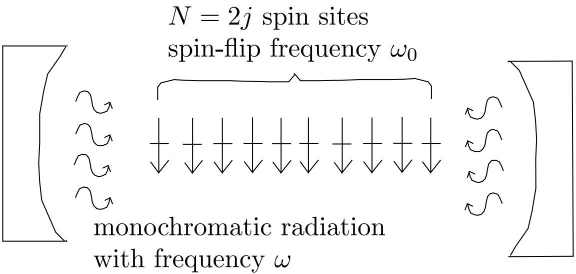
Quantum phase transitions in the Dicke model

Michal Kloc, Pavel Stránský, Pavel Cejnar
kloc@ipnp.troja.mff.cuni.cz

Institute of Particle and Nuclear Physics
Faculty of Mathematics and Physics, Charles University
Prague, Czech Republic

March 9, 2017

Model of atom-field interaction in a cavity



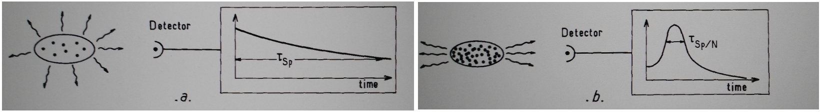
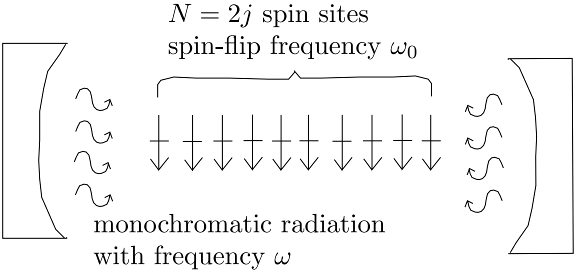


Figure: Coherent superradiant emission, taken from
Gross, Haroche, Physics reports 93.5 (1982): 301-396.

Model of atom-field interaction in a cavity



'*collective*' spin operators:

$$J_+ = \sum_{i=1}^{2j} a^\dagger_{\uparrow i} a_{\downarrow i}, \quad J_- = \sum_{i=1}^{2j} a^\dagger_{\downarrow i} a_{\uparrow i}, \quad J_0 = \frac{1}{2} \sum_{i=1}^{2j} \left(a^\dagger_{\uparrow i} a_{\uparrow i} - a^\dagger_{\downarrow i} a_{\downarrow i} \right),$$

- Supposing the size Δx of the atomic condensate (spin chain) is much smaller than the size of the cavity L.

Model Hamiltonians

The Dicke Hamiltonian

$$H_{\mathrm{D}} = \omega_0 J_0 + \omega b^\dagger b + \frac{\lambda}{\sqrt{2j}} \left(b + b^\dagger\right) \left(J_- + J_+\right)$$

R. H. Dicke, Phys. Rev. 93 (1954) 99

- Number of degrees of freedom $f = 2$, algebraically speaking $SU(2) \times HW(1)$
- Non-integrable, partly chaotic
- In principle infinite number of photons, the basis must be properly truncated for numerical calculations

Model Hamiltonians

The Dicke Hamiltonian

$$H_{\mathrm{D}} = \omega_0 J_0 + \omega b^\dagger b + \frac{\lambda}{\sqrt{2j}} \left(b + b^\dagger\right) \left(J_- + J_+\right)$$

R. H. Dicke, Phys. Rev. 93 (1954) 99

The Tavis-Cummings Hamiltonian

$$H_{\mathrm{TC}} = \omega_0 J_0 + \omega b^\dagger b + \frac{\lambda}{\sqrt{2j}} \left(b J_+ + b^\dagger J_-\right)$$

Tavis, Cummings, Phys. Rev. 170 (1968) 379

- Dicke model in a *Rotating wave approximation*
- Additional conserved quantity $M = b^\dagger b + J_0 + j$ reduces number of degrees of freedom to $f = 1$
- Integrable, regular system
- The dynamics of the system splits into mutually non-interacting subspaces numbered by M $\rightsquigarrow$ *invariant subspaces*

Extended Dicke model

The Hamiltonian

$$H(\lambda,\delta) = \omega b^\dagger b + \omega_0 J_0 + \frac{\lambda}{\sqrt{2j}}\left(bJ_+ + b^\dagger J_- + \delta b^\dagger J_+ + \delta b J_-\right)$$

Additional parameter $\delta \in [0,1]$ smoothly varying between integrable Tavis-Cummings and full Dicke regimes

T. Brandes Phys. Rev. E 88, 032133 (2013)

Extended Dicke model

The Hamiltonian

$$H(\lambda,\delta) = \omega b^\dagger b + \omega_0 J_0 + \frac{\lambda}{\sqrt{2j}}\left(bJ_+ + b^\dagger J_- + \delta b^\dagger J_+ + \delta b J_-\right)$$

Additional parameter $\delta \in [0,1]$ smoothly varying between integrable Tavis-Cummings and full Dicke regimes

T. Brandes Phys. Rev. E 88, 032133 (2013)

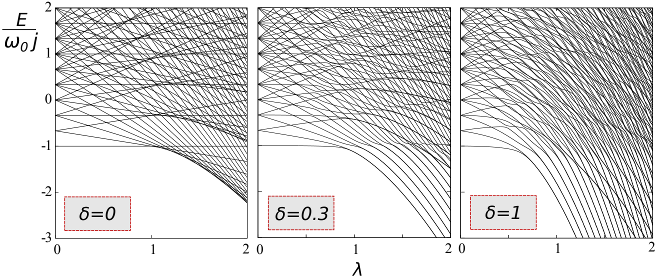


Figure: Spectra of quantum energies as a function of control parameter λ, $j = 3$, $\omega_0 = \omega = 1$, $\lambda_c = 1/(1+\delta)$

Intermezzo: entanglement

Quantum entanglement

Let's have a composite system $\mathcal{H} = \mathcal{H}_{\mathrm{l}} \otimes \mathcal{H}_{\mathrm{r}}$. We denote orthonormal bases on $\mathcal{H}_{\mathrm{l}}$ and $\mathcal{H}_{\mathrm{r}}$ as $\{|\psi_{\mathrm{l}i}\rangle\}_{i=1}^{m}$ and $\{|\psi_{\mathrm{r}j}\rangle\}_{j=1}^{n}$. Any state $|\Psi\rangle \in \mathcal{H}$ can be expressed as

$$|\Psi\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} \gamma_{ij}|\psi_{\mathrm{l}i}\rangle|\psi_{\mathrm{r}j}\rangle \neq \sum_{i=1}^{m} \alpha_i|\psi_{\mathrm{l}i}\rangle \sum_{j=1}^{n} \beta_j|\psi_{\mathrm{r}j}\rangle,$$

i.e. coefficients γ_{ij} generally cannot be factorized $\gamma_{ij} \neq \alpha_i\beta_j$.

- purely quantum correlation
- for example spin singlet $|00\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$
- EPR paradox, Deutsch algorithm (quantum information)

Entanglement measures

Entanglement measures

Atom-field entanglement

$$S(\psi) = -\frac{\mathrm{Tr}\,[\rho_{\mathrm{A}}\,\ln\rho_{\mathrm{A}}]}{\ln(2j+1)} = -\frac{\mathrm{Tr}\,[\rho_{\mathrm{F}}\,\ln\rho_{\mathrm{F}}]}{\ln(2j+1)}\,. \tag{1}$$

Atom-atom entanglement

$$C(\psi) = (N-1)\max\left\{\sqrt{\lambda_1}-\sqrt{\lambda_2}-\sqrt{\lambda_3}-\sqrt{\lambda_4}\,,0\right\}\,, \tag{2}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are eigenvalues (real and non-negative) of a non-Hermitian matrix $\varrho = \rho_{\mathrm{A}}^{kl}\,\left(\sigma_y^k \otimes \sigma_y^l\right)\rho_{\mathrm{A}}^{kl*}\left(\sigma_y^k \otimes \sigma_y^l\right)$.

$\rho_{\mathrm{A}}^{kl} \leftrightarrow$ density matrix of a pair of qubits. Due to the symmetry can be expressed through the expectation values of the collective spin operators $\langle J_z\rangle$, $\langle J_z^2\rangle$ and $\langle J_+\rangle$

X. Wang and K. Mølmer, Eur. Phys. J. D 18 385 (2002)

Entanglement in QPT

N. Lambert, C. Emary, T. Brandes, Phys. Rev. Let 92(7), 073602 (2008)

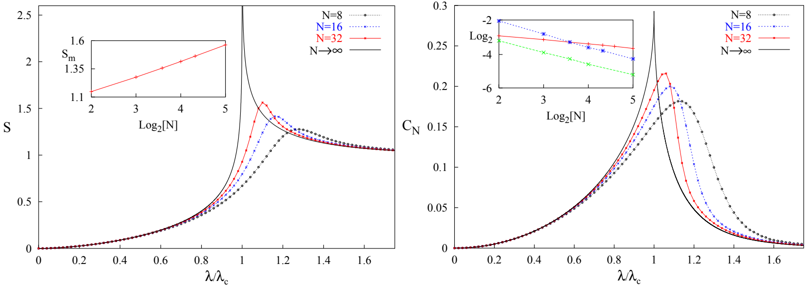


Figure: Anomalous behaviour of entanglement in QPT in Dicke model.

Extended Dicke model

Entanglement in QPT

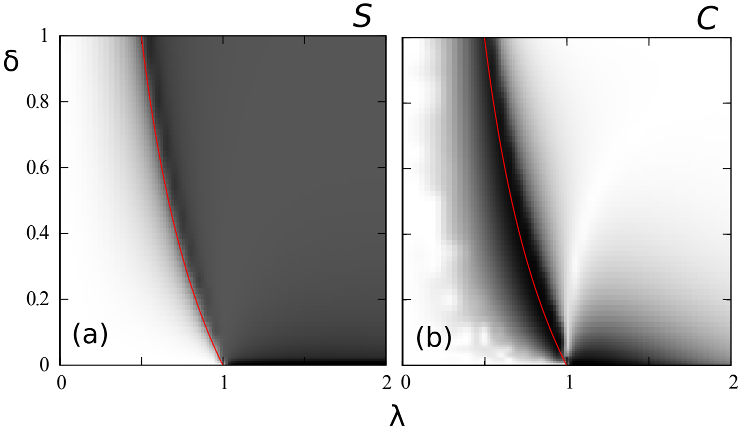


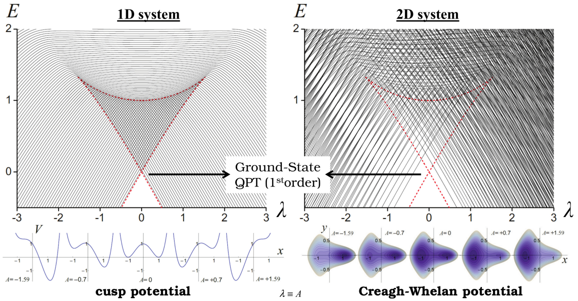
Figure: Entanglement properties of the ground state of the $j = N/2$ model in the plane of control parameters λ and δ (for $\omega = \omega_0 = 1$, $N = 40$). The red curve indicates the QPT critical coupling λ_c

Excited-state quantum phase transitions

ESQPTs are

'Singularities' in

1. energy eigenstate density as a function of excitation energy
2. flow of energy spectrum as a function of the control parameter



Level density

$H(\lambda,\delta) = \omega b^\dagger b + \omega_0 J_0 + \frac{\lambda}{\sqrt{2j}}\left(bJ_+ + b^\dagger J_- + \delta b^\dagger J_+ + \delta b J_-\right)$

- Such collective models can be treated semiclassically in thermodynamic limit after a proper scaling

  $\mathfrak{b} \equiv \frac{b}{\sqrt{N}} \rightsquigarrow [\mathfrak{b}, \mathfrak{b}^\dagger] = \frac{1}{N}, \quad \mathfrak{J}_\bullet \equiv \frac{J_\bullet}{N} \rightsquigarrow [\mathfrak{J}_x, \mathfrak{J}_y] = \frac{i}{N}\mathfrak{J}_z$
- Several ways to obtain the semiclassical Hamiltonian
  M. A. Bastarrachea-Magnani et al Phys. Rev. A 89, 032101 (2014)

$$H_{\rm cl} = \omega_0 j_z + \frac{\omega}{2}\left(p^2 + x^2\right) + \lambda\sqrt{j}\sqrt{1 - \frac{j_z^2}{j^2}}\left[(1+\delta)q\cos\phi - (1-\delta)p\sin\phi\right],$$

- Semiclassical level density $\rho(E) = \frac{\partial}{\partial E}\frac{1}{(2\pi)^f}\int d^f\mathrm{q}\; d^f\mathrm{p}\; \Theta(E - H_{\rm cl}),$

Level density

$H(\lambda,\delta) = \omega b^\dagger b + \omega_0 J_0 + \frac{\lambda}{\sqrt{2j}}\left(bJ_+ + b^\dagger J_- + \delta b^\dagger J_+ + \delta b J_-\right)$

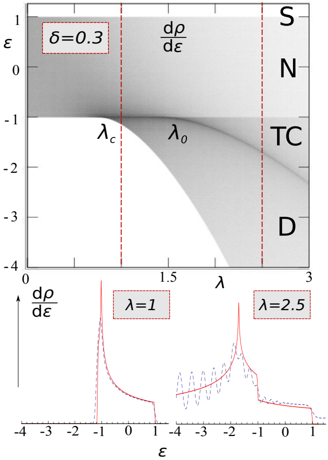


Figure: The derivative of semiclassical level density ρ with respect to $\varepsilon \equiv \frac{E}{\omega_0 j}$.

Critical couplings:

- $\lambda_c = \frac{\sqrt{\omega_0\omega}}{1+\delta}$ and $\lambda_0 = \frac{\sqrt{\omega_0\omega}}{1-\delta}$

'Phase diagram' - - →

D - Dicke phase
TC - Tavis-Cummings phase
N - Normal phase
S - Saturated phase

Specification of the phases

$H(\lambda, \delta) = \omega b^\dagger b + \omega_0 J_0 + \frac{\lambda}{\sqrt{2j}} \left(bJ_+ + b^\dagger J_- + \delta b^\dagger J_+ + \delta b J_- \right)$

Peres lattice: quantum expectation value $\langle \psi_i | \bullet | \psi_i \rangle$ in individual energy eigenstates *vs.* the energy E_i

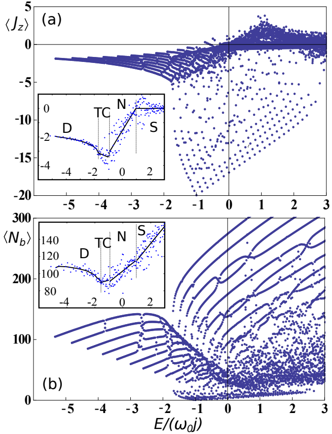


Figure: Peres lattices, $j = 20$, $\omega = \omega_0 = 1$, $\lambda = 2.5$, $\delta = 0.3$.

Inset: Averaged data over 20 neighbouring eigenstates.

Specification of the phases

$H(\lambda,\delta) = \omega b^\dagger b + \omega_0 J_0 + \frac{\lambda}{\sqrt{2j}}\left(bJ_+ + b^\dagger J_- + \delta b^\dagger J_+ + \delta b J_-\right)$

Peres lattice: quantum expectation value $\langle\psi_i|\bullet|\psi_i\rangle$ in individual energy eigenstates *vs.* the energy E_i

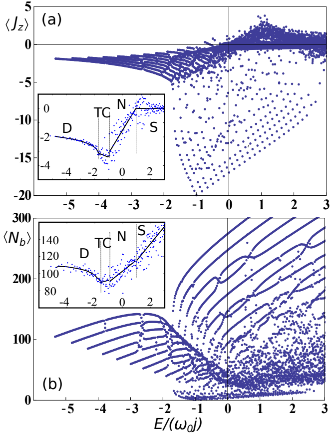


Figure: Peres lattices, $j = 20$, $\omega = \omega_0 = 1$, $\lambda = 2.5$, $\delta = 0.3$.

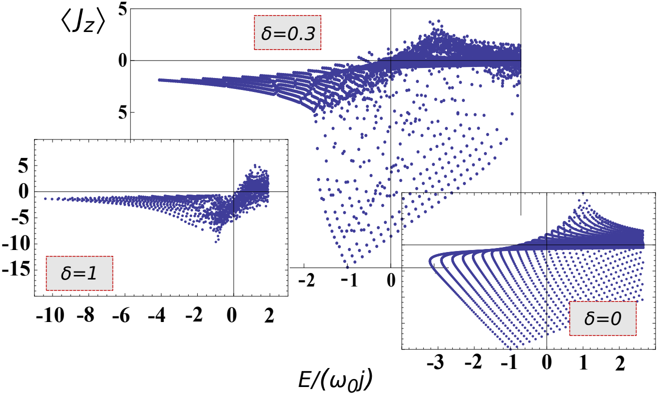
Inset: Averaged data over 20 neighbouring eigenstates.

Different energy dependence of the averaged data $\leftrightarrow$ *quantum phases.*

Specification of the phases

$H(\lambda,\delta) = \omega b^\dagger b + \omega_0 J_0 + \frac{\lambda}{\sqrt{2j}}\left(bJ_+ + b^\dagger J_- + \delta b^\dagger J_+ + \delta b J_-\right)$

Comparing the phases with the limiting regimes, i. e. the Dicke ($\delta = 1$) and Tavis-Cummings ($\delta = 0$) models



Specification of the phases

$H(\lambda,\delta) = \omega b^\dagger b + \omega_0 J_0 + \frac{\lambda}{\sqrt{2j}}\left(bJ_+ + b^\dagger J_- + \delta b^\dagger J_+ + \delta b J_-\right)$

Feynman-Hellmann theorem $\frac{dE_i}{d\lambda} = \langle \frac{dH}{d\lambda} \rangle_i = \frac{\langle H_{\text{int}} \rangle_i}{\sqrt{2j}}$

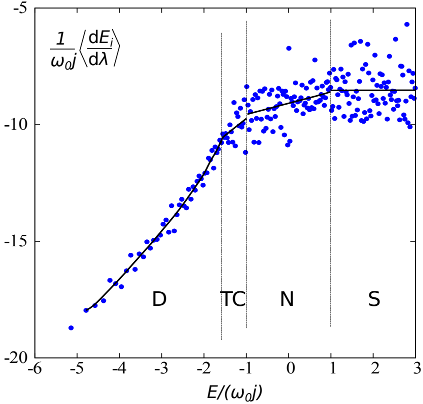


Figure: The slopes averaged over 20 neighbouring levels, $j = 20$, $\omega = \omega_0 = 1$, $\lambda = 2.5$, $\delta = 0.3$.

Atom-field entanglement

Motivation:
Known anomalous behaviour of entanglement related to ground state QPT. Do singularities in an excited spectrum affect entanglement, too?

Our approach:
We study bipartide entanglement of *atoms vs. field*

Quantification using (scaled) von Neumann entropy
$S(\rho_i) = -\mathrm{Tr}\left[\rho_i \ln \rho_i\right] / \ln\left(2j+1\right)$,
where ρ_i is a reduced density matrix related to either atomic or photonic subsystems

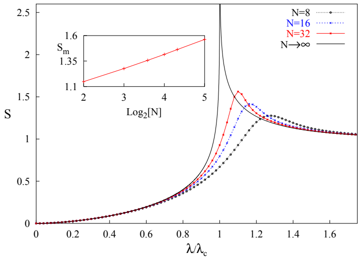


Figure: Diverging entanglement entropy at QPT for Dicke model ($\delta = 1$).

N. Lambert et al, Phys. Rev. Lett. 92, 073602 (2004)

Atom-field entanglement

Question: What happens to the entanglement entropy when varying parameter δ?

$$H(\lambda,\delta) = \omega b^{+} b + \omega_0 J_0 + \frac{\lambda}{\sqrt{2j}}\left(bJ_{+} + b^{+}J_{-} + \delta b^{+}J_{+} + \delta b J_{-}\right)$$

We follow the evolution of the '*entropic spectrum*'.

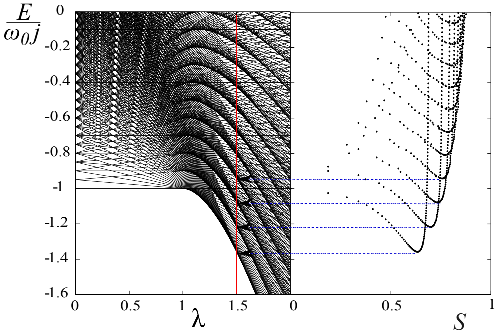


Figure: How the entropic spectrum is computed, $\delta = 0$, $\lambda = 1.5$. Parameters $j = 20, \omega = \omega_0 = 1$

Atom-field entanglement, evolution along δ

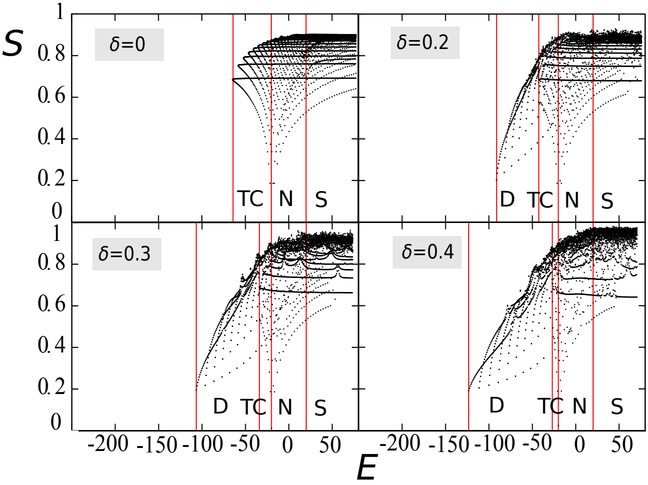


Figure: The evolution of entanglement entropy with δ, $\lambda = 2.5$. Parameters $j = 20, \omega = \omega_0 = 1$.

Atom-field entanglement, evolution along δ

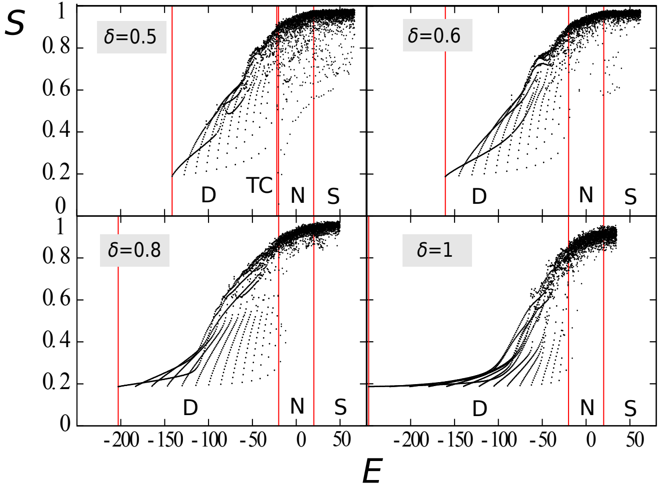


Figure: The evolution of entanglement entropy with δ, $\lambda = 2.5$. Parameters $j = 20, \omega = \omega_0 = 1$

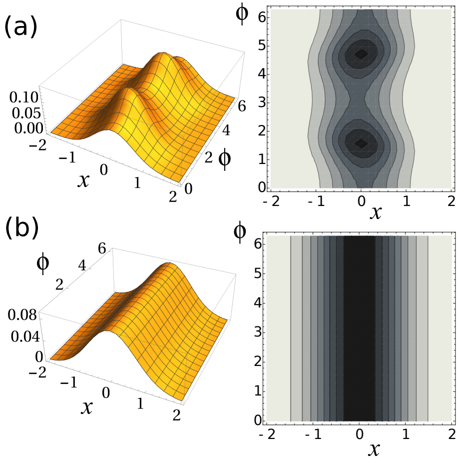


Figure: (Color online) A detail of the wave functions. In panel (a) we show the eigenstate at ESQPT between TC and N phases, in (b) we present a simple non-perturbed ground state wave function.

Atom-atom pairwise entanglement

Motivation:
As a counterpart one may wish to observe how two individual spins are entangled

Our approach:
Entanglement entropy is no longer a good measure ⇝ atoms are not in a pure state

Quantification using (scaled) *concurrence* $C_N \equiv \mathcal{C}N$ with $\mathcal{C} = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, where λ_i are eigenvalues of $\varrho_{12} = \rho_{12}(\sigma_{1y} \otimes \sigma_{2y})\rho^{\star}_{12}(\sigma_{1y} \otimes \sigma_{2y})$. ϱ_{12} is a reduced density matrix for two atoms.

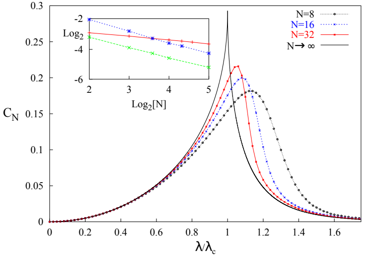


Figure: Peak in a rescaled concurrence at QPT for Dicke model ($\delta = 1$).

N. Lambert et al, Phys. Rev. Lett. 92, 073602 (2004)

Pairwise entanglement, evolution along δ

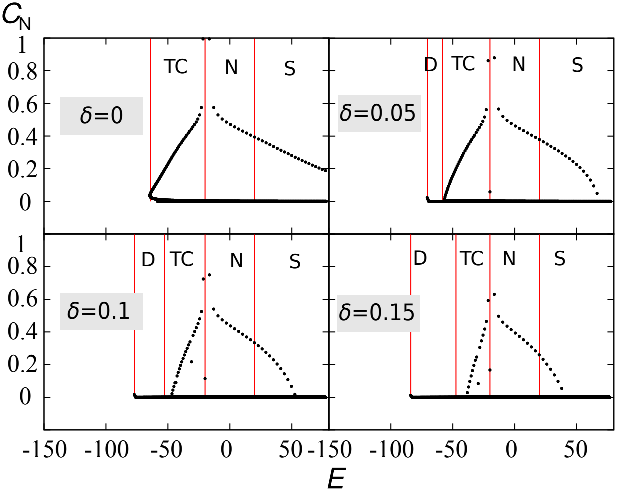


Figure: The evolution of entanglement entropy with δ, $\lambda = 2.5$. Parameters $j = 20, \omega = \omega_0 = 1$.

Pairwise entanglement, evolution along δ

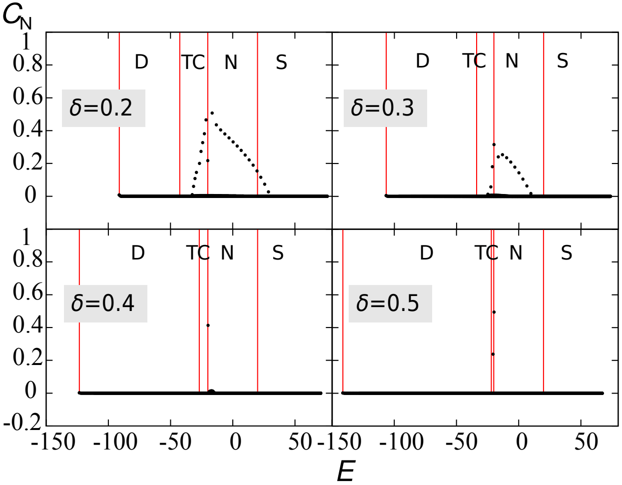


Figure: The evolution of entanglement entropy with δ, $\lambda = 2.5$. Parameters $j = 20, \omega = \omega_0 = 1$

Summary

- Despite its relative simplicity, Extended Dicke model is very rich theoretical ‘playground’ in many-body physics.
- We thoroughly studied the ESQPTs in the system and proposed how to specify different phases in the spectrum.
- We numerically studied entanglement in excited states.
- Some anomalies related to certain ESQPTs have been observed.
- Publication accepted by Annals of Physics :-)