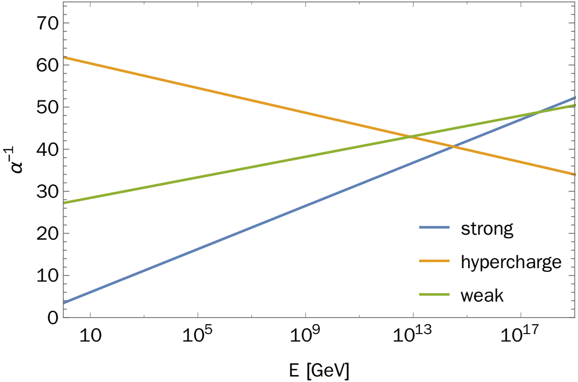
The group $SO(10)$ in the Grand Unified Theory

Kateřina Jarkovská

Výjezdní seminář ÚČJF

- Standard model gauge group $G_{SM} = SU(3) \times SU(2) \times U(1)$
- SM running coulings (one loop level, without scalar fields):



$$\alpha^{-1}(M) = \alpha^{-1}(M_0) + \frac{1}{2\pi}\left(-\frac{11}{3}C_2^{adj} + \frac{2}{3}T_2(R)\right)\ln\left(\frac{M}{M_0}\right)$$

Special orthogonal group $SO(n)$

Definition

If $B : V \times V \to \mathbb{F}$ is a symmetrical positive definite billinear form. Then we define Orthogonal group

$$O(n, \mathbb{F}) := \{M \in GL(V) : B(Mv, Mw) = B(v, w) \text{ for } v, w \in V\},$$

where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

$$O(n) \cong \left\{M \in \mathbb{R}^{n \times n} : M^T M = I_n\right\}$$

$$SO(n) = O(n) \cap \{M \in \mathbb{R}^{n \times n} : \det M = 1\}$$

- $SO(n)$ is connected and compact $\Rightarrow$ expressed by an exponential of generators from special orthogonal algebra $\mathfrak{so}(n)$.

$$\mathfrak{so}(n) := \left\{X \in \mathbb{R}^{n \times n} : X^T + X = 0, \mathrm{Tr}(X) = 0\right\}$$

Definition

Representation (ϕ, V) of a Lie algebra $\mathfrak{g}$ is a linear map

$$\phi : \mathfrak{g} \to \mathfrak{gl}(V)$$

which preserves Lie bracket:

$$[\phi(X), \phi(Y)] = \phi([X, Y])$$

for all $X, Y \in \mathfrak{g}$. If $\langle \cdot, \cdot \rangle$ is a Hermitian scalar product on V and

$$\forall X \in \mathfrak{g}, v_1, v_2 \in V : \langle v_1, \phi(X) v_2 \rangle = -\langle \phi(X) v_1, v_2 \rangle,$$

then (ϕ, V) is an unitary representation.

- Representation of $\mathfrak{so}(n)$ $\longleftrightarrow$ representation of a double cover $Spin(n)$ $\longleftrightarrow$ (single, double valued) representation of $SO(n)$

Spinor representation of $\mathfrak{so}(2n)$

Definition

Clifford algebra $Cliff(2n, \mathbb{C})$ is an associative algebra generated by an identity 1 and vectors from an orthonormal basis of $\mathbb{C}^{2n}$ satisfying

$$e_i e_j + e_j e_i = 2\delta_{ij}$$

for $i, j \in \{1, \ldots, 2n\}$

- Witt's basis is for $j \in \{1, \ldots, n\}$

$$w_j := \frac{1}{2}(e_{2j-1} + ie_{2j}) \qquad \bar{w}_j := \frac{1}{2}(e_{2j-1} - ie_{2j})$$

- $\{w_j, w_k\} = \{\bar{w}_j, \bar{w}_k\} = 0, \ \ \{w_j, \bar{w}_k\} = \delta_{jk}$

- Exterior algebra

$$\Lambda\mathbb{C}^n = \mathbb{C}^n \oplus \Lambda^1\mathbb{C}^n \oplus \Lambda^2\mathbb{C}^n \cdots \oplus \Lambda^n\mathbb{C}^n$$

where $\Lambda\mathbb{C}^j$ is generated by

$$w_{k_1} \wedge w_{k_2} \cdots \wedge w_{k_j} = sign(\sigma) w_{\sigma(k_1)} \wedge w_{\sigma(k_2)} \cdots \wedge w_{\sigma(k_j)}$$

- Spinor representation on exterior algebra $\Lambda\mathbb{C}^n$ is given by an action

$$w_j : \Lambda\mathbb{C}^n \to \Lambda\mathbb{C}^n$$
$$w_{k_1} \wedge \cdots \wedge w_{k_p} \mapsto w_j \wedge w_{k_1} \wedge \cdots \wedge w_{k_p}$$

and

$$\bar{w}_j : \Lambda\mathbb{C}^n \to \Lambda\mathbb{C}^n$$
$$w_{k_1} \wedge \cdots \wedge w_{k_p} \mapsto \sum_{s=1}^{p} \delta_{js}(-1)^{s+1} w_{k_1} \wedge \dots w_{k_{s-1}} \wedge w_{k_{s+1}} \wedge \cdots \wedge w_{k_p}$$

- $\dim \Lambda\mathbb{C}^n = 2^n$

- $\mathfrak{so}(2n) \subset Cliff(2n, \mathbb{C})$ is generated by

$$\{w_j w_k, \bar{w}_j \bar{w}_k, \bar{w}_j w_k\}_{j,k=1}^{n} = \{[e_j, e_k]\}_{j,k=1}^{n}$$

- Action of $\mathfrak{so}(2n)$ preserves parity of the grades $\Rightarrow$ spinor representation of $\mathfrak{so}(2n)$ is reducible $\Rightarrow$ decomposition into irreducible subspaces:

$$\Lambda\mathbb{C}^n = \Lambda^{even}\mathbb{C}^n \oplus \Lambda^{odd}\mathbb{C}^n$$

where

$$\Lambda^{even}\mathbb{C}^n = \mathbb{C}^n \oplus \Lambda^2\mathbb{C}^n \oplus \dots$$

and

$$\Lambda^{odd}\mathbb{C}^n = \Lambda^1\mathbb{C}^n \oplus \Lambda^3\mathbb{C}^n \dots$$

- fermionic annihilation and creation operators $\{f_j^\dagger, f_j\}_{j=1}^n$:

$$\{f_j^\dagger, f_k^\dagger\} = \{f_j, f_k\} = 0, \quad \{f_j^\dagger, f_k\} = \delta_{jk}$$

- vacuum state $|0\rangle$: $f_j|0\rangle = 0$ for all $j \in \{1, \ldots, n\}$
- fermionic Fock space

$$\mathcal{F}_n^+ = \left\langle |0\rangle, f_{j_1}^\dagger|0\rangle, f_{j_1}^\dagger f_{j_2}^\dagger|0\rangle, \ldots, f_{j_1}^\dagger f_{j_2}^\dagger \ldots f_{j_n}^\dagger|0\rangle \right\rangle$$

Fermionic realization:

$$\begin{aligned}
w_j &\longleftrightarrow f_j^\dagger \\
\bar{w}_j &\longleftrightarrow f_j \\
\mathfrak{so}(2n) \cong \langle w_j w_k, \bar{w}_j \bar{w}_k, \bar{w}_j w_k \rangle &\longleftrightarrow \left\langle f_j f_k, f_j^\dagger f_k^\dagger, f_j^\dagger f_k \right\rangle \\
\Lambda \mathbb{C}^n &\longleftrightarrow \mathcal{F}_n^+ \\
\Lambda^{even} \mathbb{C}^n &\longleftrightarrow \mathcal{F}_{even}^+ := \left\langle |0\rangle, f_{j_1}^\dagger f_{j_2}^\dagger|0\rangle, \ldots \right\rangle \\
\Lambda^{odd} \mathbb{C}^n &\longleftrightarrow \mathcal{F}_{odd}^+ := \left\langle f_{j_1}^\dagger|0\rangle, f_{j_1}^\dagger f_{j_2}^\dagger f_{j_3}^\dagger|0\rangle, \ldots \right\rangle
\end{aligned}$$

Standard model representation

- For one generation ($Q = T_3 + Y$):

$$\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \longleftrightarrow \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}_{-\frac{1}{2}} \longleftrightarrow (1, 2, -\tfrac{1}{2})$$

$$\begin{pmatrix} u_L^R & u_L^G & u_L^B \\ d_L^R & d_L^G & d_L^B \end{pmatrix} \longleftrightarrow \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}_{\frac{1}{6}} \longleftrightarrow (3, 2, \tfrac{1}{6})$$

$$(e_R) \longleftrightarrow \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}_{-1} \longleftrightarrow (1, 1, -1)$$

$$(\nu_R) \longleftrightarrow \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}_{0} \longleftrightarrow (1, 1, 0)$$

$$(u_R^R \quad u_R^G \quad u_R^B) \longleftrightarrow \mathbb{C}^3 \otimes \mathbb{C} \otimes \mathbb{C}_{\frac{2}{3}} \longleftrightarrow (3, 1, \tfrac{2}{3})$$

$$(d_R^R \quad d_R^G \quad d_R^B) \longleftrightarrow \mathbb{C}^3 \otimes \mathbb{C} \otimes \mathbb{C}_{-\frac{1}{3}} \longleftrightarrow (3, 1, -\tfrac{1}{3})$$

$$G_{SM} = SU(3) \times SU(2) \times U(1)$$

- Fermion representation of one generation:

$$F = (1, 2, -\frac{1}{2}) \oplus (3, 2, \frac{1}{6}) \oplus (1, 1, 0) \oplus (1, 1, -1) \oplus (3, 1, \frac{2}{3}) \oplus (3, 1, -\frac{1}{3})$$

- Antifermion representation of one generation:

$$F^* = (1, 2, \frac{1}{2}) \oplus (\bar{3}, 2, -\frac{1}{6}) \oplus (1, 1, 0) \oplus (1, 1, 1) \oplus (\bar{3}, 1, -\frac{2}{3}) \oplus (\bar{3}, 1, \frac{1}{3})$$

- Standard model representation of one fermion and antifermion generation:

$$F \oplus F^*$$

- $\dim F \oplus F^* = 32 = 2^5$

$SO(10)$ in the Grand Unified Theory

- $\dim F \oplus F^* = 2^5 = \dim \Lambda\mathbb{C}^5$
- spinor representation on $\Lambda^{odd}\mathbb{C}^5$, $\Lambda^{even}\mathbb{C}^5$
- Left-handed Weyl spinors:

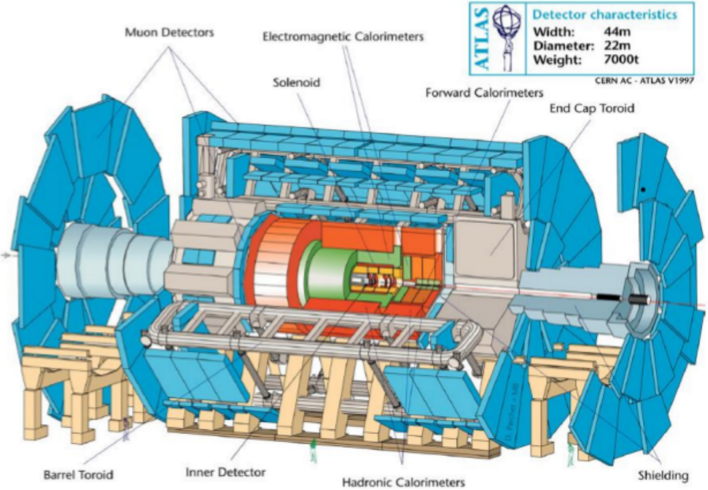
$$\Lambda^{even}\mathbb{C}^5 \cong (\bar{\nu}_L) \oplus (e_L^+) \oplus \begin{pmatrix} u_L^R & u_L^G & u_L^B \\ d_L^R & d_L^G & d_L^B \end{pmatrix} \oplus (\bar{u}_L^R \quad \bar{u}_L^G \quad \bar{u}_L^B) \oplus$$

$$\oplus \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \oplus (\bar{d}_L^R \quad \bar{d}_L^G \quad \bar{d}_L^B)$$

- Right-handed Weyl spinors:

$$\Lambda^{odd}\mathbb{C}^5 \cong (e_R^+ \quad \bar{\nu}_R) \oplus (d_R^R \quad d_R^G \quad d_R^B) \oplus (e_R) \oplus \begin{pmatrix} \bar{u}_L^R & \bar{u}_L^G & \bar{u}_L^B \\ \bar{d}_L^R & \bar{d}_L^G & \bar{d}_L^B \end{pmatrix} \oplus$$

$$\oplus (u_R^R \quad u_R^G \quad u_R^B) \oplus (\nu_R)$$


Muon Detectors
Electromagnetic Calorimeters
Solenoid
Forward Calorimeters
End Cap Toroid
ATLAS
Detector characteristics
Width: 44m
Diameter: 22m
Weight: 7000t
CERN AC - ATLAS V1997
Barrel Toroid
Inner Detector
Hadronic Calorimeters
Shielding

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