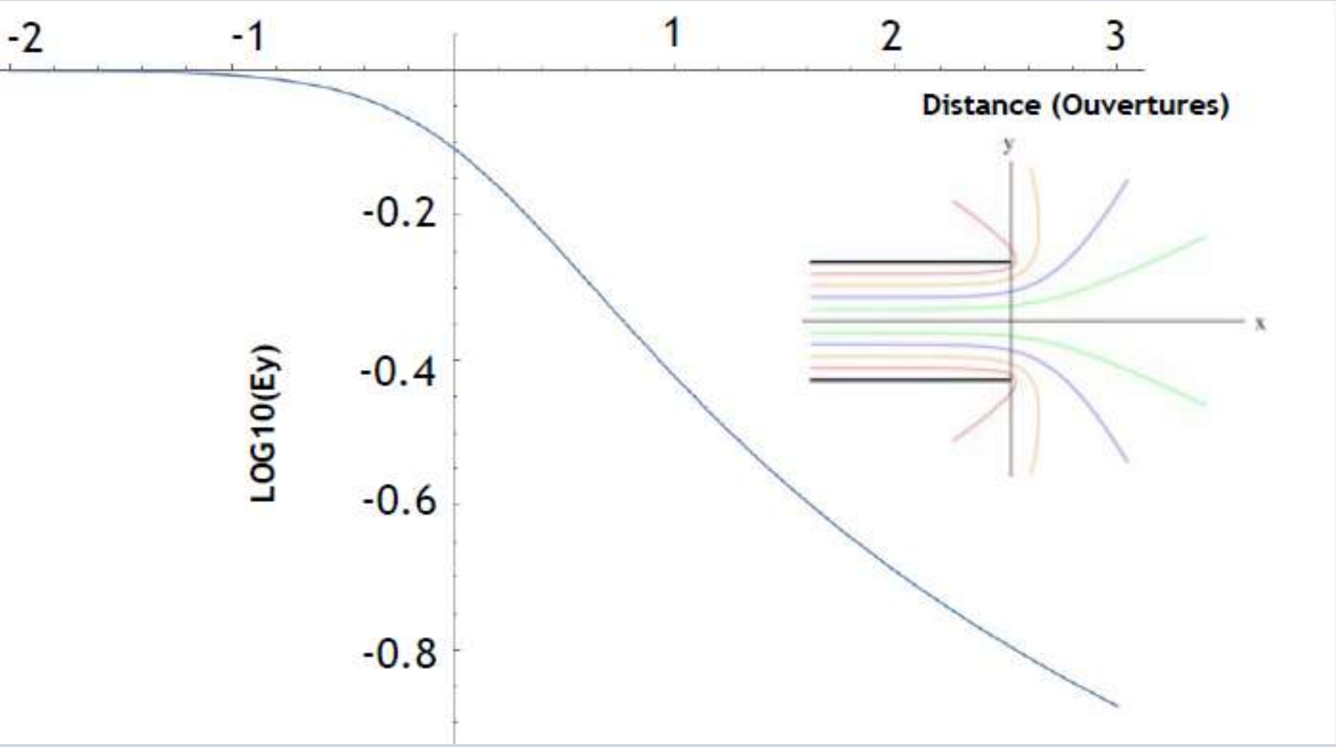
To be compiled to get a ready-to-use library for curved or straight deflectors

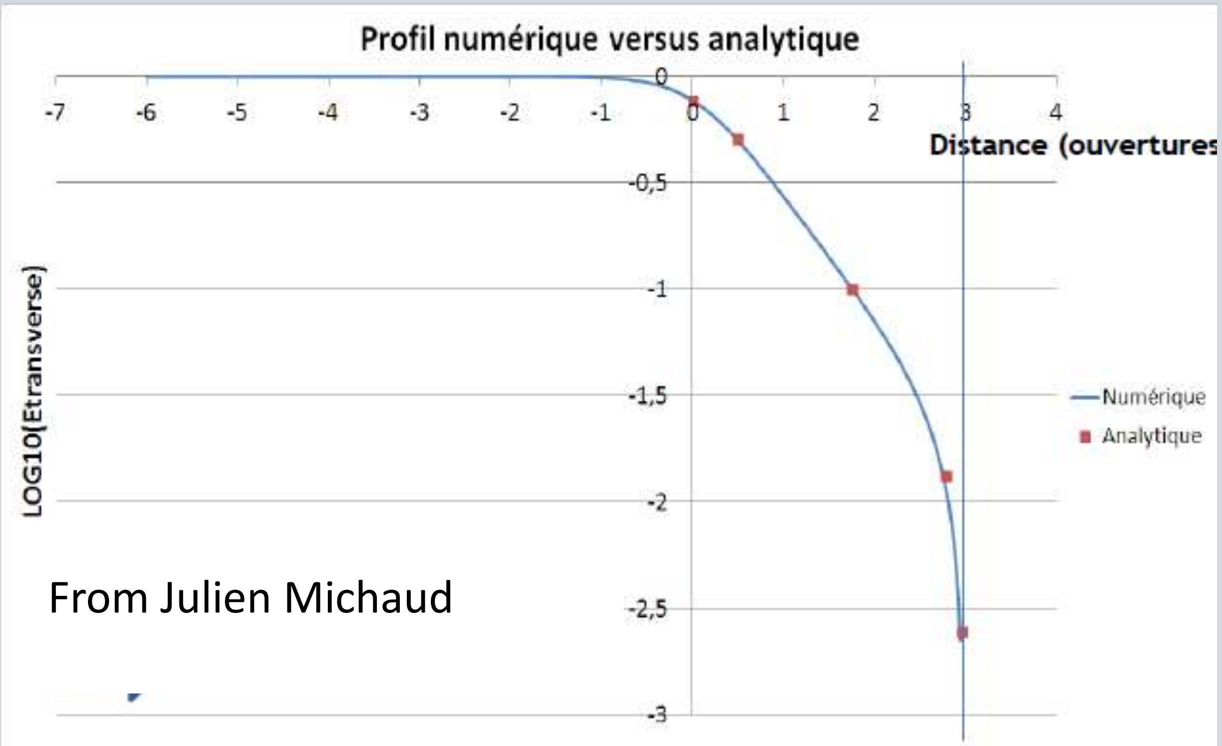
Comparison « naive » deflector / our model/ ANSYS for our model (Log10(Ey) on axis)

Left: parallel capacitor (semi-infinite)

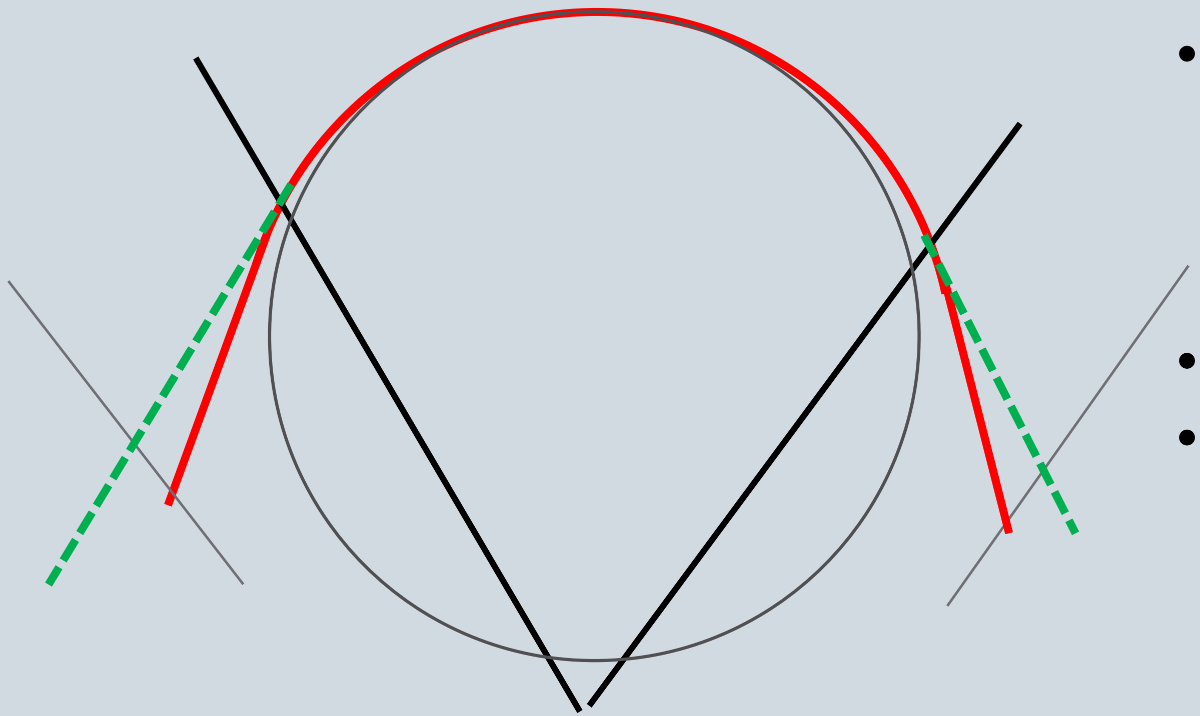
Right: ANSYS (blue line, Julien) versus analytical model (red dots)

boundary conditions at 3 gaps





Reference particle trajectory



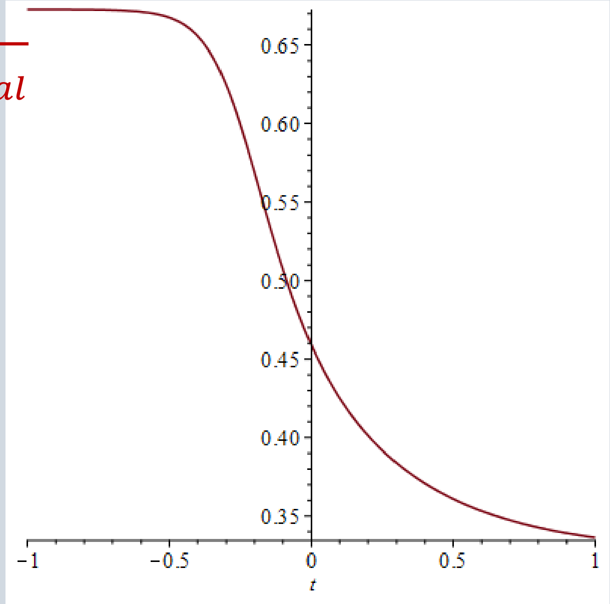
Trajectory of the reference particle with respect to the circle $\rho = \rho_0$

- Hard edge model not valid
- Trajectory up to zero equipotential has to be known
    - Transverse displacement (0.4 mm?)
    - Angle (3 mrad?)
- Calculation done by hamiltonian
- Analytical calculation
    - Non-relativistic model (or γ constant)
    - Proof of principle
    - To make a first check
    - To get the ordrers of magnitude
- Numerical integration of the –independant- equations of motion for checking

Equation of motion with respect to the nominal radius of curvature

From the hamiltonian (for $\gamma = 1$, can be done for any γ)

$$x'' = \frac{\mathcal{E}-1}{\rho_0} - \frac{\mathcal{E}+1}{\rho_0^2} \cdot x \qquad \mathcal{E} = \frac{E}{E_{nominal}}$$



Normalized equation

$$\frac{1}{\Omega} \cdot \tilde{x}'' + w(\tilde{s}) \cdot \tilde{x} = \frac{-2}{\rho_0} \cdot \frac{(s_{max} - s_{min})^2}{4\Omega}$$

Development in series vs the orthogonal polynomials of w

$$\tilde{x} = \sum_{i=0}^{M} \alpha_i \cdot P_i(\tilde{s})$$

P_i computed by recurrence relations

All integral computed by Legendre quadrature

Look for **polynomials** C and S such as

$$\begin{bmatrix} C(-1) & S(-1) \\ C'(-1) & S'(-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

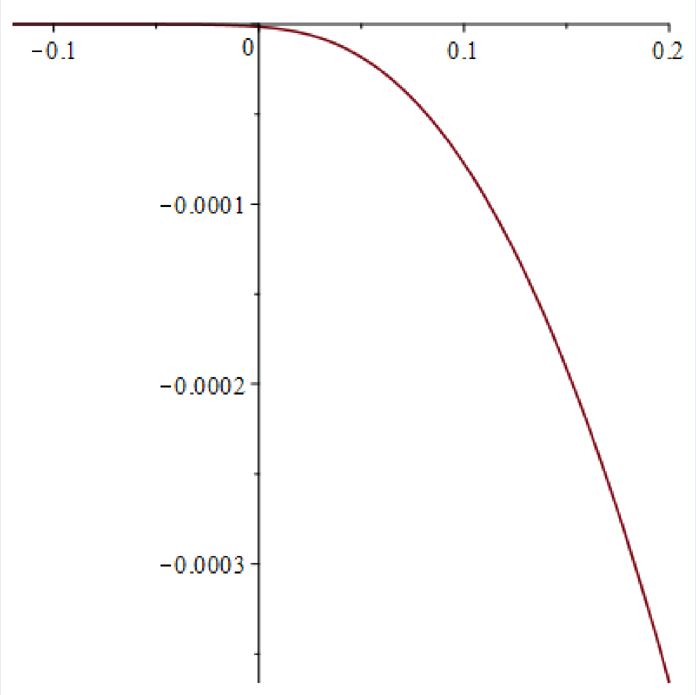
(solutions of the equation without rht)

$$x = \rho_0 \cdot \left(1 - C(\tilde{s})\right) + \left(\frac{-2}{\rho_0} \cdot \frac{(s_{max} - s_{min})^2}{4\Omega}\left[S(\tilde{s}) \cdot \int_{-1}^{\tilde{s}} C(t)dt - C(\tilde{s}) \cdot \int_{-1}^{\tilde{s}} S(t)dt\right]\right.$$

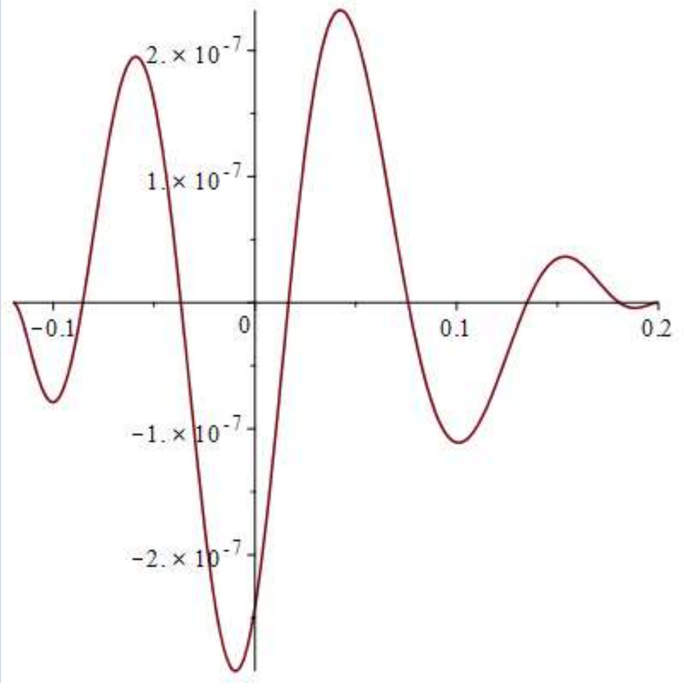
Example, for a 40m radius deflector: comparison with Rkutta integration of motion

$$-0.00127999421749403 - 0.00127998586031768\, t + 8.68991722604830\, 10^{-9}\, t^2 + 1.28195607913743\, 10^{-9}\, t^3 + 1.97697999081965\, 10^{-9}\, t^4 - 1.78560377853045\, 10^{-9}\, t^5 - 2.59980867069587\, 10^{-10}\, t^6 + 3.92287771864143\, 10^{-9}\, t^7 - 4.64397696792177\, 10^{-10}\, t^8 - 1.83388584070278\, 10^{-9}\, t^9$$

- The position deviates by 0.3 mm (not so much)
- The angle deviates by 4 mrad (not negligible)



Radial position (meters)



absolute error (meters), obtained via RK

Second order transfer function in a hard-edge deflector

$$\mathcal{H} = -\left(1+\frac{x}{\rho_0}\right)\cdot\sqrt{\frac{\gamma^2-1}{\gamma_0^2\cdot\beta_0^2}-\left(\overline{p}_x^2+\overline{p}_y^2\right)} = \mathcal{H}_{lin}+\widetilde{\mathcal{H}}$$

$$\gamma = \gamma_0\cdot\left(1-\frac{V}{E_0}+\frac{\Delta E}{E_0}\right)$$

Linear motion

$$q_0 = q_0[K_1, K_2, s] = x_0\cos(\omega s) + \frac{x'_0\sin(\omega s)}{\omega}$$
$$p_0 = p_0[K_1, K_2, s] = -x_0\,\omega\cdot\sin(\omega s) + x'_0\cos(\omega s)$$

Perturbations – variation of constants

$$q = q_0[K_1(s), K_2(s), s] = K_1\cos(\omega s) + \frac{K_2\sin(\omega s)}{\omega}$$
$$p = p_0[K_1(s), K_2(s), s] = -K_1\,\omega\cdot\sin(\omega s) + K_2\cos(\omega s)$$

$$\begin{cases} \dfrac{dK_1}{ds} = \dfrac{\partial\widetilde{\mathcal{H}}}{\partial p} \\ \dfrac{dK_2}{ds} = -\dfrac{\partial\widetilde{\mathcal{H}}}{\partial q} \end{cases}$$

Example for K1

$\gamma_0 = 1.14, \rho = 42m, S = 3.6\ m$

$$x_0 - 0.0006097550903\,x_0^2 + 0.1670449199\,x_0\,xp_0 + 0.08043763479\,\delta\,x_0 + 0.6146223085\,xp_0^2 - 15.09723022\,\delta\,xp_0 - 2.307731104\,\delta^2$$

$$\frac{\sqrt{\gamma 0^2+1}\,S\left(4\cos\left(\frac{\sqrt{\gamma 0^2+1}\,S}{\gamma 0\,\rho 0}\right)\gamma 0^2 - \gamma 0^2 - 4\cos\left(\frac{\sqrt{\gamma 0^2+1}\,S}{\gamma 0\,\rho 0}\right) + 2\right)\sin\left(\frac{\sqrt{\gamma 0^2+1}\,S}{\gamma 0\,\rho 0}\right)}{\gamma 0\,\rho 0\left(\gamma 0^2-1\right)}$$

Automatic Fortran code generation

```
doubleprecision function cg (delta, gamma0, S, rho0)
   doubleprecision delta
   integer gamma0
   doubleprecision S
   doubleprecision rho0
   cg = sqrt(dble(gamma0 ** 2 + 1)) * S * (0.4D1 * cos(sqrt(dble(ga
  #mma0 ** 2 + 1)) * S / dble(gamma0) / rho0) * dble(gamma0 ** 2) - d
  #ble(gamma0 ** 2) - 0.4D1 * cos(sqrt(dble(gamma0 ** 2 + 1)) * S / d
  #ble(gamma0) / rho0) + 0.2D1) * sin(sqrt(dble(gamma0 ** 2 + 1)) * S
  # / dble(gamma0) / rho0) / dble(gamma0) / rho0 / dble(gamma0 ** 2 -
  # 1)
   return
  end
```

Second example: spin transfer function in a hard-edge deflector for a magic ring

$$\frac{d\vec{S}}{dt} = \vec{\Omega}_t \wedge \vec{S}$$

$$\delta = \frac{\gamma_0^2 - 1}{\gamma_0^2} \cdot \frac{\Delta p}{p_0}$$

$$\vec{\Omega}_t = \frac{-q}{mc^2} \cdot \left(G - \frac{1}{\gamma^2 - 1}\right) \cdot \vec{v} \wedge \vec{E}$$

$$\Omega = \frac{-q}{mc^2} \cdot \left(G - \frac{1}{\gamma^2 - 1}\right) \cdot \left(1 + \frac{x}{\rho_0}\right) \cdot E(x)$$

$$\Omega = \frac{2}{\gamma_0 \cdot \rho_0} \cdot \left(\frac{x}{\rho_0} + \delta\right) + \frac{1}{\gamma_0^3 \cdot \rho_0} \cdot \left[(2\gamma_0^2 + 1)\frac{x^2}{\rho_0^2} + 2 \cdot (3\gamma_0^2 + 1) \cdot \frac{x}{\rho_0} \cdot \delta\right] + (3\gamma_0^2 + 1) \cdot \delta^2$$

Spin evolution can be used by integration of the known second order trajectories

Conclusion

- Tools have been developped for 2D models
- Numerical checking has been done
- A library of formulas is in preparation
- Implementation in Bmad is under way

Critics are welcome and requested

Thank you very much!