Astrophysics of Collapsing and Colliding Axion Stars

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Big Picture

Axions are light scalar fields: in QCD, masses $10^{-6} - 10^{-2}$ eV.

Interaction potential $V(\Phi) = m^2 f^2 \left[1 - \cos\left(\frac{\Phi}{f}\right)\right]$

Axion Star (ASt): macroscopic condensate of low-energy axions

- Hydrodynamic equation of state
- Supported by balance of
  - self-gravity
  - uncertainty (kinetic) pressure
  - self-interactions
- $N \sim 10^{60}$ axions, sphere of radius R
- Non-relativistic wavefunction $\psi(r)$
- QM coordinate / momentum uncertainty

Want to know

- Macroscopic properties; Stability

?: If unstable, what happens during collapse?

?: What can trigger collapse?

Variational Method: Minimize Axion Star Energy

Self-interaction energy of an ASt, in the non-relativistic limit:

$$W(\psi) = -m^2 f^2 \sum_{k=0}^{\infty} (-1)^k a_k \left[\frac{\psi^* \psi}{2\, m\, f^2}\right]^{k+2}$$

The total energy

$$E(\psi) = \int d^3 r \left[\overbrace{\frac{|\nabla \psi|^2}{2m}}^{\text{kinetic}} + \overbrace{\frac{1}{2} V_{grav}(|\psi|^2)}^{\text{gravitational}} + \overbrace{W(\psi)}^{\text{self-interaction}} \right]$$

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We make an ansatz for the wavefunction $\psi(r)$ and compute E:

Rescale: Energy e, Radius ρ, and Particle number n

$$e(\rho) \equiv \frac{E(\rho)}{m\, N\, \delta} = \frac{\alpha}{\rho^2} - \frac{\beta\, n}{\rho} - \frac{1}{\delta} \sum_{k=0}^{\infty} (-1)^k \gamma_k \left(\frac{n\, \delta}{\rho^3}\right)^{k+1}$$

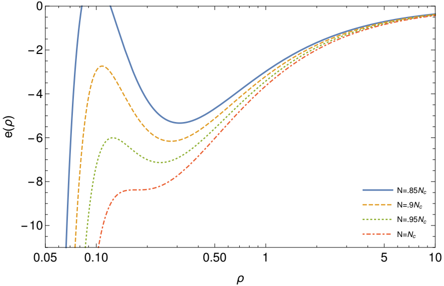
$\delta \equiv f^2 / M_P{}^2 \ll 1$ $\qquad$ α, β, γ_k ansatz-dependent constants

Minima of E w.r.t. ρ are stable bound states, axion stars!

Dilute[1] vs Dense[2] Axion Stars

Dilute: $e_0(\rho) = \frac{\alpha}{\rho^2} - \frac{\beta\, n}{\rho} - \frac{\gamma_0\, n}{\rho^3}$

Well-approximated by only leading, attractive self-interaction term. Minima at $\rho = \mathcal{O}(1)$.



Exists for $M < M_c \sim 10^{19}$ kg; $R \sim \mathcal{O}(100)$ km

[1] Barranco/Bernal (1001.1769); Chavanis/Delfini (1103.2050, 1103.2054); (1412.3430); Guth/Hertzberg/Prescod-Weinstein (1412.5930)

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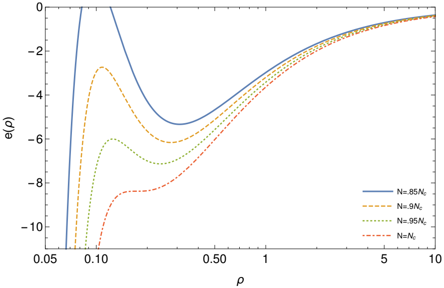

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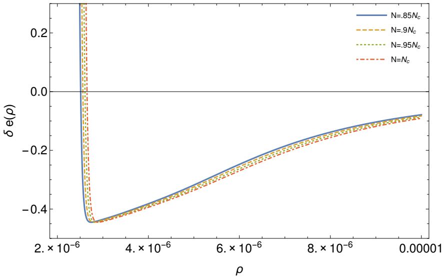
Dense: $e_K(\rho) = e_0(\rho) + \frac{\gamma_1\, n^2\, \delta}{\rho^6} - \frac{\gamma_2\, n^3\, \delta^2}{\rho^9} + \ldots$

Use full self-interaction potential; higher-order terms, some repulsive, are important at $\rho \ll 1$.



Exists for any M; $R \sim$ few meters

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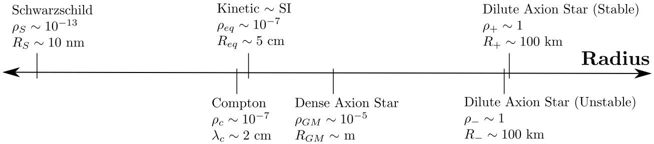
Collapse of Dilute ASts

A dilute ASt can collapse if e.g. its mass exceeds M_c. Variational approximation of the total energy $E(\rho)$ lends itself directly to a classical collapse analysis:[3]

- LO: Collapse to black hole.
- Full axion potential: Collapse from dilute to dense ASt state!



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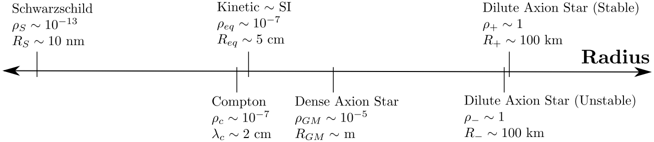

However: Possible quantum mechanical effects lead to decay of ASts.
Axions are real scalars $\Rightarrow$ No symmetry protects axion number

- Can decay through $a \to 2\gamma$ (in free or condensed state)
- Inside the condensate, novel decay mechanisms are allowed through self-interactions:[4]
  - Microscopic picture: $N\, a_c \to N'\, a_c + j\, a_f$
  - Macroscopic picture: $\mathcal{A}_N \to \mathcal{A}_{N'} + j\, a_f$

Leading order process: Emission of single relativistic axion, momentum $p = \sqrt{9\,E_0^2 - m^2} \approx \sqrt{8}\, m$

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Bosenova[6]

!: Decay rate increases as the density / binding energy increases[5]; it is an increasing function of parameter

$$\Delta \equiv \sqrt{1 - \frac{{E_0}^2}{m^2}}$$

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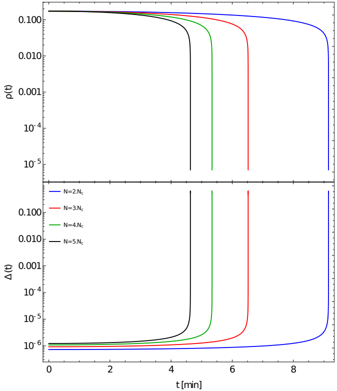

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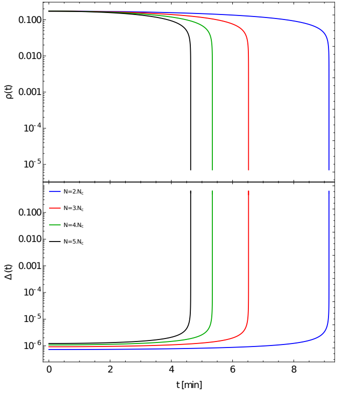

- Collapsing axion stars have rapidly increasing binding energy, leading to fast emission of relativistic free axions
- As yet, not clear if a significant fraction of the ASt mass remains to form a dense ASt, or if such states are stable against decay on long timescales

Upshot: Collapse → Bosenova!

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Induced Instability

?: What might induce collapse of otherwise stable axion stars?

- Accretion to large masses $M > M_c$
- Interactions which change energy functional, e.g. astrophysical collisions?
- We analyzed three possibilities: collisions between...

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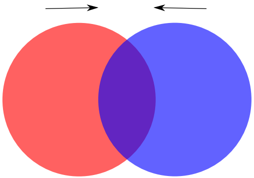
Axion Star - Axion Star

Two roughly equal-size objects
Important axion self-interactions



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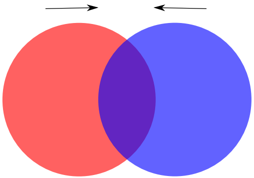
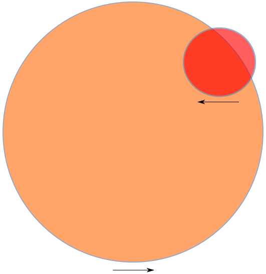


Axion Star - Ordinary Star

$R_\odot \gg R_{AS}$
Gravitational interaction



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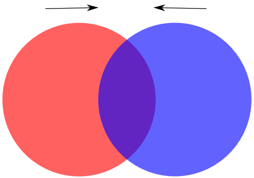

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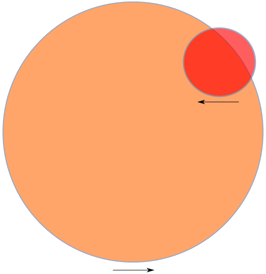
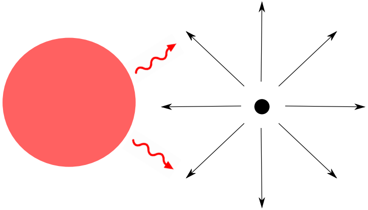


Axion Star - Neutron Star

$a F \tilde{F}$ coupling converts $a \to \gamma$



Source of Fast Radio Bursts?

Collision Rates

The rate of astrophysical collisions is

$$\Gamma = \underbrace{\frac{1}{S}}_{\substack{S=2 \\ \text{if} \\ i=AS}} \int d^3r \quad \underbrace{n_{AS}(\vec{r})}_{\substack{\text{Number Density} \\ \text{of AS}}} \underbrace{n_i(\vec{r})}_{\substack{\text{Number Density} \\ \text{of “i”}}} \underbrace{\langle \sigma\, v \rangle_i}_{\substack{\text{Cross section} \\ \text{times} \\ \text{relative velocity}}}$$

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$$n_{AS}(r) = \frac{\rho_{NFW}(r)}{m} = \frac{\rho_0/m}{\frac{r}{R_0}\left(1 + \frac{r}{R_0}\right)^2}$$

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Total Number of AS:

$$N_{AS} = \frac{\mathcal{F}_{DM}\, M_{DM}}{\mathcal{F}_{AS}\, M_{AS}}$$

$\mathcal{F}_{AS}\, M_{AS} \equiv$ average ASt mass
$\mathcal{F}_{DM}\, M_{DM} \equiv$ ASt contribution to DM

$M_{AS} = 10^{19}$ kg (QCD) and
$M_{DM} = 10^{12} M_\odot$ (MW) give

$$N_{AS} \sim 10^{23}\frac{\mathcal{F}_{DM}}{\mathcal{F}_{AS}}$$

Collisions: 2 Axion Stars

Energy functional changes to

$$\frac{E_{2AS}}{m\,(N_1+N_2)\,\delta}=\frac{\alpha}{\rho^2}-\frac{\beta\,(n_1+n_2)}{\rho}-\frac{\gamma_0\,(n_1+n_2)}{\rho^3}+\ldots$$

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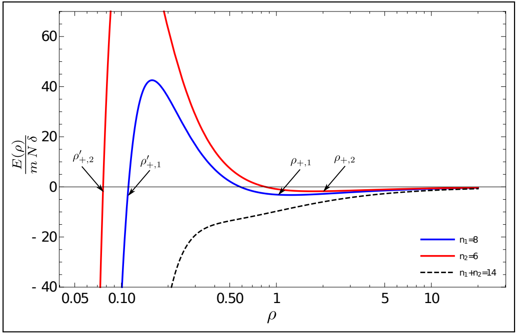

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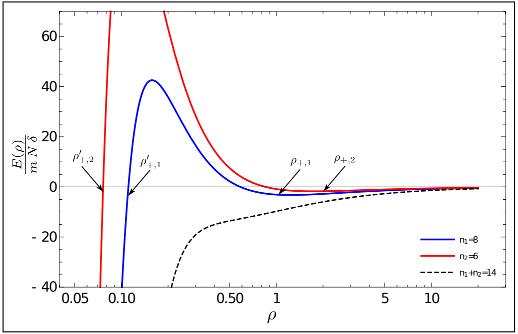
If $N_1 + N_2 > N_c$, *collapse!*

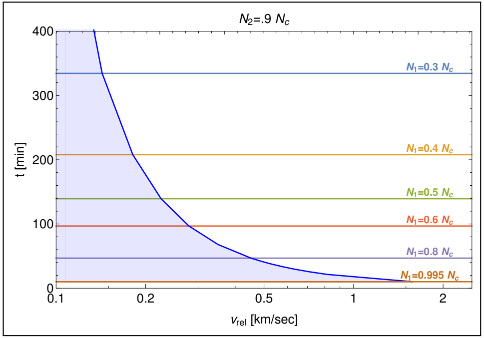
Long time for collapse to be completed:

Total $\Gamma \sim 10^7 \left(\frac{\mathcal{F}_{DM}}{\mathcal{F}_{AS}}\right)^2 \frac{\text{collisions}}{\text{year}\cdot\text{galaxy}}$

But $P(v_{rel} \lesssim 1\ \text{km/sec}) \sim 10^{-8}$

$\Rightarrow$ Small total collapse rate



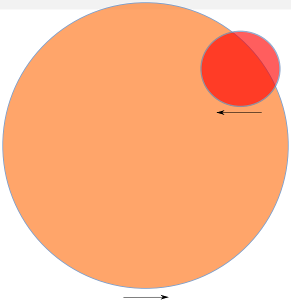


Collisions: Axion Star and Ordinary Star

- Energy functional acquires additional term

$$\frac{E_{GS}(\rho)}{m\,N\,\delta} = \frac{E(\rho)}{m\,N\,\delta} + \frac{\mu_s\,F}{\rho_s^3}\rho^2 + c_1$$

- Effect of star's gravity can decrease $N_c \Rightarrow$ *collapse*
- Large number of collisions: $\Gamma \sim 3000\,\frac{\mathcal{F}_{DM}}{\mathcal{F}_{AS}}\,\frac{\text{collisions}}{\text{year} \cdot \text{galaxy}}$



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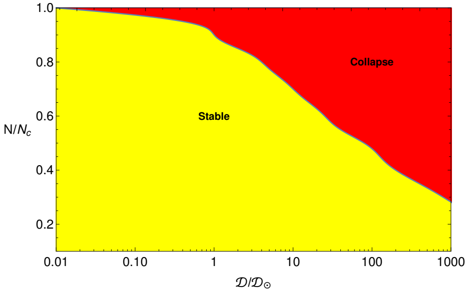
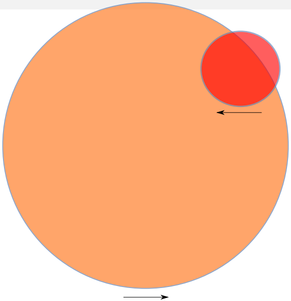
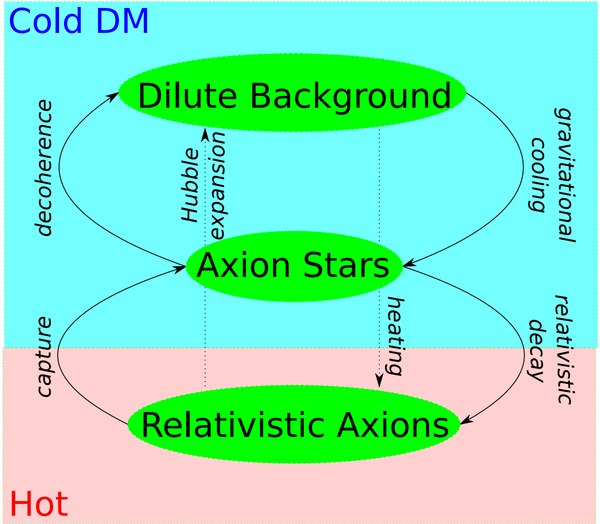




- Depends on type of star (density $\mathcal{D} = M_s/{R_s}^3$)
- Typically $R_s \gg R_{AS}$
  - Lots of time to collapse
- Each induced collapse could convert $\mathcal{O}(1)$ fraction of axion star mass to relativistic particles in Bosenova

Disappearing Dark Matter



Estimate of conversion rate to relativistic axions through collisions with ordinary stars:

- *If* an $\mathcal{O}(1)$ fraction of DM is in axion stars,
- *If* an $\mathcal{O}(1)$ fraction of axion stars have $M \sim M_c = 10^{19}$ kg,
- *If* an $\mathcal{O}(1)$ fraction of $\mathcal{O}(1000)$ collisions/year/galaxy lead to collapse,
- *If* an $\mathcal{O}(1)$ fraction of a collapsing axion star's mass is converted to relativistic particles,

Then:

$$M_{converted} \sim \mathcal{O}(10^{23})\,\frac{\text{kg}}{\text{year}\cdot\text{galaxy}}$$

Collisions: Axion Star and Neutron Star

Fast Radio Bursts (FRBs) are characterized by

- Narrow Bandwidth: $1.2 - 1.6$ GHz $\leftarrow$ Axion mass is $m \sim \left(\frac{m_{eV}}{10^{-6}}\right)$ GHz

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Note:

This event rate has been estimated before us. Our contribution:

- Updated phenomenological fit to NS distribution
- Characterization of uncertainties by $\mathcal{F}$ factors

Interpretation as source of FRBs remains interesting proposal!

Conclusions

- There are energetically allowed **dilute** and **dense** axion star states
- The dense state radius is larger than the Schwarzschild radius
- Gravitationally unstable dilute axion stars collapse toward the dense state, but emit a large number of relativistic axions along the way
  - ASt collapse $\Rightarrow$ Bosenova
- Collisions between ASts can induce collapse.
  - Collapses induced by ASt-ASt collisions likely negligible
  - Possible measurable effect from ASt-Star collisions
- Still need to understand the mass distribution of ASts, both in early universe and how it evolves today

Acknowledgements

- M. Leembruggen, J. Leeney, P. Suranyi, L.C.R. Wijewardhana
- University of Cincinnati and Fermi National Accelerator Laboratory
- Conference Organizers
- All of you!

Thank you!

Backup Slides

Relativistic Classical Field Analysis

This is the framework used by Kaup (for Ł $= 0$) and by Colpi et al. (for Ł $\gg 1$):

$$\frac{A'}{A^2 x} + \frac{1}{x^2}\left(1 - \frac{1}{A}\right) = \left(\frac{\Omega^2}{B} + 1\right)\sigma^2 + \frac{\Lambda}{2}\sigma^4 + \frac{(\sigma')^2}{A}$$

$$\frac{B'}{B^2 x} + \frac{1}{x^2}\left(1 - \frac{1}{A}\right) = \left(\frac{\Omega^2}{B} + 1\right)\sigma^2 - \frac{\Lambda}{2}\sigma^4 + \frac{(\sigma')^2}{A}$$

$$\sigma'' + \left(\frac{2}{x} + \frac{B'}{2B} - \frac{A'}{2A}\right)\sigma' + A\left[\left(\frac{\Omega^2}{B} - 1\right)\sigma - \Lambda\sigma^3\right] = 0,$$

where the rescaled variables are $x = m\, r$, $\sigma = \sqrt{4\pi\, G}\,\phi$, $\Omega = \mu_0/m$ (μ_0 the eigenenergy of one boson), and $\Lambda = \lambda\, {M_P}^2/(4\,\pi\, m^2)$

- Kaup: Non interacting theories have $M_{max}^{NI} = .633\,\frac{{M_P}^2}{m}$
- Colpi et al.: Repulsively interacting theories have $M_{max}^{rep} = .22\sqrt{\frac{\lambda}{4\,\pi}}\,\frac{{M_P}^3}{m^2}$

RB for Axions

Expectation values of EKG:

$$\langle N | G_\mu{}^\nu | N \rangle = 8\,\pi\,G\,\langle N | T_\mu{}^\nu | N \rangle$$
$$\langle N-1 | \left[\mathcal{D}\,\mathcal{A} - V'(\mathcal{A}) \right] | N \rangle = 0$$

The resulting equations of motion take the form

$$\frac{A'}{A^2\,r} + \frac{A-1}{A\,r^2} = \frac{8\,\pi\,f^2}{{M_P}^2}\left[\frac{{\mu_0}^2\,N\,R^2}{B\,f^2} + \frac{N\,R'^2}{A\,f^2} + m^2\left[1 - J_0\left(\frac{2\sqrt{N}\,R}{f}\right)\right]\right],$$
$$\frac{B'}{A\,B\,r} - \frac{A-1}{A\,r^2} = \frac{8\,\pi\,f^2}{{M_P}^2}\left[\frac{{\mu_0}^2\,N\,R^2}{B\,f^2} + \frac{N\,R'^2}{A\,f^2} - m^2\left[1 - J_0\left(\frac{2\sqrt{N}\,R}{f}\right)\right]\right],$$
$$\sqrt{N}\,R'' + \sqrt{N}\left(\frac{2}{r} + \frac{B'}{2\,B} - \frac{A'}{2\,A}\right)R' + A\left[\frac{\sqrt{N}\,{\mu_0}^2}{B}R - f\,m^2\,J_1\left(\frac{2\sqrt{N}\,R}{f}\right)\right] = 0$$

RB for Axions (2)

Metric functions:

$$A = 1 + \delta\, a, \qquad B = 1 + \delta\, b$$

EKG at leading order in $\delta = 8\pi\, f^2/{M_P}^2$:

$$\begin{aligned}
a' &= -\frac{a}{z} + z\left[\frac{1}{4}\,{\epsilon_\mu}^2\, Z^2 + \frac{1}{4} Z'^2 + 1 - J_0(Z)\right],\\
b' &= \frac{a}{z} + z\left[\frac{1}{4}\,{\epsilon_\mu}^2\, Z^2 + \frac{1}{4} Z'^2 - 1 + J_0(Z)\right],\\
Z'' &= \left[-\frac{2}{z} + \frac{\delta}{2}(a' - b')\right] Z' - {\epsilon_\mu}^2 (1 + \delta\, a - \delta\, b)\, Z + 2\,(1 + \delta\, a)\, J_1(Z)
\end{aligned}$$

RB for Axions (3)

EKG at leading order in δ and $\Delta = \sqrt{1 - {\epsilon_\mu}^2}$:

$$
\begin{aligned}
a'(x) &= \frac{x}{2}\, Y(x)^2 - \frac{a(x)}{x}, \\
b'(x) &= \frac{a(x)}{x}, \\
Y''(x) &= -\frac{2}{x} Y'(x) - \frac{1}{8}\, Y(x)^3 + [1 + \kappa\, b(x)]\, Y(x)
\end{aligned}
$$

Particle Number in Real Scalar Field Theory

Recall that the conjugate momentum for a scalar field Φ can be written as

$$\Pi = D_t\,\Phi = \frac{1}{B}\left[-i\,\mu_n\,R_n(\vec{r})\,e^{-i\,\mu_n\,t}\,a_n + i\,\mu_n\,R_n^{\dagger}(\vec{r})\,e^{i\,\mu_n\,t}\,a_n^{\dagger}\right],$$

which is implicitly summed over n. D_t is a covariant time derivative which gives rise to the metric function B in the denominator. Then

$$[\Phi(\vec{r}), \Pi(\vec{r}\,')] = \frac{2}{B}\,R_n^{\dagger}(\vec{r})\,R_n(\vec{r}\,').$$

The requirement that the commutator be canonically normalized, $[\Phi, \Pi] = \delta^3(\vec{r} - \vec{r}\,')$, is equivalent to a completeness relation on the R_n functions:

$$\sum_n \frac{2}{B}\,\mu_n\,R_n^{\dagger}(\vec{r})\,R_n(\vec{r}\,') = \delta^3(\vec{r} - \vec{r}\,')$$

Given that the R_n functions form a complete set, we can write down a related normalization condition,

$$\int \frac{2}{B}\,\mu_n\,R_n^{\dagger}(\vec{r})\,R_n(\vec{r})\,\sqrt{|g|}\,d^3r = 1$$

Binding Energy Corrections to the mass M

$$\langle N|T_{00}|N\rangle = f^2\left[\frac{{\mu_0}^2\,N\,R^2}{B\,f^2}+\frac{N\,R'^2}{A\,f^2}-m^2\,[1-J_0(X)]\right]$$
$$= f^2\left[\frac{{\mu_0}^2\,X^2}{4\,B}+\frac{m^2\,X'^2}{4\,A}+\frac{m^2}{4}X^2-\frac{m^2}{64}X^4+\ldots\right]$$
$$= f^2\,m^2\left[\frac{{\epsilon_\mu}^2}{4\,B}\Delta^2\,Y^2+\frac{1}{4}\Delta^2\,Y^2+\frac{\Delta^4\,Y'^2}{4\,A}-\frac{\Delta^4\,Y^4}{64}\right]$$

Thus the mass is

$$M = \int\langle N|T_{00}|N\rangle\sqrt{|g|}\,d^3r$$
$$\approx \frac{f^2}{m\,\Delta}\pi\int\left[(1-\Delta^2)(1-\delta\,b)\,Y^2+Y^2+\frac{\Delta^2\,Y'^2}{A}-\frac{\Delta^2\,Y^4}{16}\right]\left[1+\frac{\delta}{2}(a+b)\right]\,x^2dx$$
$$= \frac{f^2}{m\,\Delta}\,2\,\pi\int\left[Y_0^2+2\,Y_0\,Y_1+\frac{\delta\,a}{2}\,Y_0^2-\frac{\Delta^2}{2}\,Y_0^2+\frac{\Delta^2}{2}\,Y_0'^2-\frac{\Delta^2\,Y_0^4}{32}\right]\,x^2dx$$

Binding Energy Corrections to N

The particle number is

$$
\begin{aligned}
N &= \int \langle N | J^0 | N \rangle \sqrt{|g|}\, d^3 r \\
&= \int \sqrt{\frac{A}{B}}\, 2\, \mu_0\, N\, R(r)^2\, d^3 r \\
&= \frac{f^2}{m^2\, \Delta}\, 2\, \pi \sqrt{1 - \Delta^2} \int \sqrt{\frac{1 + \delta a(x)}{1 + \delta b(x)}}\, Y(x)^2\, x^2\, dx \\
&\approx \frac{f^2}{m^2\, \Delta}\, 2\, \pi \left[1 - \frac{\Delta^2}{2}\right] \int \left[1 + \frac{\delta}{2}(a - b)\right] Y(x)^2\, x^2\, dx \\
&= \frac{f^2}{m^2\, \Delta}\, 2\, \pi \left[Y_0^2 + 2\, Y_0\, Y_1 - \frac{\Delta^2}{2}\, Y_0^2 + \frac{\delta}{2}(a - b)\, Y_0^2\right] x^2\, dx
\end{aligned}
$$

Leading Binding Energy Corrections

$$M = \frac{2\,\pi\,f^2}{m\,\Delta}\left[I_0 + 2\,I_1 + \frac{\delta}{2}\,I_a - \frac{\Delta^2}{2}\,I_0 + \frac{\Delta^2}{2}\,I_p - \frac{\Delta^2}{32}\,I_4\right], \quad N = \frac{2\,\pi\,f^2}{m^2\,\Delta}\left[I_0 + 2\,I_1 + \frac{\delta}{2}\,I_a - \frac{\delta}{2}\,I_b - \frac{\Delta^2}{2}\,I_0\right]$$

in terms of the integrals

$$I_0 = \int {Y_0}^2\,x^2\,dx \qquad I_p = \int {Y_0'}^2\,x^2\,dx \qquad I_4 = \int {Y_0}^4\,x^2\,dx$$

$$I_a = \int a\,{Y_0}^2\,x^2\,dx \qquad I_b = \int b\,{Y_0}^2\,x^2\,dx \qquad I_1 = \int Y_0\,Y_1\,x^2\,dx$$

The binding energy, at leading order in δ, Δ^2, is

$$E_B = M - m\,N = \frac{2\,\pi\,f^2}{m\,\Delta}\left[\frac{\delta}{2}\,I_b + \frac{\Delta^2}{2}\,I_p - \frac{\Delta^2}{32}\,I_4\right]$$

Numerical Values

κ	M [kg]	R_{99} [km]	d [kg / m^3]	$\frac{E_B}{m\,N}$ [10^{-13}]
0.01	2.01×10^{18}	115	311	141
0.09	6.91×10^{18}	386	28.6	2.93
0.16	1.02×10^{19}	593	11.6	-3.24
0.25	1.27×10^{19}	854	4.85	-4.18
0.29	1.31×10^{19}	972	3.41	-3.99
0.34	1.33×10^{19}	1077	2.53	-3.71
0.38	1.32×10^{19}	1183	1.90	-3.39
0.64	1.20×10^{19}	1652	0.633	-2.25
1	1.03×10^{19}	2145	0.248	-1.49
4	5.56×10^{18}	4499	0.0146	$-.384$
16	2.85×10^{18}	9062	0.000913	$-.109$
100	1.15×10^{18}	22849	0.000023	$< 10^{-2}$

Table: Macroscopic parameters describing a dilute axion star: mass M, radius R_{99}, average density d, and reduced binding energy per particle $E_B/m\,N$, as a function of $\kappa = \delta/\Delta^2$. To set the numerical scale we have fixed the QCD parameters $m = 10^{-5}$ eV and $f = 6 \times 10^{11}$ GeV.

Computing the $3a_c \to a_p$ Decay Rate

We modify the axion field expansion to include a free axion term

$$\mathcal{A} = R(r)e^{-i\,\mu_0\,t}a_0 + \int \frac{d^3p}{\sqrt{2\,\mu_p}} e^{i\vec{p}\cdot\vec{r}-i\,\mu_p\,t}\,a_p(\vec{p}) + h.c.$$

This leads to the leading-order matrix element

$$\begin{aligned}
\mathcal{M}_3 &\equiv \mathcal{M}[N \to (N-3) + 1 \text{ emitted}] \\
&= m^2 f^2 \int dt\, d^3r \langle N-3, p|1 - cos\left(\frac{\mathcal{A}}{f}\right)|N\rangle \\
&= -i\frac{f}{m}\frac{1}{\sqrt{2\mu_p}} \int dt\, d^3y\, J_3[Z(y)] e^{i\vec{p}\cdot\vec{r}} e^{i(3\mu_0-\mu_p)t} \\
&= -i\frac{4\pi^2}{\sqrt{2\mu_p}}\frac{f}{p}\,\delta(3\mu_0-\mu_p) \underbrace{\int_{-\infty}^{\infty} y\, sin\left(\frac{p\,y}{m}\right) J_3[Z(y)]\,dy}_{I_3(p)}
\end{aligned}$$

The Decay Rate at Weak Binding

Note that the dimensionless momentum $k = p/m$ of the ejected axion is sharply peaked for $E - m \ll m$

$$k_3 = \sqrt{9E^2/m^2 - 1} \approx \sqrt{8}$$

We find the decay rate for $3a_c \to a_p$ is

$$\Gamma_3 = \frac{1}{T} \int \frac{d^3 p}{(2\pi)^3 (2\,\mu_p)} |\mathcal{M}_3|^2 = \frac{2\,\pi\, f^2}{m\, k_3} \Big| I_3(k_3) \Big|^2$$

$$I_3(k) = \int_{-\infty}^{\infty} y\, \sin\left(\frac{p\, y}{m}\right) J_3[Z(y)]\, dy$$

$$= \frac{1}{\Delta^2} \int_{-\infty}^{\infty} x\, \sin\left(\frac{k\, x}{\Delta}\right) J_3[\Delta\, Y(x)]\, dx$$

$$\underbrace{\approx}_{\Delta \ll 1} \frac{\Delta}{48} \int_{-\infty}^{\infty} x\, \sin\left(\frac{k\, x}{\Delta}\right) Y(x)^3\, dx$$

In principle, we can use our solutions $Y(x)$ and integrate directly.
In practice, this is made difficult by the rapidly oscillating *sin* term

Contour Integration of $I_3(k)$

$$I_3(k_3) = \frac{\Delta}{48} \int_{-\infty}^{\infty} x \, sin\left(\frac{k_3 \, x}{\Delta}\right) Y(x)^3 \, dx$$

Consider the contour integral instead:

- $Y(x)$ has no singularities along the real axis
- We show that the leading singular term is of the form $Y_s(x) = \dfrac{8 \, y_I}{x^2 + {y_I}^2}$ with $y_I \approx .603$
- Deforming the contour of integration until we reach $i \, y_I$, the contribution of this pole dominates the integral. The result is

$$I_3(k_3) = i \frac{32\pi \, y_I}{3\Delta} \exp\left(-\frac{k_3 \, y_I}{\Delta}\right)$$

Alternative Derivation: Spherical Waves

For our complete set of scattering states, we could have used

$$\phi_s(t,r) = \frac{1}{2\,\pi^2} \sum_{\ell=0}^{\infty} \sum_{\ell_z=-\ell}^{\ell} Y_{\ell,\ell_z}(\hat{r}) \int_0^{\infty} \frac{dp\,p}{2\,\mu_p}\, j_\ell(p\,r)\, e^{-i\,\mu_p\,t} a_{\ell,\ell_z}(p)$$

The spherical wave annihilation operators satisfy the commutation relation

$$\left[a_{\ell,\ell_z}(p), a_{\ell',\ell'_z}(p')\right] = (2\pi)^3\, 2\,\mu_p\,\delta(p-p')\,\delta_{\ell,\ell'}\,\delta_{\ell_z,\ell'_z}.$$

Note that the annihilation operators in the two bases are related as

$$a_{\ell,\ell_z}(p) = i^{\ell}\, p \int d\Omega_p\, Y^*_{\ell,\ell_z}(\hat{p})\, a(\vec{p}),$$

which can also be inverted,

$$a(\vec{p}) = \frac{1}{p} \sum_{\ell=0}^{\infty} \sum_{\ell_z=-\ell}^{\ell} i^{-\ell}\, Y_{\ell,\ell_z}(\hat{p})\, a_{\ell,\ell_z}(p).$$

Both sets of scattering states are *precisely equal*. One can be derived from the other by using the expansion of the exponential in spherical harmonics.

Alternative Derivation: Spherical Waves (2)

The transition matrix element in this basis has the form

$$\mathcal{M}_3^{sph} = -i\, m^2 f \int dt\, d^3r\, J_3\left(\frac{2\sqrt{N}\,R(r)}{f}\right) \langle 0|\phi_s(t,r)|\vec{p}\rangle$$
$$= -i\, m^2 f \int dt\, d^3r\, J_3\left(\frac{2\sqrt{N}\,R(r)}{f}\right) \sqrt{4\pi}\, j_0(p\,r)\, e^{-i\,\mu_p\, t}$$

Although this matrix element is different when using the spherical waves, the decay rate is calculated using a different integration over phase space,

$$\Gamma_3 = \frac{1}{T}\int \frac{1}{(2\pi)^3}\frac{dp}{2\,\mu_p}|\mathcal{M}_3^{sph}|^2$$

and the final answer is the same.

The Nonrelativistic Limit for Axions

Expand the axion field in the nonrelativistic limit as

$$\mathcal{A}(t,r) = \frac{1}{\sqrt{2\,m}}\left[e^{-i\,m\,t}\,\psi(t,r) + e^{i\,m\,t}\,\psi^*(t,r)\right].$$

In the axion potential, the nth term is proportional to the factor

$$\mathcal{A}^{2n} = \frac{{}^{2n}C_n}{(2m)^n}(\psi^*\,\psi)^n + \mathcal{O}(e^{\pm i\,m\,t}),$$

where ${}^{2n}C_n$ are binomial coefficients. Dropping the rapidly oscillating pieces, we obtain

$$\begin{aligned} W(\psi) &= m^2\,f^2\left[1-\cos\left(\frac{\mathcal{A}}{f}\right)\right] - \frac{m^2}{2}\,\mathcal{A}^2 \\ &\to m^2\,f^2\left[1-\sum_{n=0}^{\infty}\frac{{}^{2n}C_n\,(-1)^n}{(2n)!}\left(\frac{\psi^*\,\psi}{2mf^2}\right)^n\right] - \frac{m}{2}\,\psi^*\,\psi \\ &= m^2\,f^2\left[1-\frac{\psi^*\,\psi}{2mf^2} - J_0\left(\sqrt{\frac{2\psi^*\,\psi}{mf^2}}\right)\right] \end{aligned}$$