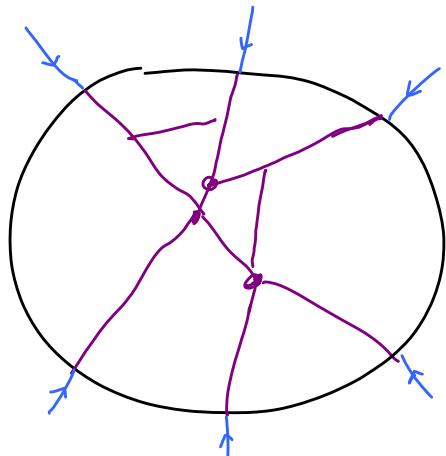
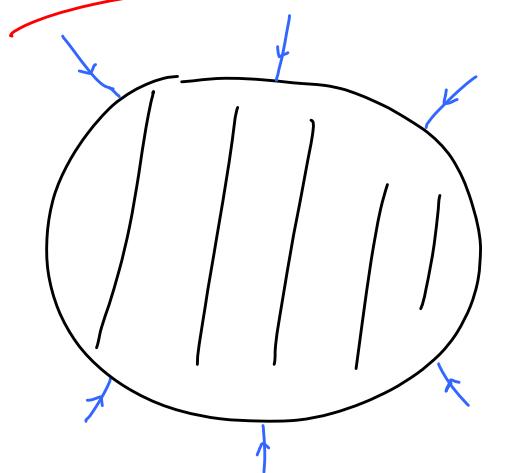


Scattering Amplitudes in the Real World



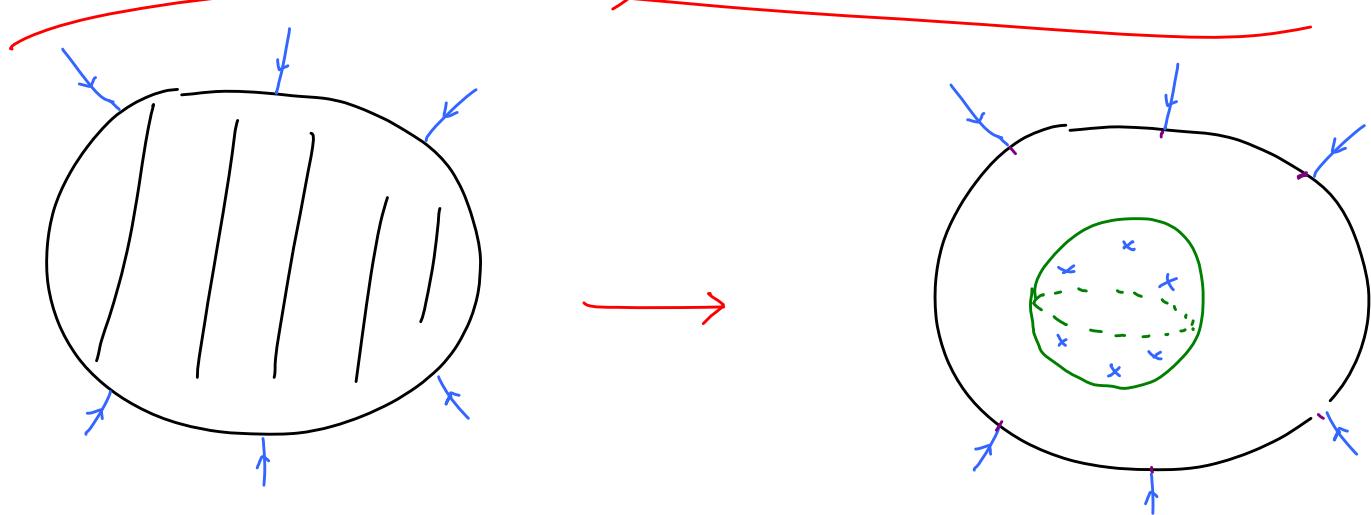
What is the Q to which A is the Answer?



$$? \quad \text{---} = \text{---} - \text{---}$$

Local, Unitary Evolution
in Space time

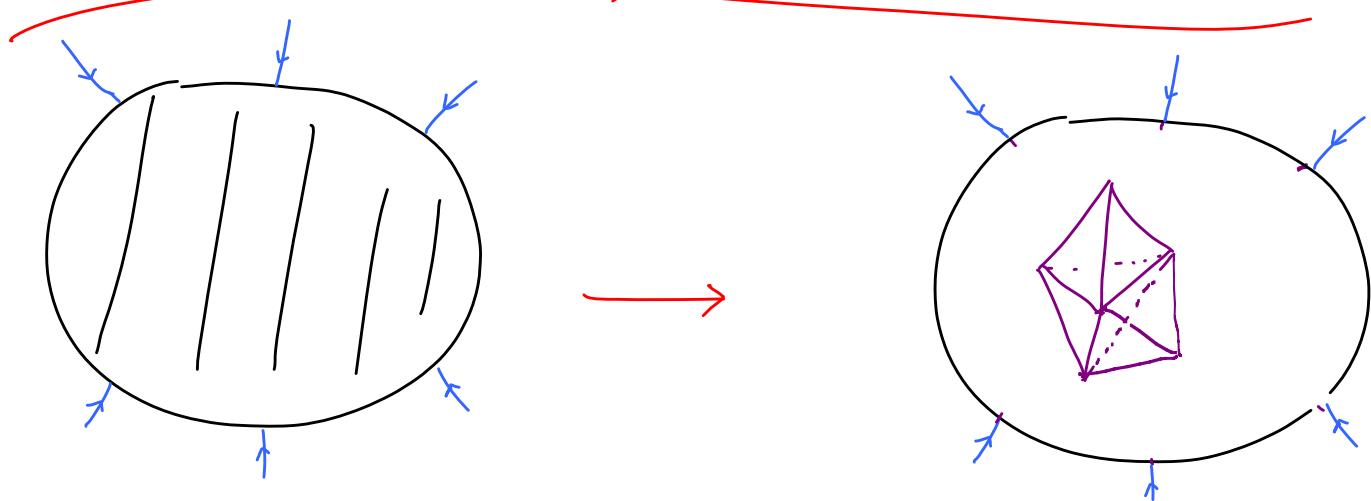
What is the Q to which A the answer?



$$\Im \quad \text{sun} = \text{circle} - \text{circle}$$

World-sheet correlators
Pert. Strings "Scattering Eqs"

What is the Q to which A the answer?



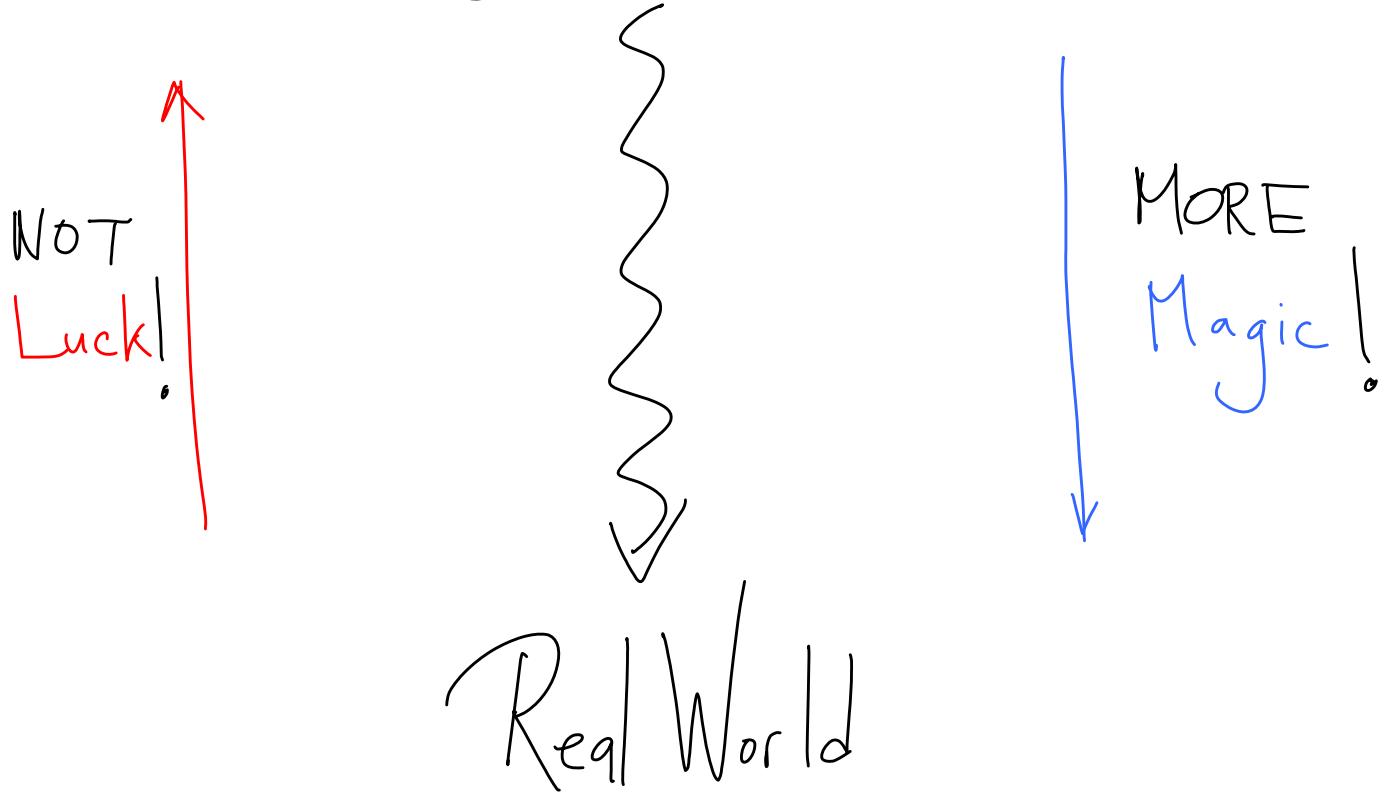
$$2 \text{ } \text{ } \text{ } \text{ } = \text{ } \text{ } \text{ } \text{ }$$

Canonical Forms of
Positive Geometries:
Combinatorial origin
of Locality + Unitarity

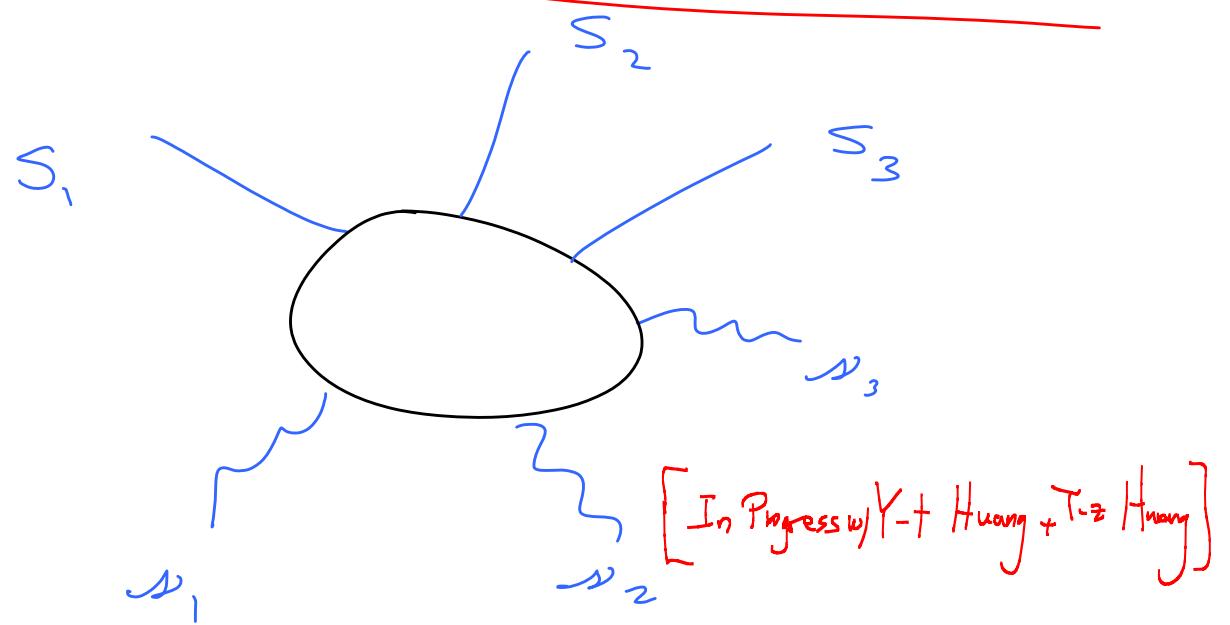
Toy Models



Toy Models



S-Matrix Rules For General m, S :



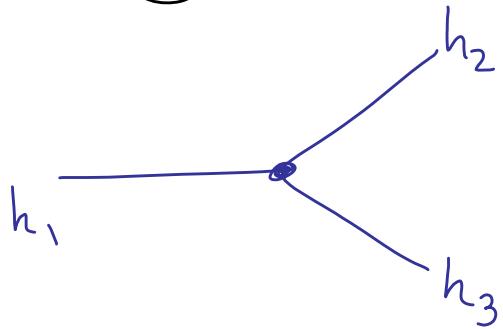
Manifesting the Little Group

Massless: $P_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$

$$(L_p) = p \rightarrow \begin{array}{l} \lambda_\alpha \rightarrow t \lambda_\alpha \\ \tilde{\lambda}_{\dot{\alpha}} \rightarrow \bar{t}' \tilde{\lambda}_{\dot{\alpha}} \end{array}$$

$$M[t\lambda, \bar{t}\tilde{\lambda}] = \bar{t}^{2h} M[\lambda, \tilde{\lambda}]$$

Massless Particle Dynamics: Fixed by Poincaré + Unitarity

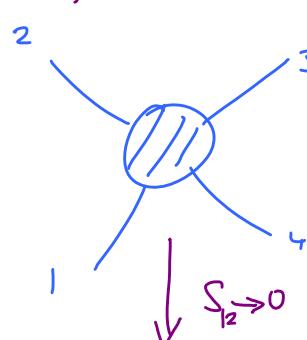


$$\lambda_1 \propto \lambda_2 \propto \lambda_3, \quad \begin{matrix} [12]^{h_1+h_2-h_3} & [23]^{h_2+h_3-h_1} & [31]^{h_3+h_1-h_2} \\ \langle 12 \rangle^{h_3-h_1-h_2} & \langle 23 \rangle^{h_1-h_2-h_3} & \langle 31 \rangle^{h_2-h_3-h_1} \end{matrix} \quad h_1 + h_2 + h_3 > 0$$

$$\tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3 \quad h_1 + h_2 + h_3 < 0$$

Unitarity @ Weak Coupling

Amplitude has only poles [\sim "tree level"]; Unitarity = Factorization; 4-particle test



$$\frac{1}{S_{12}} \sum_h \begin{array}{c} 2 \\[-1ex] h \\[-1ex] 1 \end{array} \begin{array}{c} -h \\[-1ex] 3 \\[-1ex] 4 \end{array}$$

$$\begin{array}{c} 2^- \\[-1ex] 1^- \end{array} \begin{array}{c} 3^+ \\[-1ex] 4^+ \end{array} = \left\{ \begin{array}{l} \frac{\langle 12 \rangle^2 [34]^2}{t} \omega=1 \\ + \frac{\langle 12 \rangle^4 [34]^4}{t^2} \omega=2 \end{array} \right.$$

$$A^{N=1} \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} = \frac{\langle 12 \rangle^2 [34]^2}{st}$$

$$A^{N=2} \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} = - \frac{\langle 12 \rangle^4 [34]^4}{stu}$$

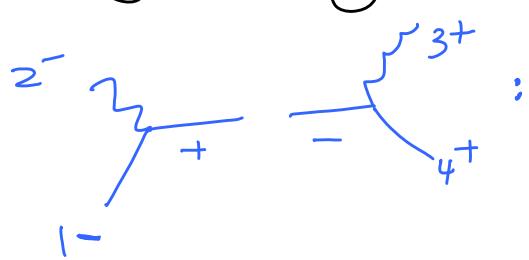
Tension Between Locality + Unitarity For Spin 1,2

$$A^{\text{grav}} \left[\bar{1} \bar{2} \bar{3}^+ \bar{4}^+ \right] = \frac{\langle 12 \rangle [34]^4}{stu} \quad \leftarrow \frac{\text{Product of}}{s,t,u}$$

Can't be written as a sum
over s, t, u channels without breaking
manifest Lorentz Invariance

(Massless) Higher Spin Ruled out by GR/YM

e.g. charged spin ν :



$$\frac{\langle 12 \rangle^2 [34]^2 \langle 11 (2-3) | + \rangle}{u^{2\nu-1}}$$

impossible
for $\nu \geq \frac{3}{2}$

Only consistent 4pt: Spins $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$
 YM \uparrow \uparrow \uparrow
 $N \leq 8$ SUSY unique, GR

- * Obviously no Lagrangian, no Quantum Fields...
- * Analyticity (here = only does!) imprint of Causality
- * CPT + Antiparticles hardwired by
- * Spin-Statistics, Weinberg-Witten +
Coleman-Mandula elementary consequences

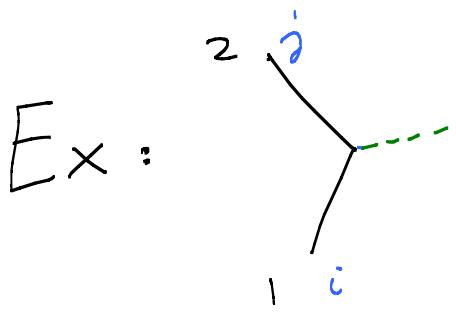
- * Modern S-Matrix program: exploit locality + unitarity to determine amplitudes
(Responsible: Symmetries → Simplicity)
- * But conversely: look for a new picture for what amplitudes "really are", where space-time + Hilbert space don't appear — see locality + unitarity as derived notions
(Magic)

Manifesting the Little Group

Massive: $P_{\alpha i} = \gamma^{\dot{c}} \tilde{\gamma}^i \epsilon_{ij}$ $i: SU(2)$
little grp.

$$P \gamma^i = m \tilde{\gamma}^i ; \quad \det \gamma \det \tilde{\gamma} = m^2$$

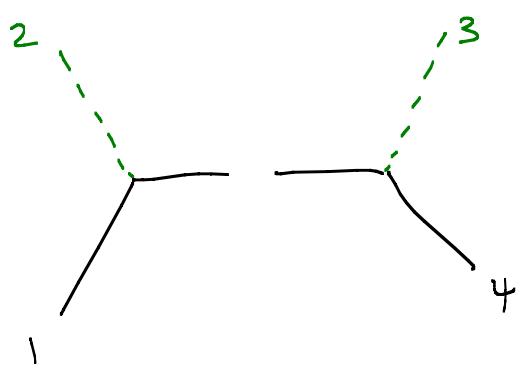
$$M^{\{i_1, \dots, i_{25}\}} [\gamma^i, \tilde{\gamma}^i] \quad \left\{ \begin{array}{l} \psi_{i_1}, \dots, \psi_{i_{25}} \\ \text{instead of} \\ \langle s, m_2 | \psi \rangle \end{array} \right.$$



$$K \langle 1^i 2^j \rangle + K' [1^i 2^j]$$

or

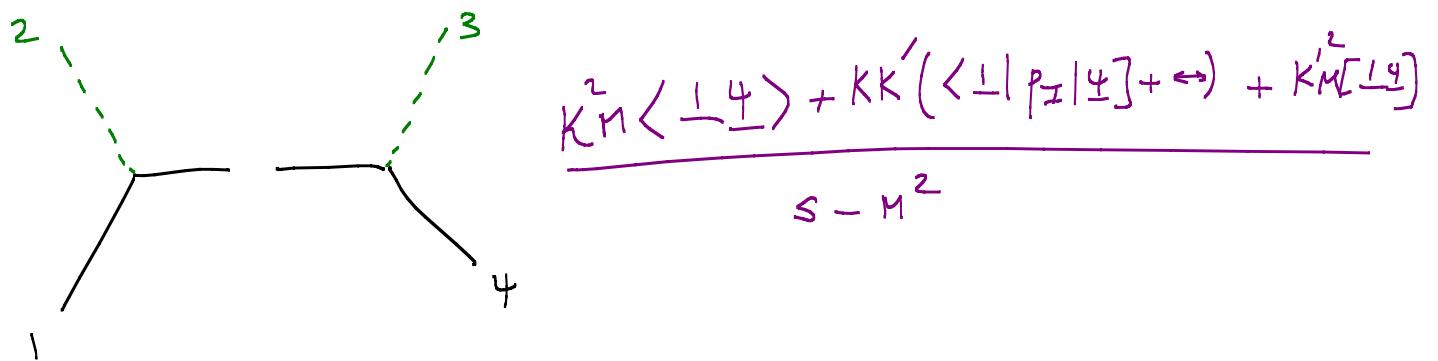
$$K \langle \perp \Xi \rangle + K' [\perp \Xi]$$



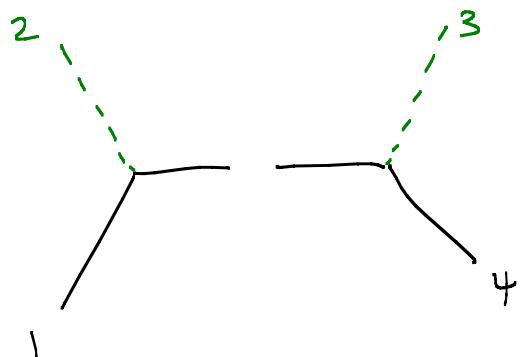
$$\frac{K_M^2 \langle \perp \Xi \rangle + K K' (\langle \perp | p_I | \Xi \rangle + \text{c.c.}) + K'_M [\perp \Xi]}{s - M^2}$$

[No γ matrices, Dirac eqn,
 $\bar{u} + v$'s etc: artefact of
field picture]

Massive \rightarrow Massless @ High- E is trivial



Massive \rightarrow Massless @ High- E is trivial



$$\frac{K_M^2 \langle 1|4\rangle + K\bar{K}' (\langle 1|p_2|4\rangle + \langle 1|p_1|4\rangle) + K_M' \langle 1|4\rangle}{s}$$

Just erase the "bars" —
and identify the helicity
components!

2 Massless, 1 Massive

$$\langle 12 \rangle [12] = m^2$$

$$g \lambda_1$$

$$s + s_2 - s_1$$

$$s + s_1 - s_2$$

$$\langle 12 \rangle$$

$$\{ \alpha_1 \dots \alpha_{2s} \}$$

{Note e.g., Yang's theorem!}

↓
Pd sums
are easy

$$\sum_{N_{ij}} \langle 13 \rangle^{N_{13}} \langle 14 \rangle^{N_{14}} \langle 23 \rangle^{N_{23}} \langle 24 \rangle^{N_{24}}$$

$$\frac{N_1! N_2! N_3! N_4!}{N_{13}! N_{14}! N_{23}! N_{24}!}$$

$$\{ N_1 = s + s_2 - s_1 \text{ etc.} \}$$

2 massive + 1 massless

$$\lambda_\alpha = u_\alpha, (p_1 \bar{\lambda})_\alpha = - (p_2 \bar{\lambda})_\alpha = v_\alpha$$

$$\langle uv \rangle = m_1^2 - m_2^2$$

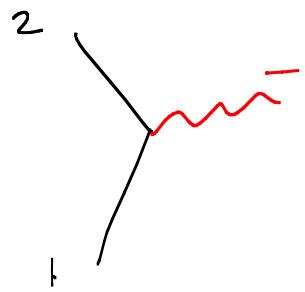
$m_1 \neq m_2$ u, v a basis

$$u_\alpha^{N_1} u_\beta^{N_2} v_\alpha^{M_1} v_\beta^{M_2}$$

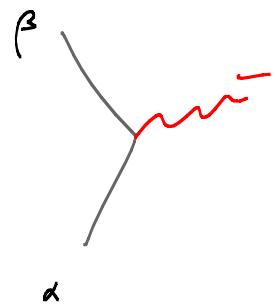
$$m_1 = m_2; v_\alpha = \cancel{u}_\alpha$$

$$\epsilon_{\alpha_1 \beta_1} \dots \epsilon_{\alpha_k \beta_k} u_{\alpha \dots \beta \dots}^{2[s_1+s_2-k]} X^{2[1+k-s_1-s_2]}$$

Note: "X" is not local!

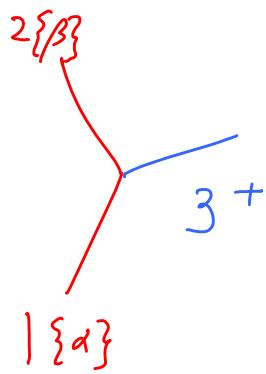


$$e^m x^2, \left(\frac{m}{M_{pl}}\right) m x^2$$



$$e^m x^\beta E_{\alpha\beta} + (g-2) \lambda_\alpha \lambda_\beta$$

Minimal Coupling



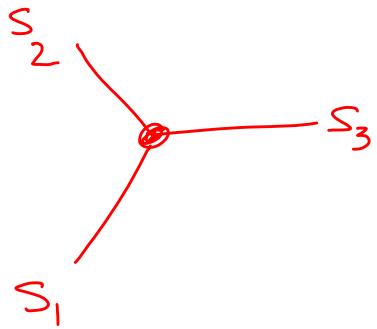
$$m_X \epsilon_{\{\alpha_1 \beta_1 \dots \alpha_{2S} \beta_{2S}\}} \cdot \left(\frac{m}{M_{Pl}} \kappa \right)$$

Nailed by having
only ^+h

Feynman diagram showing a vertex with two external lines. One line emerges from the top-right, labeled $+h$. One line enters from the bottom-right, labeled $-h$.

in high-energy limit

3 Massive



$$\in_{\alpha\beta}, (P_i P_j + P_j P_i)_{\alpha\beta}$$

for one of $(ij) = (12), (23), (31)$ provides basis

4 pt Challenge

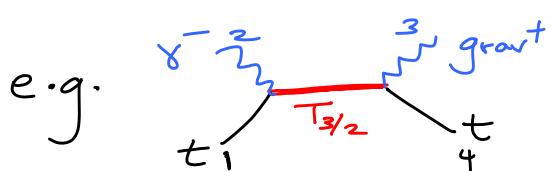
* Given spectrum $\{m_i, \nu_i\}$, couplings

find $\{\nu_1 \nu_2 \nu_3 \nu_4, \nu_1 \nu_2 \nu_3 \xi \alpha_3\}$

which factorize correctly

- * Most naively: just take

$$\frac{1}{s-m^2} \times \text{Y} \rightarrow \text{t, u channels}$$



$$\frac{1}{s-m^2} \times \left\{ \begin{array}{l} gg' \lambda_{2\alpha} \lambda_{3\beta} \langle 2| p_4 | 3 \rangle^3 \\ + gg'' \lambda_{2\alpha} p_4 \lambda_{3\beta}^2 \langle 23 \rangle \langle 2| p_4 | 3 \rangle^2 \end{array} \right. + 1 \leftrightarrow 4 \quad (+\text{contact})$$

Diagram showing a 3-point vertex with two external lines of mass m_1 and one internal line of mass m_2 . The text "m_1 = m_2" is written above the diagram. Below the diagram, the text " $\langle X \rangle$ not local" is written in red.

- * Only challenge:

- * But can easily be done for any 3pt couplings

\Rightarrow Explicit expression for 4pt given 3pt

Example: Compton Scattering

Feynman diagram for Compton scattering:

Amplitude:

$$\frac{1}{(s-m^2)(u-m^2)} \times \left\{ \begin{array}{l} \langle 2 | p_1 - p_4 | 3 \rangle^2 \\ \langle 2 | p_1 - p_4 | 3 \rangle \left(\langle \downarrow 2 \rangle [\pm 3] + \langle \mp 2 \rangle [\pm 3] \right) \\ \left(\langle \downarrow 2 \rangle [\pm 3] + \langle \mp 2 \rangle [\pm 3] \right)^2 \end{array} \right\}$$

spin 0

spin $\frac{1}{2}$

spin 1

Massive Higher Spins Can't be Pointlike!

$$A^{(-n, +\infty)} \xrightarrow[\text{SU}]{e^2} \frac{\langle 12 \rangle^2 [34]^2}{\langle 11 | p_3 - p_2 | 4 \rangle} \times \left(\frac{\langle 11 | p_3 - p_2 | 4 \rangle}{m^2} \right)^{2s-2}$$

↑
Blows up for $E \gg m$

It is possible to do **everything**

we normally cover in intro QFT,

(e.g. $(g-2)_e$, β -function, Higgs mechanism ...)

in this way : Direct Route from Principles:

Poincaré + Locality + Causality \Rightarrow Physics

Lets consider the $e^+, e^- \rightarrow \gamma$ at one loop. The diagram we want to build is:

$$\sim e^3 m^3 x_a \varepsilon_{\alpha\beta} \left[\varepsilon^{\beta\gamma} \frac{x_b}{x_c} \left(\varepsilon + x_c \frac{\lambda_\ell \lambda_\ell}{m} \right)^{\alpha\delta} + \varepsilon^{\alpha\delta} \frac{x_c}{x_b} \left(\varepsilon - x_b \frac{\lambda_\ell \lambda_\ell}{m} \right)^{\beta\gamma} \right] \quad (1)$$

where we've glued the three-point vertices according to the two possible helicity configurations in the internal photon lines. Now the piece which is independent of λ_s will be present for a scalar particle, and thus corresponds to the minimal coupling. The magnetic moment piece is then given by:

$$e^2 m^2 x_a (x_b - x_c) \lambda_\ell^\delta \lambda_\ell^\gamma = -m x_a q^\delta {}_{\dot{\alpha}} \ell^{\dot{\alpha}\beta}. \quad (2)$$

This gives us the following integrand:

$$-m x_a \int \frac{d^4 \ell}{(2\pi)^4} \frac{q^\delta {}_{\dot{\alpha}} \ell^{\dot{\alpha}\beta}}{\ell^2 ((\ell - p_2)^2 - m^2)((\ell + p_1)^2 - m^2)} = \frac{e^2}{(4\pi)^2} 2 x_a \frac{q^\delta {}_{\dot{\alpha}} p_1^{\dot{\alpha}\beta}}{m} = \frac{\alpha}{2\pi} x^a \lambda_q^\gamma \lambda_q^\delta.$$

The loop amplitude will be constructed by gluing the $2 \rightarrow 2$ amplitude involving the scalar probe particle exchanging a photon with X . Assuming that the mass of X is identical with that of the scalar probe, the relevant tree amplitudes can be easily constructed:¹

Diagrams and equations for the three cases:

- Scalar Case:** Diagram shows two external lines (1, 2) and two internal lines (a, b). The internal lines are labeled with γ and X . The equation is:

$$X \in \text{scalar} \frac{m^2}{s} \left(\frac{x_a}{x_b} + \frac{x_b}{x_a} \right) = \frac{2p_1 \cdot p_3}{s}$$
- Fermion Case:** Diagram shows two external lines (1, 2) and two internal lines (a, b). The internal lines are labeled with γ and X . The equation is:

$$X \in \text{fermion} \frac{m^2}{s} \left[\frac{x_a}{x_b} \left(\varepsilon - x_b \frac{\lambda\lambda}{m} \right)_{\alpha\beta} + \frac{x_b}{x_a} \varepsilon_{\alpha\beta} \right]$$

$$= \frac{2p_1 \cdot p_3}{s} \varepsilon_{\alpha\beta} - \frac{p_{1\dot{\alpha}}(\alpha} P^{\dot{\alpha}}_{\beta)}{2s}$$
- Vector Case:** Diagram shows two external lines (1, 2) and two internal lines (a, b). The internal lines are labeled with γ and X . The equation is:

$$X \in \text{vector} \frac{m^2}{s} \left[\frac{x_a}{x_b} \left(\varepsilon - x_b \frac{\lambda\lambda}{m} \right)_{(\alpha_3\alpha_4} \left(\varepsilon - x_b \frac{\lambda\lambda}{m} \right)_{\beta_3)\beta_4} + \frac{x_b}{x_a} \varepsilon_{(\alpha_3\alpha_4} \varepsilon_{\beta_3)\beta_4} \right]$$

$$= \frac{2p_1 \cdot p_3}{s} \varepsilon_{(\alpha_3\alpha_4} \varepsilon_{\beta_3)\beta_4} - \varepsilon_{(\alpha_3\alpha_4} \frac{p_{1\dot{\alpha}\beta_3)} P^{\dot{\alpha}}_{\beta_4}}{2s} - \varepsilon_{(\alpha_3\alpha_4} \frac{p_{1\dot{\alpha}\beta_4} P^{\dot{\alpha}}_{\beta_3)}}{2s} - \varepsilon_{(\beta_3\beta_4} \frac{p_{1\dot{\alpha}\alpha_3)} P^{\dot{\alpha}}_{\alpha_4}}{2s}$$

$$- \varepsilon_{(\beta_3\beta_4} \frac{p_{1\dot{\alpha}\alpha_4} P^{\dot{\alpha}}_{\alpha_3)}}{2s} + \frac{1}{4!} \left(\frac{p_{1\dot{\alpha}\alpha_4} P^{\dot{\alpha}}_{\alpha_3} p_{1\dot{\beta}\alpha_4} P^{\dot{\beta}}_{\alpha_3}}{sm^2} + \text{perm}(\alpha_3, \alpha_4, \beta_3, \beta_4) \right)$$

(9)

We can now glue the tree amplitudes into the one-loop integrand. The beta function can be readily read off by picking out the divergent piece that is proportional to the tree-amplitude $\frac{2(p_1 \cdot p_3)}{2}$. Let us use the scalar correction as an example. The one-loop amplitude is now

$$= A_4^{\text{scalar}}(p_1, \ell_1) A_4^{\text{scalar}}(\ell_2, p_3) = \frac{4(p_1 \cdot \ell_1)(p_3 \cdot \ell_2)}{s^2}. \quad (10)$$

The one-loop integrand is then simply:

$$\frac{4}{s^2} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^4} \frac{(p_1 \cdot \ell_1)(p_3 \cdot \ell_2)}{(\ell^2 - m^2)((\ell - P)^2 - m^2)} = -\frac{1}{(4\pi)^2\epsilon} \frac{1}{6} \frac{(2p_1 \cdot p_3)}{s} + \dots \quad (11)$$

where \dots represent terms that are purely functions of s , or finite. For fermions, there are now two pieces that are relevant, the square of the scalar piece, and the square of the $p_i P$ piece. All other contributions cannot generate the $p_1 \cdot p_3$ tensor structure. We find:

$$A_4^{\text{fermion}}(p_1, \ell_1) A_4^{\text{fermion}}(\ell_2, p_3) = \frac{8(p_1 \cdot \ell_1)(p_3 \cdot \ell_2)}{s^2} - 2 \frac{(p_1 \cdot p_3)}{s} + \dots. \quad (12)$$

The relevant part of the one-loop integrand is then:

$$\frac{1}{s} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^4} \frac{8(p_1 \cdot \ell_1)(p_3 \cdot \ell_2)/s - 2(p_1 \cdot p_3)}{(\ell^2 - m^2)((\ell - P)^2 - m^2)} = -\frac{1}{(4\pi)^2\epsilon} \frac{4}{3} \frac{(2p_1 \cdot p_3)}{s} + \dots. \quad (13)$$

Finally, similar analysis for vectors yield:

$$A_4^{\text{vector}}(p_1, \ell_1) A_4^{\text{vector}}(\ell_2, p_3) = \frac{12(p_1 \cdot \ell_1)(p_3 \cdot \ell_2)}{s^2} + 8 \frac{(p_1 \cdot p_3)}{s} \quad (14)$$

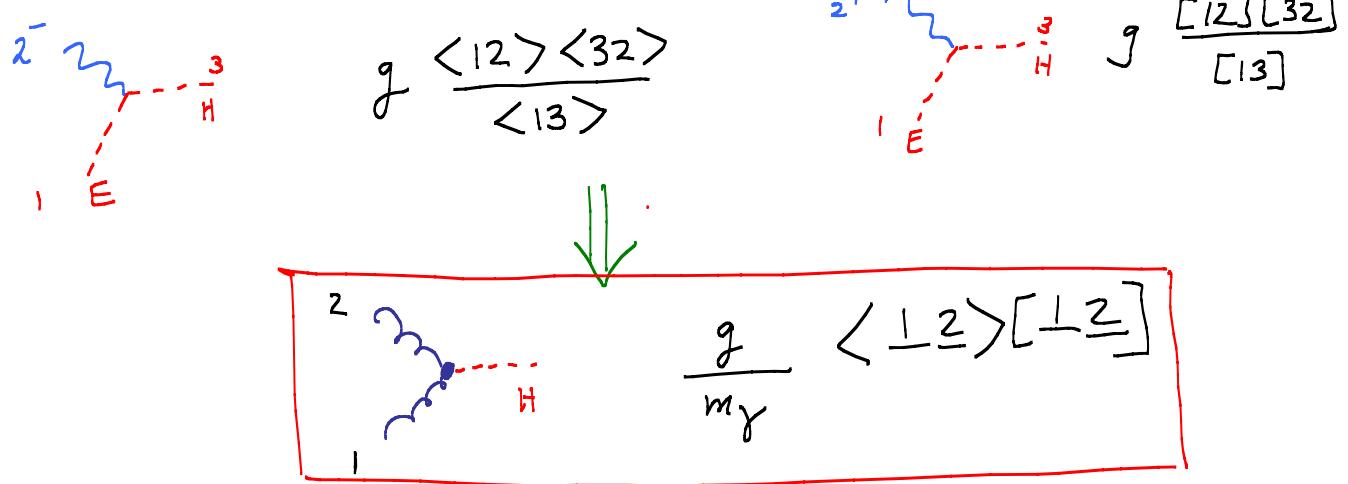
which leads to

$$\frac{1}{s} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^4} \frac{12(p_1 \cdot \ell_1)(p_3 \cdot \ell_2)/s + 8(p_1 \cdot p_3)}{(\ell^2 - m^2)((\ell - P)^2 - m^2)} = \frac{1}{(4\pi)^2\epsilon} \frac{7}{2} \frac{(2p_1 \cdot p_3)}{s} + \dots. \quad (15)$$

Thus we've found that the beta function for a scalar is $\frac{1}{6}$ a Dirac fermion $\frac{4}{3}$ and a massless vector being $-\frac{7}{2} + \frac{1}{6} = -\frac{11}{3}$, where we've subtracted the scalar "eaten" by the massive vector.

Higgs Mechanism: "IR Deformation"

E_x : Photon + Charged Scalar in UV:



$$\frac{g^2}{m_\gamma^2} \left\{ \frac{\langle \underline{1} \underline{2} \rangle [\underline{1} \underline{2}] \langle \underline{3} \underline{4} \rangle [\underline{3} \underline{4}]}{s - m_H^2} \right. + t \left. + u \right\}$$

$\xrightarrow{\text{HE for long. modes}}$

$$\frac{g^2}{m_\gamma^2} m_H^2 = \cancel{\lambda} \Rightarrow \boxed{\frac{m_H^2}{\lambda} = \frac{m_\gamma^2}{g^2}}$$

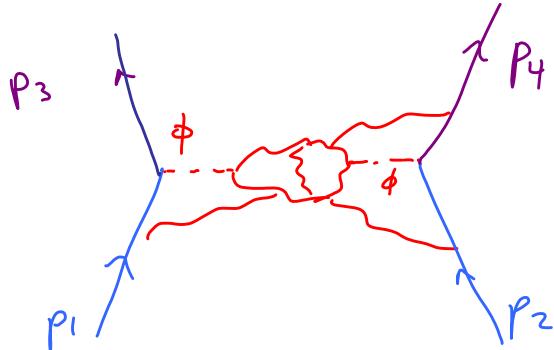
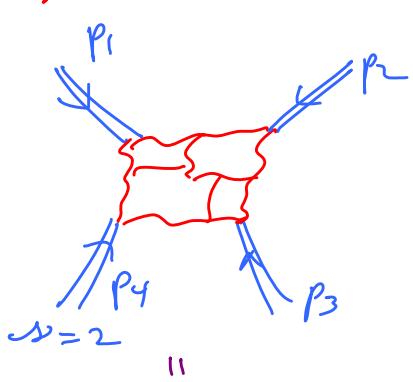
Ex. for students: Derive from $T_g - D_{\text{work}}$, +
also do the "Super-Higgs" mechanism

{ Time is Ripe to

Compute all Planar $\mathcal{N} = 4$ amps on

the Conformal Branch, w/ Rutger Boels
+ Yutin

Massive Amps For Any Spin \longleftrightarrow Correlation Fns



$$\langle T_{(p)} T_{(p_2)} T_{(p_3)} T_{(p_4)} \rangle$$

Correlators

$$\langle W \phi^*(x) \phi(y) W' \rangle$$

Wilson loops

All "Local" Observables on Same Footing

* It will be interesting to systematically understand SM from this purely on-shell perspective — and at the same time look for “magic” — like some hidden positive geometry underlying it.

* Note that [trivially + unsurprisingly] we never encounter any aspect of the hierarchy problem in this way of doing things.

"Hierarchy Problem is meaningful only in theories where the Weak Scale is Calculable"

Toy Models



HUGE amount of WORK to realize this!
FUN

