Canonical quantization of homogeneous cosmological models in the presence of conditional symmetries

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Motivation of my work

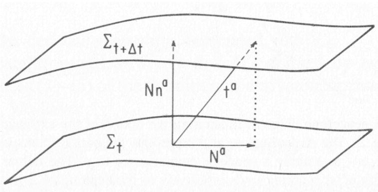
- Study conceptual and technical problems of quantum gravity
  - operator ordering problem
  - Hilbert space problem, positive definite inner product, probabilities
  - use the symmetries to reduce to a finite degrees of freedom problem
- Study physical problems of quantum cosmology (e.g. initial singularities)

Here, I will describe how the existence of conditional symmetries can

- make the integration of the system of equations of motion easier
- be used to refine canonical quantization by promoting the associated charges to operators

and I will use a semiclassical approximation to derive some physical insight.

3+1 decomposition of spacetime



$$ds^2 = \left(N^\alpha N_\alpha - N^2\right) dt^2 + 2N_\alpha \sigma^\alpha_i dx^i dt + \gamma_{\alpha\beta}\sigma^\alpha_i \sigma^\beta_j dx^i dx^j$$

where

$$\sigma^\alpha_{i,j} - \sigma^\alpha_{j,i} = C^\alpha_{\beta\gamma}\sigma^\beta_j \sigma^\gamma_i$$

$$\gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta$$

$$N = -t^\alpha n_\alpha$$

$$N_\alpha = h_{\alpha\beta} t^\beta$$

From now on we set $N^\alpha = 0$

Lagrangian formulation

Inserting in the vacuum gravitational action

$$S = \int d^4x\sqrt{-g}R$$

or coupled to a scalar field

$$S_{tot} = S_{grav} + S_{mat} = \int d^4x\sqrt{-g}\left(R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)$$

this ansatz for the metric, we obtain Lagrangians of the form

$$L = \frac{1}{2N}G_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta - N\mathcal{V}(q),\ \alpha = 0, 1, \ldots n$$

The resulting equations of motion are

$$\mathcal{G}_{\mu\nu} = -\frac{1}{4}g_{\mu\nu}g^{\kappa\lambda}\partial_\kappa\phi\partial_\lambda\phi + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi,$$

The Lagrangian is invariant under $t = f(\tilde{t})$ and the following transformation in the dependent variables

$$N(t) \to \tilde{N}(\tilde{t}) = N(f(\tilde{t}))f'(\tilde{t}), \quad q^\alpha(t) \to \tilde{q}^\alpha(\tilde{t}) = q^\alpha(f(\tilde{t})).$$

Hamiltonian formulation

- We start by finding the canonical momenta

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} = \frac{1}{N} G_{\alpha\beta} \dot{q}^\beta, \quad p_N = \frac{\partial L}{\partial \dot{N}} = 0$$

  Thus we have a primary constraint

- The Hamiltonian is found by a Legendre transform to be

$$H = N \left(\frac{1}{2N^2} G^{\alpha\beta} p_\alpha p_\beta + \mathcal{V}(q) \right) \equiv N\mathcal{H}$$

- Consistency demands no time evolution for the constraints

$$\dot{p}_N = \{p_N, H\} \approx 0 \Rightarrow \mathcal{H} \approx 0$$

  This is a secondary constraint (Hamiltonian constraint). Its consistency does not lead to further constraints.

The constraints represent the t-reparametrization invariance of the action

Conditional symmetries

- The definition of the conditional symmetries is

$$\mathcal{L}_\xi G^{\alpha\beta} = \rho(q) G^{\alpha\beta}, \quad \mathcal{L}_\xi V(q) = \rho(q)\mathcal{V}(q)$$

  where ξ are conformal Killing vector fields.
- Each of the conditional symmetry corresponds to a phase space quantity

$$Q_i := \xi_i^\alpha p_\alpha$$

  which satisfies

$$\{Q, H\} = N\mathcal{H} \approx 0$$

  i.e. they are symmetries on the constraint surface.
- It can be shown that the variational symmetries of the singular action and the Lie point symmetries of the equations of motion are the conditional symmetries (which are conformal Killing fields) plus the scaling symmetry generator plus the time reparametrization generator.

Constant potential parametrization

- Taking advantage of the reparametrization invariance of the action we can always perform a lapse rescaling and write

$$L = \frac{1}{2n}\bar{G}_{\alpha\beta}(q)\dot{q}^{\alpha}\dot{q}^{\beta} - n\bar{V}(q)$$

  such that the conditional symmetries become

$$\mathcal{L}_{\xi}\bar{G}^{\alpha\beta} = 0, \quad \mathcal{L}_{\xi}\bar{V}(q) = 0$$

- This happens when the potential $\bar{V}$ is constant and we choose it equal to 1.
- Under this change, the quantities Q retain their form, i.e. they are still conserved on the constraint surface.
- The dependent variables are the same as in the initial action
- Now we have the following changes in the generators of the action and EOM symmetries
  - the conformal Killing fields $\rightarrow$ Killing fields
  - the scaling symmetry $\rightarrow$ homothecy
  - time reparametrization is intact

Integrals of motion

Therefore, for the Lagrangian

$$L = \frac{1}{2N} G_{\alpha\beta}(q) \dot{q}^{\alpha} \dot{q}^{\beta} - N$$

the integrals of motion reduce to the following

- for the Killing vector fields $Q_i = \xi_i^{\alpha} p_{\alpha} = \kappa_i$
- for the homothetic field $Q_h = \kappa_h + \int dt N(t)$
- Since they are of first order in the velocities/momenta, they can be solved much easier than the EOM which are of second order.

In addition, the symmetry generators satisfy a Lie algebra of the form

$$[\xi_i, \xi_j] = c_{ij}^{k} \xi_k,$$

where c_{ij}^{k} are the structure constants. The corresponding Poisson bracket satisfy the same algebra as their generators.

Quantization procedure

Following Dirac's procedure for the canonical quantization of constrained systems we have

- Promotion of the canonical variables to operators according to the rule

$$\hat{p}_\alpha = -i\frac{\partial}{\partial q^\alpha}, \quad \hat{p}_N = -i\frac{\partial}{\partial N}$$

  and of the Poisson brackets to commutators $\{.,.\} \to -\frac{i}{\hbar}[.,.]$.

- We promote the constraints *as well as the conserved quantities* to operators and impose them as conditions on the wave function

$$\hat{p}_\alpha \Psi := -i\frac{\partial}{\partial q^\alpha}\Psi = 0$$

$$\hat{\mathcal{H}}\Psi(q) := \left(-\frac{1}{2}\Box_c^2 + 1\right)\Psi(q) = 0, \quad \Box_c^2 \equiv \Box^2 + \frac{d-2}{4(d-1)}R,$$

$$\hat{Q}_i\Psi := -\frac{i}{2\mu}(\mu\xi_i^\alpha\partial_\alpha + \partial_\alpha\mu\xi_i^\alpha)\Psi = \kappa_i\Psi,$$

The choice for the factor ordering for the kinetic part of the Hamiltonian implied by the conformal Laplacian stems from the demands of (i) general covariance of the Lagrangian, (ii) hermiticity of the operators under the inner product of the form $\int d^n q \mu \psi_1^* \psi_2$, with μ a proper measure and (iii) conformal invariance of the action.

- The classical algebra becomes

$$[\hat{Q}_i, \hat{Q}_j] = ic_{ij}^m \hat{Q}_m$$

- Acting on the wave function

$$[\hat{Q}_i, \hat{Q}_j]\Psi = (\kappa_i \kappa_j - \kappa_j \kappa_i)\Psi = 0$$

  gives a condition for the constants: $c_{ij}^m \kappa_m = 0$.

- This relation prohibits the simultaneous realization of all the operators simultaneously.

Semiclassical analysis

- The wave function is assumed to have the form $\Psi(q) = \Omega(q)e^{iS(q)}$.
- Inserting it in the Wheeler-DeWitt equation, we get the following equation

$$G^{\alpha\beta}\partial_\alpha S \partial_\beta \Omega + \frac{\Omega}{2\mu}\partial_\alpha(\mu G^{\alpha\beta}\partial_\beta S) = 0$$

and a modified Hamilton-Jacobi equation of the form

$$\frac{1}{2}G^{\alpha\beta}\partial_\alpha S \partial_\beta S - \frac{1}{2}\frac{\Box\Omega}{\Omega} + V = 0,$$

where

$$\mathcal{Q}(q) \equiv -\frac{1}{2\Omega}\Box\Omega = -\frac{1}{2\mu}\partial_\alpha(\mu G^{\alpha\beta}\partial_\beta)\Omega.$$

- The equations of motion are

$$\frac{\partial S}{\partial q^\alpha} = \frac{\partial L}{\partial \dot{q}^\alpha}.$$

- We apply these equations for each subalgebra and examine whether their solution for each algebra approximates the classical one.
- It turns out that the semiclassical solution does not coincide with the classical when $\mathcal{Q} \neq 0$

Example: A massless scalar field in the spatially curved FLRW

Classical solution of a FLRW spacetime in the presence of a massless scalar field I

- The Robertson-Walker metric is

$$ds^2 = -N^2(t)dt^2 + a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2\sin\theta^2 + r^2\sin^2\theta d\varphi^2\Big]$$

- The scalar field is of the form $\phi = \phi(t)$
- The rescaled Lagrangian after a reparametrisation of the lapse function $N = \frac{n}{6ka} \equiv \frac{n}{\mathcal{V}(a,\phi)}$ becomes

$$L = n - \frac{36ka^2\dot{a}^2}{n} + \frac{3ka^4\dot{\phi}^2}{n} \equiv n - \frac{1}{2n}G_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta$$

where the supermetric is

$$G_{ab} = \begin{pmatrix} -72ka^2 & 0 \\ 0 & 6ka^4 \end{pmatrix}$$

- The Killing vector fields and the homothetic of this supermetric are

$$\xi_1 = (\frac{e^{\phi/\sqrt{3}}}{a}, -\frac{2\sqrt{3}e^{\phi/\sqrt{3}}}{a^2}), \quad \xi_2 = (\frac{e^{-\phi/\sqrt{3}}}{a}, \frac{2\sqrt{3}e^{-\phi/\sqrt{3}}}{a^2})$$

$$\xi_3 = (0,1), \qquad \xi_h = (\frac{a}{4}, 0)$$

Classical solution of a FLRW spacetime in the presence of a massless scalar field II

- The conserved quantities are found to be

$$Q_1 = -\frac{12e^{\frac{\phi}{\sqrt{3}}}k(6a\dot{a}^2 + \sqrt{3}a^2\dot{\phi})}{n^2}$$

$$Q_2 = \frac{12e^{-\frac{\phi}{\sqrt{3}}}k(-6a\dot{a}^2 + \sqrt{3}a^2\dot{\phi})}{n^2}$$

$$Q_3 = \frac{6ka^4\dot{\phi}}{n}$$

$$Q_h = -\frac{18ka^3\dot{a}}{n}$$

Solution for the spacetime element

The final spacetime element is

$$ds^2 = -\frac{\lambda}{4\sqrt{T}(1+T\epsilon)^3}dT^2 + \frac{\lambda\sqrt{T}}{(1+T\epsilon)}\left(\frac{dr^2}{1-r^2\epsilon} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right).$$

To find it we used the system $Q_i = \kappa_i, i = 1, 2, 3$ and $Q_h = k_h + \omega$ where $\omega(t) = \int dtn$ and the conditions that $\partial_t a = \dot{a},\ \dot{\omega} = n$ This geometry has one essential constant. The Ricci scalar is

$$R = -\frac{3(T\epsilon+1)^3}{2T^{3/2}\lambda},$$

where we have set $\lambda = -\frac{\kappa_3}{\sqrt{3}k^{3/2}}$ rendering the metric element singular for both $T \to 0$ and $T \to \infty$.

Quantization I

- We impose the quantum constraints and the conserved quantities on the wave function with quantum measure $\mu(a,\phi) = 6\sqrt{3}a^3k$,

$$\hat{Q}_1\Psi = -\frac{ie^{\phi/\sqrt{3}}(-6\partial_\phi\Psi + \sqrt{3}a\partial_a\Psi)}{\sqrt{3}a^2} = \kappa_1\Psi,$$

$$\hat{Q}_2\Psi = -\frac{ie^{-\phi/\sqrt{3}}(6\partial_\phi\Psi + \sqrt{3}a\partial_a\Psi)}{\sqrt{3}a^2} = \kappa_2\Psi,$$

$$\hat{Q}_3\Psi = -i\partial_\phi\Psi = \kappa_3\Psi,$$

$$\hat{\mathcal{H}}\Psi = \frac{-144ka^4\Psi - 12\partial_{\phi\phi}\Psi + a(\partial_a\Psi + a\partial_{aa}\Psi)}{144ka^4} = 0$$

The condition $c^m_{ij}\kappa_m = 0$ gives as allowed subalgebras the two dimensional $\{Q_1, Q_2\}$ and the one dimensional $\{Q_1\}, \{Q_2\}, \{Q_3\}$.

The subalgebras wave functions

We thus have the following cases

- For the two-dimensional subalgebra $\{Q_1, Q_2\}$ the wave function is found by applying $\hat{Q}_1\Psi = 0, \hat{Q}_2\Psi = 0, \hat{\mathcal{H}}\Psi = 0$

$$\Psi_{12}(a,\phi) = ce^{i\frac{1}{4}a^2 e^{-\frac{\phi}{\sqrt{3}}}(\kappa_1+\kappa_2 e^{\frac{2\phi}{\sqrt{3}}})}$$

- Similarly for the Q_1 subalgebra $\hat{Q}_1\Psi(a,\phi) = 0, \hat{\mathcal{H}}\Psi(a,\phi) = 0$ The wave function is

$$\Psi_1(a,\phi) = ce^{i(-\frac{36ka^2 e^{\frac{\phi}{\sqrt{3}}}}{\kappa_1}+\frac{1}{4}a^2\kappa_1 e^{-\frac{\phi}{\sqrt{3}}})}$$

- For the Q_2 algebra, we impose the relations $\hat{Q}_2\Psi(a,\phi) = 0,\ \hat{\mathcal{H}}\Psi(a,\phi) = 0$ and the wave function is found to be

$$\Psi_2(a,\phi) = ce^{i(\frac{36ka^2 e^{-\frac{\phi}{\sqrt{3}}}}{\kappa_2}+\frac{1}{4}a^2\kappa_2 e^{\frac{\phi}{\sqrt{3}}})}$$

- Finally, for the Q_3 algebra $\hat{Q}_3\Psi(a,\phi) = 0,\ \hat{\mathcal{H}}\Psi(a,\phi) = 0$, the wave function is given by

$$\Psi_{cl}(a,\phi) = e^{i\phi\kappa_3}(A_1 I_{-i\sqrt{3}\kappa_3}(6a^2) + B_1 I_{i\sqrt{3}\kappa_3}(6a^2)),$$
$$\Psi_{op}(a,\phi) = e^{i\phi\kappa_3}(A_2 J_{-i\sqrt{3}\kappa_3}(6a^2) + B_2 J_{i\sqrt{3}\kappa_3}(6a^2)),$$

Semiclassical analysis I

- The calculation of the quantum potential $\mathcal{Q}$ for the first three cases i.e. $\{Q_1, Q_2\}$, $\{Q_1\}$, $\{Q_2\}$ gives zero. Solving the equations of motion

$$\frac{\partial L}{\partial \dot{a}} = \frac{\partial S}{\partial a}, \quad \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial S}{\partial \phi}$$

  we get the classical solution.

- In the case of the algebra $\{Q_3\}$ things are different. First we make some approximations, in order to write the wave function in polar form.

- We find Ψ for the small and large limits of a, using the simplifying assumption $A_1 = B_1, \ A_2 = B_2$. For small arguments it is

$$\Psi_{sm} \approx c_1 e^{i\kappa_3\phi} \cos \ln a.$$

  for the large values, the wave function becomes

$$\Psi_{la}^{cl} \approx \frac{e^{a^2}}{a} e^{i\kappa_3\phi}, \quad \Psi_{la}^{op} \approx \frac{\sin(6a^2)}{a} e^{i\kappa_3\phi}. \tag{1}$$

Semiclassical analysis II

- The phase function is $S = \kappa_3 \phi$ and the solution of the semiclassical equations with respect to (a, n) is

$$a = c, \quad n = \frac{6ka^4}{\kappa_3}\dot{\phi}, \tag{2}$$

Choosing a gauge for $\phi(t)$, such that the lapse function $N(t)$ of the semiclassical element is the same as for the classical and inserting the solution in the 4-dimensional element we find

$$ds^2 = -\frac{\lambda}{4\sqrt{T}(1+T\epsilon)^3}dt^2 + \frac{1}{1-\epsilon r^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2,$$

where $c^2 = \frac{\lambda^2}{16}$. This spacetime has constant Ricci scalar $R = 6k$, all higher derivatives of its Riemann tensor zero and constant all curvature scalars constructed from its Riemann tensor. Hence, there is no curvature and/or higher derivative curvature singularity.

Conclusions

- We defined the conditional symmetries for singular systems
- We showed that their use simplify the way to obtain the solution for the variables classically
- The wave function of each case can be found more easily because of the additional condition imposed.
- The semiclassical analysis works well for this model, since when $\mathcal{Q} = 0$, the solution coincides with the classical one.
- However, we found that in the case that the superpotential is not zero, these do not coincide.
- For the case of the spatially curved FLRW coupled to a scalar field we found that in the semiclassical approximation the singularity can be resolved.

Thank you!