Unitarity Constraints on Dimension–Six Operators

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TC, O. J. P. Eboli and M. C. Gonzalez-Garcia, Phys. Rev. D 91, no. 3, 035014 (2015), arXiv: 1505.05516

Partial Wave Unitarity

Dimension–six operators → growth of amplitude w/ E_{COM}
→ violation of S–matrix unitarity

We will impose the unitarity of the S–Matrix to constrain the operators:

- bounds derived from the optical theorem
- Obtain bounds on the COM energy ($\sqrt{s}$) as a function of f_i/Λ (for $2 \to 2$ processes)
  → we can combine these with Higgs fits!
- imply the necessity of new physics
  → new fundamental particles (SM Higgs fixes unitarity issues of GB scattering)
  → new composite degree of freedom (ρ meson fixes unitarity issues of π scattering)

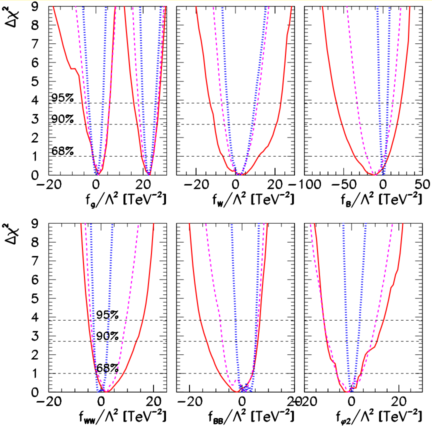
Dimension–6 Operators for Unitarity

The relevant operators are: (Reduced via EOM & precision data in arXiv: 1211.4580)

$$\mathcal{O}_{BB} = \Phi^\dagger \hat{B}_{\mu\nu} \hat{B}^{\mu\nu} \Phi \qquad \mathcal{O}_{WW} = \Phi^\dagger \hat{W}_{\mu\nu} \hat{W}^{\mu\nu} \Phi \qquad \mathcal{O}_{GG} = \Phi^\dagger \Phi G^a_{\mu\nu} G_a^{\mu\nu}$$

$$\mathcal{O}_{B} = (D_\mu \Phi)^\dagger \hat{B}^{\mu\nu} (D_\nu \Phi) \qquad \mathcal{O}_{W} = (D_\mu \Phi)^\dagger \hat{W}^{\mu\nu} (D_\nu \Phi) \qquad \mathcal{O}_{WWW} = \mathrm{Tr}[\hat{W}_\mu{}^\nu \hat{W}_\nu{}^\rho \hat{W}_\rho{}^\mu]$$

$$\mathcal{O}_{\Phi,2} = \frac{1}{2} \partial_\mu (\Phi^\dagger \Phi) \partial^\mu (\Phi^\dagger \Phi) \qquad \mathcal{O}_{\Phi,4} = (D_\mu \Phi)^\dagger (D^\mu \Phi)(\Phi^\dagger \Phi)$$



Partial Wave Unitarity

Decomposing the amplitudes for $VV \to VV$ & $f\bar{f} \to VV$ into partial waves:

$$\mathcal{M}(V_{1\lambda_1}V_{2\lambda_2} \to V_{3\lambda_3}V_{4\lambda_4}) = 16\pi \sum_J \left(J+\frac{1}{2}\right)\sqrt{1+\delta^{V_2\lambda_2}_{V_1\lambda_1}}\sqrt{1+\delta^{V_4\lambda_4}_{V_3\lambda_3}}\, d^J_{\lambda\mu}(\theta)e^{iM\phi}T^J(V_1V_2 \to V_3V_4)$$

$$\mathcal{M}(f_{1\sigma_1}\bar{f}_{2\sigma_2} \to V_{3\lambda_3}V_{4\lambda_4}) = 16\pi \sum_J (J+\frac{1}{2})\delta_{\sigma_1,-\sigma_2} d^J_{\sigma_1-\sigma_2,\lambda_3-\lambda_4}(\theta)T^J(f_1\bar{f}_2 \to V_3V_4)$$

Using the optical theorem one may derive the unitarity limit for the T^J's:

$$|T^J(V_{1\lambda_1}V_{2\lambda_2} \to V_{1\lambda_1}V_{2\lambda_2})| \le 2,$$

$$\sum_{V_{3\lambda_3},V_{4\lambda_4}} \left|T^J(f_{1\sigma_1}\bar{f}_{2\sigma_2} \to V_{3\lambda_3}V_{4\lambda_4})\right|^2 \le 1$$

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$$\mathcal{M}(f_{1\sigma_1} \bar{f}_{2\sigma_2} \to V_{3\lambda_3} V_{4\lambda_4}) = 16\pi \sum_J (J + \frac{1}{2}) \delta_{\sigma_1, -\sigma_2} d^J_{\sigma_1 - \sigma_2, \lambda_3 - \lambda_4}(\theta) T^J (f_1 \bar{f}_2 \to V_3 V_4)$$

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$VV \to VV$ Unitarity Violating Amplitudes: $\mathcal{O}_{\Phi,2}$ & $\mathcal{O}_{\Phi,4}$

The high energy ($\sqrt{s} \gg M_W, M_Z, M_H$) behavior for $\mathcal{O}_{\Phi,2}$ & $\mathcal{O}_{\Phi,4}$ $\to$ same up to a sign.

We define: $f_{\Phi 2,4} \equiv f_{\Phi,2} - f_{\Phi,4}$

	$(\times \frac{f_{\Phi,2,4}}{\Lambda^2} \times s)$
$W^+W^+ \to W^+W^+$	-1
$W^+Z \to W^+Z$	$-\frac{1}{2}X$
$W^+H \to W^+H$	$-\frac{1}{2}X$
$W^+W^- \to W^+W^-$	$\frac{1}{2}Y$
$W^+W^- \to ZZ$	1
$W^+W^- \to HH$	-1
$ZZ \to HH$	-1
$ZH \to ZH$	$-\frac{1}{2}X$

$\mathcal{M}(s, M_W, M_Z, M_H) \to \mathcal{M}(s \gg M_W^2, M_Z^2, M_H^2)$

$X \equiv 1 - \cos\theta$

$Y \equiv 1 + \cos\theta$

Only grow as s, result of gauge symmetry!

Same Lorentz structures as SM

$\to$violation only in $V_L V_L \to V_L V_L$

$VV \to VV$ Unitarity Violating Amplitudes: $\mathcal{O}_{WWW}$

Even more Lorentz structures → some more helicity combinations violating unitarity...

	$(\times 2e^4 \frac{f_{WWW}}{\Lambda^2} \times s)$							
	$00++$	$0+0-$	$0+-0$	$+00-$	$+0-0$	$++00$	$+++-$ $+---$ $+-++$ $++-+$	$++--$
$W^+W^+ \to W^+W^+$	0	$-\frac{3(2+Y)}{32s_W^4}$	$\frac{3(2+X)}{32s_W^4}$	$\frac{3(2+X)}{32s_W^4}$	$-\frac{3(2+Y)}{32s_W^4}$	0	$-\frac{3}{4s_W^4}$	$\frac{3}{2s_W^4}$
$W^+Z \to W^+Z$	$\frac{3(Y-X)c_W}{32s_W^4}$	0	$\frac{3(X+2)c_W}{32s_W^4}$	$\frac{3(X+2)c_W}{32s_W^4}$	0	$\frac{3(Y-X)c_W}{32s_W^4}$	$-\frac{3c_W^2}{8s_W^4}X$	$\frac{3c_W^2}{4s_W^4}X$
$W^+\gamma \to W^+\gamma$	–	0	–	–	–	–	$-\frac{3}{8s_W^2}X$	$\frac{3}{4s_W^2}X$
$W^+Z \to W^+\gamma$	$-\frac{3(Y-X)}{32s_W^3}$	0	–	$\frac{3(X+2)}{32s_W^3}$	–	–	$-\frac{3c_W}{8s_W^3}X$	$\frac{3c_W}{4s_W^3}X$
$W^+Z \to W^+H$	–	–	$\frac{3(X+2)c_W}{32s_W^4}$	–	$\frac{3(2+Y)}{32s_W^4}$	$-\frac{3(Y-X)c_W}{32s_W^4}$	–	–
$W^+\gamma \to W^+H$	–	–	$\frac{3(X+2)}{32s_W^3}$	–	–	$-\frac{3(Y-X)}{32s_W^3}$	–	–
$W^+W^- \to W^+W^-$	$\frac{3(Y-X)}{32s_W^4}$	$\frac{3(2+Y)}{32s_W^4}$	0	0	$\frac{3(2+Y)}{32s_W^4}$	$\frac{3(Y-X)}{32s_W^4}$	$\frac{3}{8s_W^4}Y$	$-\frac{3}{4s_W^4}Y$
$W^+W^- \to ZZ$	0	$\frac{3(2+Y)c_W}{32s_W^4}$	$-\frac{3(X+2)c_W}{32s_W^4}$	$-\frac{3(X+2)c_W}{32s_W^4}$	$\frac{3(2+Y)c_W}{32s_W^4}$	0	$\frac{3c_W^2}{4s_W^4}$	$-\frac{3c_W^2}{2s_W^4}$
$W^+W^- \to \gamma\gamma$	0	–	–	–	–	–	$\frac{3}{4s_W^2}$	$-\frac{3}{2s_W^2}$
$W^+W^- \to Z\gamma$	0	$\frac{3(2+Y)}{32s_W^3}$	–	$-\frac{3(2+X)}{32s_W^3}$	–	–	$\frac{3c_W}{4s_W^3}$	$-\frac{3c_W}{2s_W^3}$
$W^+W^- \to ZH$	–	–	$-\frac{3(2+X)c_W}{32s_W^4}$	–	$-\frac{3(2+Y)c_W}{32s_W^4}$	$\frac{3(Y-X)}{32s_W^4}$	–	–
$W^+W^- \to \gamma H$	–	–	$-\frac{3(X+2)}{32s_W^3}$	–	$-\frac{3(2+Y)}{32s_W^3}$	–	–	–

All Couplings Simultaneously

In paper & poster we give bounds for one operator different from zero at a time.

With no model of NP to guide us, we must consider more than one coupling non–zero.

- search for largest allowed value for each coefficient while varying others
  → Most conservative constraints on a parameter allowing for cancellations in others

$$\left|\frac{f_{\Phi 2,4}}{\Lambda^2}s\right| \leq 105$$

$$\left|\frac{f_{W}}{\Lambda^2}s\right| \leq 205$$

$$\left|\frac{f_{B}}{\Lambda^2}s\right| \leq 640$$

$$\left|\frac{f_{WW}}{\Lambda^2}s\right| \leq 200$$

$$\left|\frac{f_{BB}}{\Lambda^2}s\right| \leq 880$$

$$\left|\frac{f_{WWW}}{\Lambda^2}s\right| \leq 82$$

Combined Results:

We did not constrain f_{WWW} from Higgs data, but TGC bounds:

$$-0.041 \leq \lambda_\gamma \leq -0.003 \quad \lambda_\gamma = \frac{3g^2 M_W^2}{2\Lambda^2} f_{WWW}$$

Combining the Higgs data fits with the unitarity bounds obtained we find:

$$-10 \leq \frac{f_{\Phi,2}}{\Lambda^2}(\mathrm{TeV}^{-2}) \leq 8.5 \quad \Rightarrow \quad \sqrt{\mathbf{s}} \leq \mathbf{3.2}\ \mathrm{TeV}$$

$$-5.6 \leq \frac{f_W}{\Lambda^2}(\mathrm{TeV}^{-2}) \leq 9.6 \quad \Rightarrow \quad \sqrt{s} \leq 4.6\ \mathrm{TeV}$$

$$-29 \leq \frac{f_B}{\Lambda^2}(\mathrm{TeV}^{-2}) \leq 8.9 \quad \Rightarrow \quad \sqrt{s} \leq 4.7\ \mathrm{TeV}$$

$$-3.2 \leq \frac{f_{WW}}{\Lambda^2}(\mathrm{TeV}^{-2}) \leq 8.2 \quad \Rightarrow \quad \sqrt{s} \leq 4.9\ \mathrm{TeV}$$

$$-7.5 \leq \frac{f_{BB}}{\Lambda^2}(\mathrm{TeV}^{-2}) \leq 5.3 \quad \Rightarrow \quad \sqrt{s} \leq 11\ \mathrm{TeV}$$

$$-15 \leq \frac{f_{WWW}}{\Lambda^2}(\mathrm{TeV}^{-2}) \leq 3.9 \quad \Rightarrow \quad \sqrt{\mathbf{s}} \leq \mathbf{2.4}\ \mathrm{TeV}$$

Conclusions:

We have:

- Analyzed the large $\sqrt{s}$ behavior of $VV \to VV$ and $f\bar{f} \to VV$ scattering to $\mathcal{O}(f_i/\Lambda^2)$
- Determined the lowest energies at which unitarity may be violated, requiring NP:

  $\Rightarrow$ Operators affecting Higgs couplings do not violate unitarity for $\sqrt{s} \leq 3.2$ TeV

  $\Rightarrow$ $\mathcal{O}_{WWW}$ naive TGC bounds indicate unitarity may be violated for $\sqrt{s} \geq 2.4$ TeV

In the paper (arXiv: 1505.05516) and poster there's a lot more!

Best Fit and 90% CL regions, Collider + TGC data

For Tevatron+LHC+TGC:

	Best fit	90% CL allowed range
f_g/Λ^2 (TeV^{-2})	1.1, 22	$[-3.3, 5.1] \cup [19, 26]$
f_{WW}/Λ^2 (TeV^{-2})	1.5	$[-3.2, 8.2]$
f_{BB}/Λ^2 (TeV^{-2})	-1.6	$[-7.5, 5.3]$
f_W/Λ^2 (TeV^{-2})	2.1	$[-5.6, 9.6]$
f_B/Λ^2 (TeV^{-2})	-10	$[-29, 8.9]$
$f_{\phi,2}/\Lambda^2$ (TeV^{-2})	-1.0	$[-10, 8.5]$

SM lays well within 1σ CL regions

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(Reduced via EOM & precision data in arXiv: 1211.4580)

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$$\mathcal{O}_{B} = (D_\mu\Phi)^\dagger \hat{B}^{\mu\nu}(D_\nu\Phi) \qquad \mathcal{O}_{W} = (D_\mu\Phi)^\dagger \hat{W}^{\mu\nu}(D_\nu\Phi) \qquad \mathcal{O}_{WWW} = \mathrm{Tr}[\hat{W}_\mu{}^\nu \hat{W}_\nu{}^\rho \hat{W}_\rho{}^\mu]$$

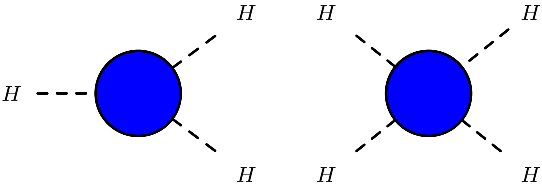
$$\mathcal{O}_{\Phi,2} = \frac{1}{2}\partial_\mu(\Phi^\dagger\Phi)\partial^\mu(\Phi^\dagger\Phi) \qquad \mathcal{O}_{\Phi,4} = (D_\mu\Phi)^\dagger(D^\mu\Phi)(\Phi^\dagger\Phi)$$

Operators in blue contain derivatives of H → finite wavefunction renormalizations:

$$H = h\left[1 + \frac{v^2}{2\Lambda^2}(2f_{\phi,2} + f_{\phi,4})\right]^{1/2} .$$

→ rescaling of all SM Higgs couplings.

As well as new Lorentz forms for 3 and 4 H vertices:



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$\mathcal{O}_{B} = (D_\mu \Phi)^\dagger \hat{B}^{\mu\nu} (D_\nu \Phi)$ $\quad \mathcal{O}_{W} = (D_\mu \Phi)^\dagger \hat{W}^{\mu\nu} (D_\nu \Phi)$ $\quad \mathcal{O}_{WWW} = \mathrm{Tr}[\hat{W}_\mu{}^\nu \hat{W}_\nu{}^\rho \hat{W}_\rho{}^\mu]$

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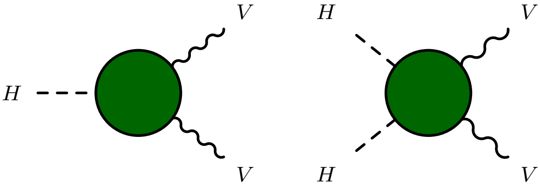
Green $\mathcal{O} \leftrightarrow$ Higgs–Gauge couplings:



Dimension–6 Operators for Unitarity

The relevant Higgs–gauge boson operators are:
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$\mathcal{O}_{B} = (D_\mu \Phi)^\dagger \hat{B}^{\mu\nu} (D_\nu \Phi)$ $\quad \mathcal{O}_{W} = (D_\mu \Phi)^\dagger \hat{W}^{\mu\nu} (D_\nu \Phi)$ $\quad \mathcal{O}_{WWW} = \mathrm{Tr}[\hat{W}_\mu{}^\nu \hat{W}_\nu{}^\rho \hat{W}_\rho{}^\mu]$

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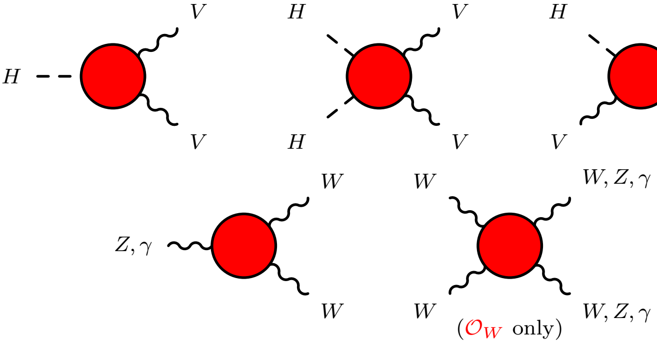
Red $\mathcal{O}$ $\leftrightarrow$ Higgs–Gauge, Triple–Gauge, and Quartic–Gauge couplings:



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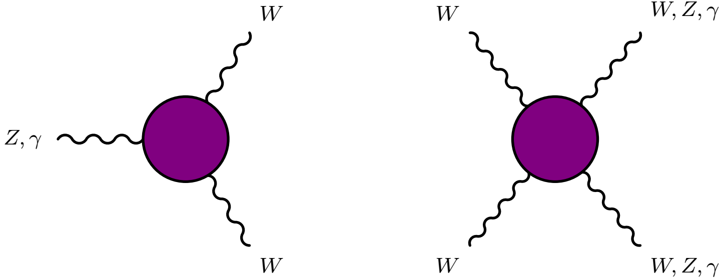
$\mathcal{O}_{\Phi,2} = \frac{1}{2} \partial_\mu (\Phi^\dagger \Phi) \partial^\mu (\Phi^\dagger \Phi)$ $\quad \mathcal{O}_{\Phi,4} = (D_\mu \Phi)^\dagger (D^\mu \Phi)(\Phi^\dagger \Phi)$

Purple $\mathcal{O} \leftrightarrow$ Triple–Gauge and Quartic–Gauge couplings:



$f\bar{f} \to WV$ Unitarity Violating Amplitudes:

Process	$\sigma_1, \sigma_2, \lambda_3, \lambda_4$	Amplitude
$e^+e^- \to W^-W^+$	$-+00$	$-\frac{ig^2 s\sin\theta}{8}\frac{c_W^2 f_W + s_W^2 f_B}{c_W^2\Lambda^2}$
	$+-00$	$-\frac{ig^2 s\sin\theta}{4}\frac{s_W^2 f_B}{c_W^2\Lambda^2}$
	$-+--$	$-\frac{3ig^4 s\sin\theta}{8}\frac{f_{WWW}}{\Lambda^2}$
	$-+++$	$-\frac{3ig^4 s\sin\theta}{8}\frac{f_{WWW}}{\Lambda^2}$
$\nu\bar{\nu} \to W^-W^+$:	$-+00$	$\frac{ig^2 s\sin\theta}{8}\frac{c_W^2 f_W - s_W^2 f_B}{c_W^2}$
	$+-00$	0
	$-+--$	$\frac{3ig^4 s\sin\theta}{8}\frac{f_{WWW}}{\Lambda^2}$
	$-+++$	$\frac{3ig^4 s\sin\theta}{8}\frac{f_{WWW}}{\Lambda^2}$
$u\bar{u} \to W^-W^+$	$-+00$	$\frac{ig^2 N_c s\sin\theta}{8}\frac{3c_W^2 f_W + s_W^2 f_B}{3c_W^2}$
	$+-00$	$\frac{ig^2 N_c s\sin\theta}{6}\frac{s_W^2}{c_W^2} f_B$
	$-+--$	$\frac{3ig^4 N_c s\sin\theta}{8}\frac{f_{WWW}}{\Lambda^2}$
	$-+++$	$\frac{3ig^4 N_c s\sin\theta}{8}\frac{f_{WWW}}{\Lambda^2}$
$d\bar{d} \to W^-W^+$	$-+00$	$-\frac{ig^2 N_c s\sin\theta}{8}\frac{3c_W^2 f_W - s_W^2 f_B}{3c_W^2}$
	$+-00$	$-\frac{ig^2 N_c s\sin\theta}{12}\frac{s_W^2 f_B}{c_W^2\Lambda^2}$
	$-+--$	$-\frac{3ig^4 N_c s\sin\theta}{8}\frac{f_{WWW}}{\Lambda^2}$
	$-+++$	$-\frac{3ig^4 N_c s\sin\theta}{8}\frac{f_{WWW}}{\Lambda^2}$
$e^+\bar{\nu} \to W^+Z$	$-+00$	$\frac{ig^2 s\sin\theta}{4\sqrt{2}}\frac{f_W}{\Lambda^2}$
	$+-00$	0
	$-+--$	$\frac{3ic_W g^4 s\sin\theta}{4\sqrt{2}}\frac{f_{WWW}}{\Lambda^2}$
	$-+++$	$\frac{3ic_W g^4 s\sin\theta}{4\sqrt{2}}\frac{f_{WWW}}{\Lambda^2}$
$e^+\bar{\nu} \to W^+A$:	$-+00$	0
	$+-00$	0
	$-+--$	$\frac{3is_W g^4 s\sin\theta}{4\sqrt{2}}\frac{f_{WWW}}{\Lambda^2}$
	$-+++$	$\frac{3is_W g^4 s\sin\theta}{4\sqrt{2}}\frac{f_{WWW}}{\Lambda^2}$

One Coupling at a Time: $VV \to VV$

Considering only one coupling at a time gives the bounds:

$$\left|\frac{3}{16\pi}\frac{f_{\Phi 2,4}}{\Lambda^2}s\right| \leq 2 \Rightarrow \left|\frac{f_{\Phi 2,4}}{\Lambda^2}s\right| \leq 33$$

$$\left|1.4\frac{g^2}{8\pi}\frac{f_W}{\Lambda^2}s\right| \leq 2 \Rightarrow \left|\frac{f_W}{\Lambda^2}s\right| \leq 87$$

$$\left|\frac{g^2 s_{\rm W}(\sqrt{9+7c_{\rm W}^2}+3s_{\rm W})}{128c_{\rm W}^2\pi}\frac{f_B}{\Lambda^2}s\right| \leq 2 \Rightarrow \left|\frac{f_B}{\Lambda^2}s\right| \leq 617$$

$$\left|\sqrt{\frac{3}{2}}\frac{g^2}{8\pi}\frac{f_{WW}}{\Lambda^2}s\right| \leq 2 \Rightarrow \left|\frac{f_{WW}}{\Lambda^2}s\right| \leq 99$$

$$\left|.20\frac{g^2}{8\pi}\frac{f_{BB}}{\Lambda^2}s\right| \leq 2 \Rightarrow \left|\frac{f_{BB}}{\Lambda^2}s\right| \leq 603$$

$$\left|(1+\sqrt{17-16c_{\rm W}^2 s_{\rm W}^2})\frac{3g^4}{32\pi}\frac{f_{WWW}}{\Lambda^2}s\right| \leq 2 \Rightarrow \left|\frac{f_{WWW}}{\Lambda^2}s\right| \leq 82$$

Grouping the amplitudes:

Group based on charge, Q, and partial wave, J, as:

(Q,J)	States					Total
$(2,0)$	$W^+_\pm W^+_\pm$	$W^+_0 W^+_0$				3
$(2,1)$	$W^+_\pm W^+_\pm$	$W^+_\pm W^+_0$	$W^+_0 W^+_\pm$			6
$(1,0)$	$W^+_\pm Z_\pm$	$W^+_0 Z_0$	$W^+_\pm \gamma_\pm$	$W^+_0 H$		6
$(1,1)$	$W^+_0 Z_0$	$W^+_\pm Z_0$	$W^+_0 Z_\pm$	$W^+_\pm Z_\pm$		
	$W^+_0 \gamma_\pm$	$W^+_\pm \gamma_\pm$	$W^+_0 H$	$W^+_\pm H$		14
$(0,0)$	$W^+_\pm W^-_\pm$	$W^+_0 W^-_0$	$Z_\pm Z_\pm$	$Z_0 Z_0$		
	$Z_\pm \gamma_\pm$	$\gamma_\pm \gamma_\pm$	$Z_0 H$	HH		12
$(0,1)$	$W^+_0 W^-_0$	$W^+_\pm W^-_0$	$W^+_0 W^-_\pm$	$W^+_\pm W^-_\pm$	$Z_\pm Z_0$	
	$Z_0 Z_\pm$	$Z_0 \gamma_\pm$	$Z_0 H$	$Z_\pm H$	$\gamma_\pm H$	18

Then form matrices of same Q and J, diagonalize for the most stringent bounds.

Example: $(Q,J)=(2,0)\rightarrow 3\times 3$ matrix:

$$\frac{s}{8\pi}\begin{pmatrix} 0 & 0 & \frac{3}{s_W^2}e^4 f_{WWW} \\ 0 & -\frac{3}{8c_W^2}e^2 f_B - \frac{3}{8s_W^2}e^2 f_W - \frac{1}{2}f_{\Phi,2} & 0 \\ \frac{3}{s_W^2}e^4 f_{WWW} & 0 & 0 \end{pmatrix}$$

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(Q,J)	States					Total
$(2,0)$	$W^+_\pm W^+_\pm$	$W^+_0 W^+_0$				3
$(2,1)$	$W^+_\pm W^+_\pm$	$W^+_\pm W^+_0$	$W^+_0 W^+_\pm$			6
$(1,0)$	$W^+_\pm Z_\pm$	$W^+_0 Z_0$	$W^+_\pm \gamma_\pm$	$W^+_0 H$		6
$(1,1)$	$W^+_0 Z_0$	$W^+_\pm Z_0$	$W^+_0 Z_\pm$	$W^+_\pm Z_\pm$		
	$W^+_0 \gamma_\pm$	$W^+_\pm \gamma_\pm$	$W^+_0 H$	$W^+_\pm H$		14
$(0,0)$	$W^+_\pm W^-_\pm$	$W^+_0 W^-_0$	$Z_\pm Z_\pm$	$Z_0 Z_0$		
	$Z_\pm \gamma_\pm$	$\gamma_\pm \gamma_\pm$	$Z_0 H$	HH		12
$(0,1)$	$W^+_0 W^-_0$	$W^+_\pm W^-_0$	$W^+_0 W^-_\pm$	$W^+_\pm W^-_\pm$	$Z_\pm Z_0$	
	$Z_0 Z_\pm$	$Z_0 \gamma_\pm$	$Z_0 H$	$Z_\pm H$	$\gamma_\pm H$	18

Then form matrices of same Q and J, diagonalize for the most stringent bounds.

Example: $(Q,J) = (2,0) \to 3 \times 3$ matrix:

$$\frac{s}{8\pi}\begin{pmatrix} 0 & 0 & \frac{3}{s_{\rm W}^2} e^4 f_{WWW} \\ 0 & -\frac{3}{8c_{\rm W}^2} e^2 f_B - \frac{3}{8 s_{\rm W}^2} e^2 f_W - \frac{1}{2} f_{\Phi,2} & 0 \\ \frac{3}{s_{\rm W}^2} e^4 f_{WWW} & 0 & 0 \end{pmatrix}$$

One Coupling at a Time: $f\bar{f} \to VV$

Recalling we form the strongest bounds from some combination $|X\rangle$ of states, we find the strongest bounds come from:

$$|x1\rangle = \frac{1}{\sqrt{24}} |N_f \left(-e^-_- e^+_+ + \nu_{e-} \bar{\nu}_{e+} + N_c u_- \bar{u}_+ - N_c d_- \bar{d}_+ \right)\rangle$$

$$|x2\rangle = \frac{1}{\sqrt{21}} |N_f \left(-\nu_{e+} e^+_- + N_c u_+ \bar{u}_- - N_c d_+ \bar{d}_- \right)\rangle$$

Which result in the bounds:

$$\frac{1}{24}\left[\left| 6 \frac{g^4}{8\pi} \frac{f_{WWW}}{\Lambda^2} s \right|^2 + \left| 1.41 \frac{g^2}{8\pi} \frac{f_W}{\Lambda^2} s \right|^2 \right] \leq 1 \quad \Rightarrow \quad \left| \frac{f_{WWW}}{\Lambda^2} s \right| \leq 122$$

$$\text{and} \quad \left| \frac{f_W}{\Lambda^2} s \right| \leq 211$$

$$\frac{1}{21} \left| \sqrt{2} \frac{s_w^2}{c_w^2} \frac{g^2}{8\pi} \frac{f_B}{\Lambda^2} s \right|^2 = \left| 0.053 \frac{g^2}{8\pi} \frac{f_B}{\Lambda^2} s \right|^2 \leq 1 \quad \Rightarrow \quad \left| \frac{f_B}{\Lambda^2} s \right| \leq 664$$

Backup: Optical Theorem

The optical theorem states:

$$\mathrm{Im}T^J(12\to 34) = \sum_{12\to 1'2'} \frac{|\vec{p}_{1'2'}|}{\sqrt{s}} T^{J*}(12\to 1'2')T^J(1'2'\to 34),$$

with:

$$|\vec{p}_{ij}| = \frac{\sqrt{[s-(m_i+m_j)^2][s-(m_i-m_j)^2]}}{2\sqrt{s}}.$$

Which we may rewrite as:

$$\mathrm{Im}T^J(12\to 12) = \frac{|\vec{p}_{12}|}{\sqrt{s}}\left|T^J(12\to 12)\right|^2 + \sum_{1'2'\neq 12}\frac{|\vec{p}_{1'2'}|}{\sqrt{s}}\left|T^J(12\to 1'2')\right|^2.$$

For only one intermediate channel we obtain:

$$\mathrm{Im}T^J(12\to 12) = \frac{|\vec{p}_{12}|}{\sqrt{s}}|T^J(12\to 12)|^2.$$

Rewriting T^J as,

$$T^J(12\to 12) = \frac{\sqrt{s}}{|\vec{p}_{12}|}e^{i\delta}\sin\delta,$$

Gives the result for elastic scattering:

$$|T^J(12\to 12)| \leq \frac{\sqrt{s}}{|\vec{p}_{12}|}\to 2.$$

Backup: Optical Theorem (II)

Alternatively considering fermion scattering we obtain from the optical theorem:

$$
\begin{aligned}
2\mathrm{Im}[T^J(f_{1\sigma_1}\bar{f}_{2\sigma_2} \to f_{1\sigma_1}\bar{f}_{2\sigma_2})] &= \left|T^J(f_{1\sigma_1}\bar{f}_{2\sigma_2} \to f_{1\sigma_1}\bar{f}_{2\sigma_2})\right|^2 \\
&+ \sum_{V_{3\lambda_3},V_{4\lambda_4}} \left|T^J(f_{1\sigma_1}\bar{f}_{2\sigma_2} \to V_{3\lambda_3}V_{4\lambda_4})\right|^2 \\
&+ \sum_N \left|T^J(f_{1\sigma_1}\bar{f}_{2\sigma_2} \to N)\right|^2 ,
\end{aligned}
$$

Defining,

$$
T^J(12 \to 12) \equiv y + ix, \qquad d \equiv \sum_{1'2' \neq 12} \frac{|\vec{p}_{1'2'}|}{\sqrt{s}} \left|T^J(12 \to 1'2')\right|^2 ,
$$

Allows us to rewrite the above as:

$$
x = \frac{|\vec{p}_{12}|}{\sqrt{s}}(x^2 + y^2) + d.
$$

Finally in order for the assumption that x is real to hold, the quadratic equation requires:

$$
2\sum_{1'2' \neq 12} \frac{|\vec{p}_{1'2'}|}{\sqrt{s}} \left|T^J(12 \to 1'2')\right|^2 \leq 1 \to \sum_{V_{3\lambda_3},V_{4\lambda_4}} \left|T^J(f_{1\sigma_1}\bar{f}_{2\sigma_2} \to V_{3\lambda_3}V_{4\lambda_4})\right|^2 \leq 1
$$