Lattice QCD at finite baryon density

Vitaly Bornyakov

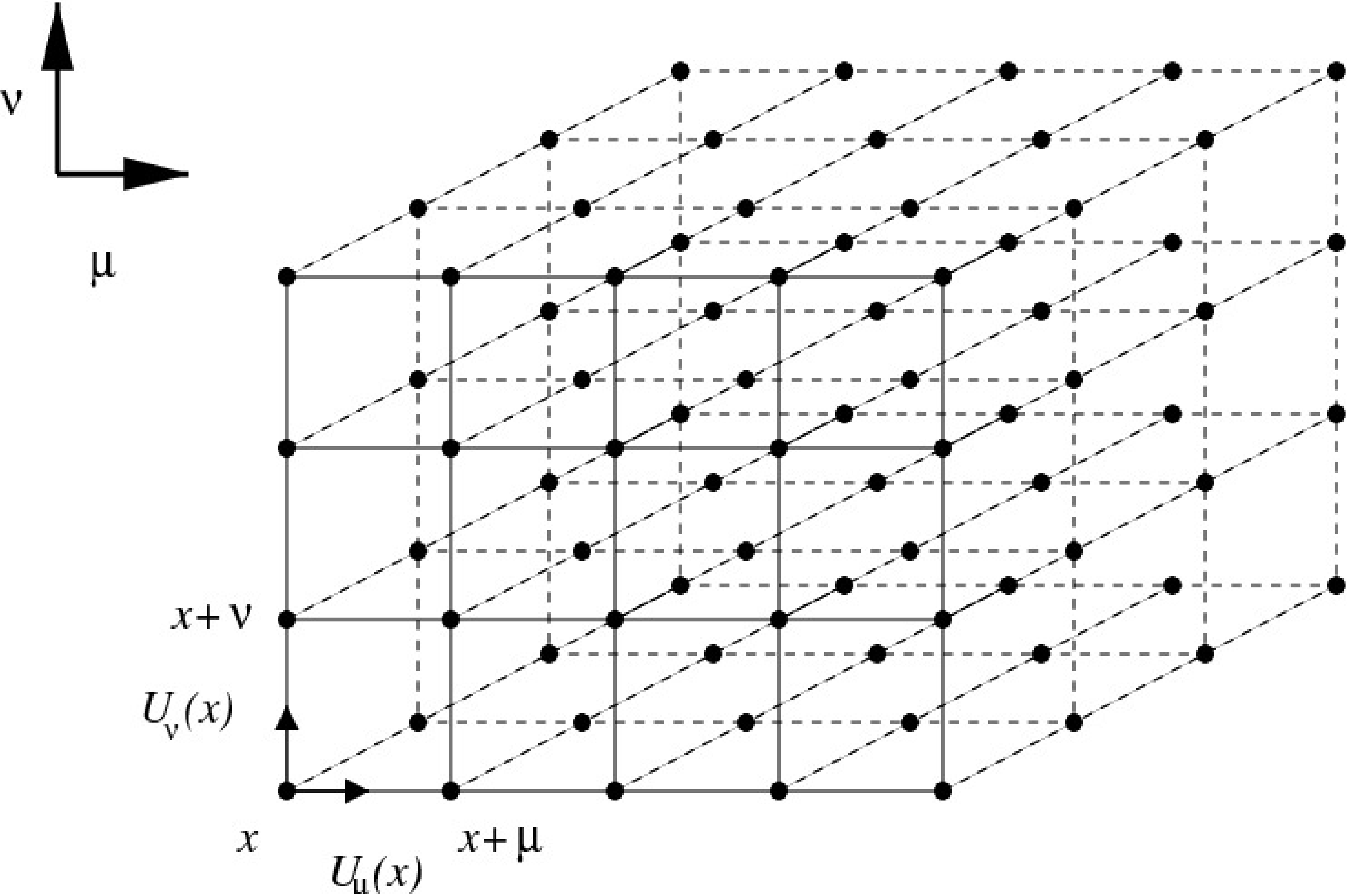
07.07.2017
„Critical points in the modern particle physics“, IHEP, Protvino

Outline

- QCD in lattice regularization

- Problem at $\mu > 0$ and methods to solve it

- Results at $\mu > 0$



ν
μ
x+ν
U_ν(x)
x
x+μ
U_μ(x)

Lattice QCD action

$$S_W^G = \beta \sum_P \left(1 - \frac{1}{3} \mathrm{Re}\ \mathrm{Tr} U_P \right)$$

$$U_{\mu\nu}(s) = U_\mu(s) U_\nu(s + \hat{e}_\mu) U_\mu^\dagger(s + \hat{e}_\nu) U_\nu^\dagger(s)\ .$$

$$S_W^F = \bar{\psi}\, M(U)\, \psi$$

$$S_W^G \xrightarrow{a\to 0} \frac{1}{2\,g^2}\int \mathrm{Tr} F_{\mu\nu}^2(x)\, d^4x + O(a^2)\,,$$

$$S_W^F \xrightarrow{a\to 0} \int \bar{\psi}_f(x)(\gamma_\mu D_\mu + m_f)\psi_f(x)\, d^4x + O(a)$$

Fermion field integration

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-\bar{\psi}M(U)\psi} = \det M(U)$$

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\ \bar{\psi}^a(s')\psi^b(s)\ e^{-\bar{\psi}M(U)\psi}$$

$$= (M^{-1}_{s,s'}(U))^{ab} \det M(U)$$

$$< \mathcal{O} > = \frac{1}{\mathcal{Z}} \int \mathcal{D}U \, \mathcal{O}(U) \, e^{-S_{\text{eff}}(U)}$$

$$\mathcal{Z} = \int \mathcal{D}U \, e^{-S_{\text{eff}}(U)}$$

$$S_{\text{eff}}(U) = S_W^G(U) - \sum_f \ln \det M_f(U)$$

$$\mathcal{D}U = \prod_{s,\mu} dU_\mu(s)$$

Gauge fields $U_\mu(x)$ are generated with probability

$$P \sim e^{-S_{eff}(U)}$$

By means of the Markov chain Monte Carlo method

Then a physical observable is computed as

$$< \mathcal{O} > \approx \frac{1}{N_{\text{conf}}} \sum_{i=1}^{N_{\text{conf}}} \mathcal{O}_i(U)$$

Results obtained in QCD in lattice regularization (aka Lattice QCD) at $\mu=0$ have statistical as well as number of systematic uncertainties.

These are controlled uncertainties.

They can be estimated and decreased.

QCD at T>0, μ>0

$$S_G[A] = \int_0^{1/T} dx_4 \int d^3x \, \frac{1}{2} \mathrm{Tr}\, F_{\mu\nu}(x) F_{\mu\nu}(x)$$

$$S_F[\bar{\psi}, \psi, A]$$
$$= \int_0^{1/T} dx_4 \int d^3x \sum_f \bar{\psi}_f(x) \left(\gamma_\mu D_\mu + m_f - \mu_f \gamma_0\right) \psi_f(x)$$

$T = 1/L_4,$ L_4 - length in 4th direction

$L_4 = aN_4,$ a – lattice spacing

Sign problem

$\det(\not{D} + m + \mu\gamma_0)$ -- In the integral

We use $\gamma_5 \not{D} \gamma_5 = \not{D}^\dagger$

$$\gamma_5(\not{D} + m + \mu\gamma_0)\gamma_5 = \not{D}^\dagger + m - \mu\gamma_0$$
$$= (\not{D} + m - \mu^*\gamma_0)^\dagger$$

$$\det(\not{D} + m + \mu\gamma_0) = \det{}^*(\not{D} + m - \mu^*\gamma_0)$$

$$\det(\not{D} + m + \mu\gamma_0) = \det{}^*(\not{D} + m - \mu^*\gamma_0)$$

Determinant is real only for μ =0 and $\mu = i\mu_I$

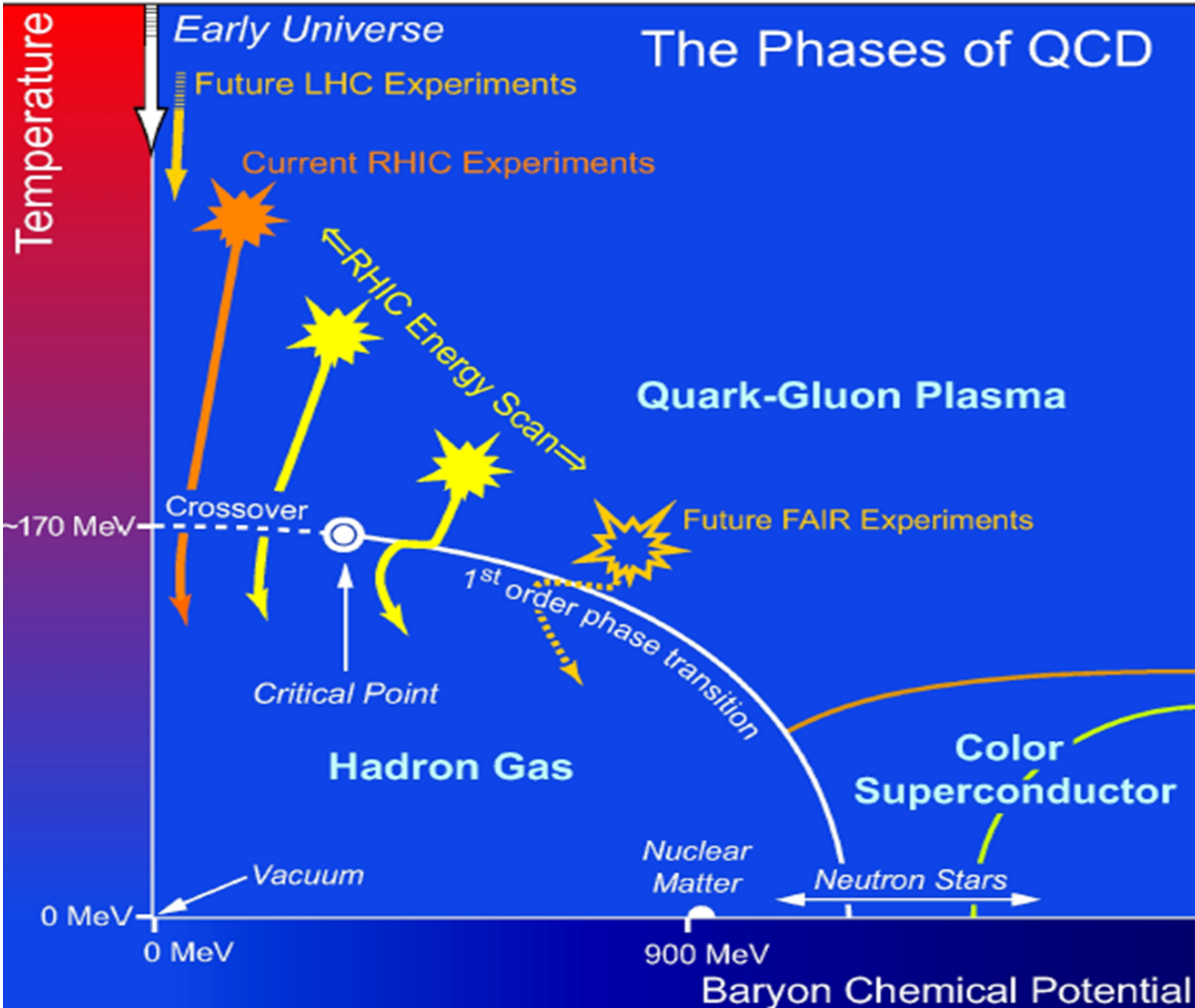
This makes impossible to apply usual MCMC algorithm In case of real μ

Note, that for imaginary μ this problem is absent

LQCD action with μ:

$$S_F(\psi, \bar{\psi}, U) = \bar{\psi}\, M(U)\psi$$

where $U_4 \rightarrow U_4 e^{a\mu}, \quad U_4^{+} \rightarrow U_4^{+} e^{-a\mu}$



The Phases of QCD
Temperature
Early Universe
Future LHC Experiments
Current RHIC Experiments
⇐RHIC Energy Scan⇒
Quark-Gluon Plasma
~170 MeV
Crossover
Future FAIR Experiments
1st order phase transition
Critical Point
Hadron Gas
Color Superconductor
Nuclear Matter
Neutron Stars
Vacuum
0 MeV
0 MeV
900 MeV
Baryon Chemical Potential

The expectation is that Tc is less than 160 MeV and μc is greater than a few hundred MeV.

Results of LQCD for $\mu = 0$ (T_c, EoS) are used to fix parameters in various phenomenological models.

But results at $\mu > 0$ are desirable

Methods to solve sign problem

- Multi-Parameter Reweighting

Fodor, Katz, 2002

- Taylor expansion

Gottlieb et al. Phys.Rev.Lett. 59, 2247 (1987) (up to μ^2)
Allton et al., Phys.Rev. D71, 054508 (2005) (up to μ^6)

- Imaginary Chemical Potential

D'Elia, Lombardo, 2002

- Canonical ensemble approach

de Forcrand, Philipsen, 2002

- Complex Langevin

- Density of states

- Dual formulation

- Lefschetz thimble

Notations

pressure

$$\frac{p}{T^4} = \frac{1}{VT^3} \log Z(V,T,\mu)$$

Quark number density

$$n_f/T^3 = \frac{\partial p/T^4}{\partial \mu_f/T}$$

Susceptibility

$$\chi_{ff}/T^2 = \frac{\partial n_f/T^3}{\partial \mu_f/T}$$

For free quark-gluon gas (Stefan-Boltzmann limit):

$$\frac{p_{SB}}{T^4} = \frac{8\pi^2}{45} + \sum_{f=u,d,..} \left[\frac{7\pi^2}{60} + \frac{1}{2}\left(\frac{\mu_f}{T}\right)^2 + \frac{1}{4\pi^2}\left(\frac{\mu_f}{T}\right)^4\right]$$

This is valid for very high T

For low T – Hadron resonance gas (HRG) model

$$\frac{p}{T^4} = G(T) + F(T)\cosh\left(\frac{3\mu_q}{T}\right)$$

Taylor expansion for pressure:

$$\frac{p}{T^4} = \sum_{n=0}^{\infty} c_n(T) \left(\frac{\mu_q}{T}\right)^n$$

$$\mu_u = \mu_d = \mu_q$$

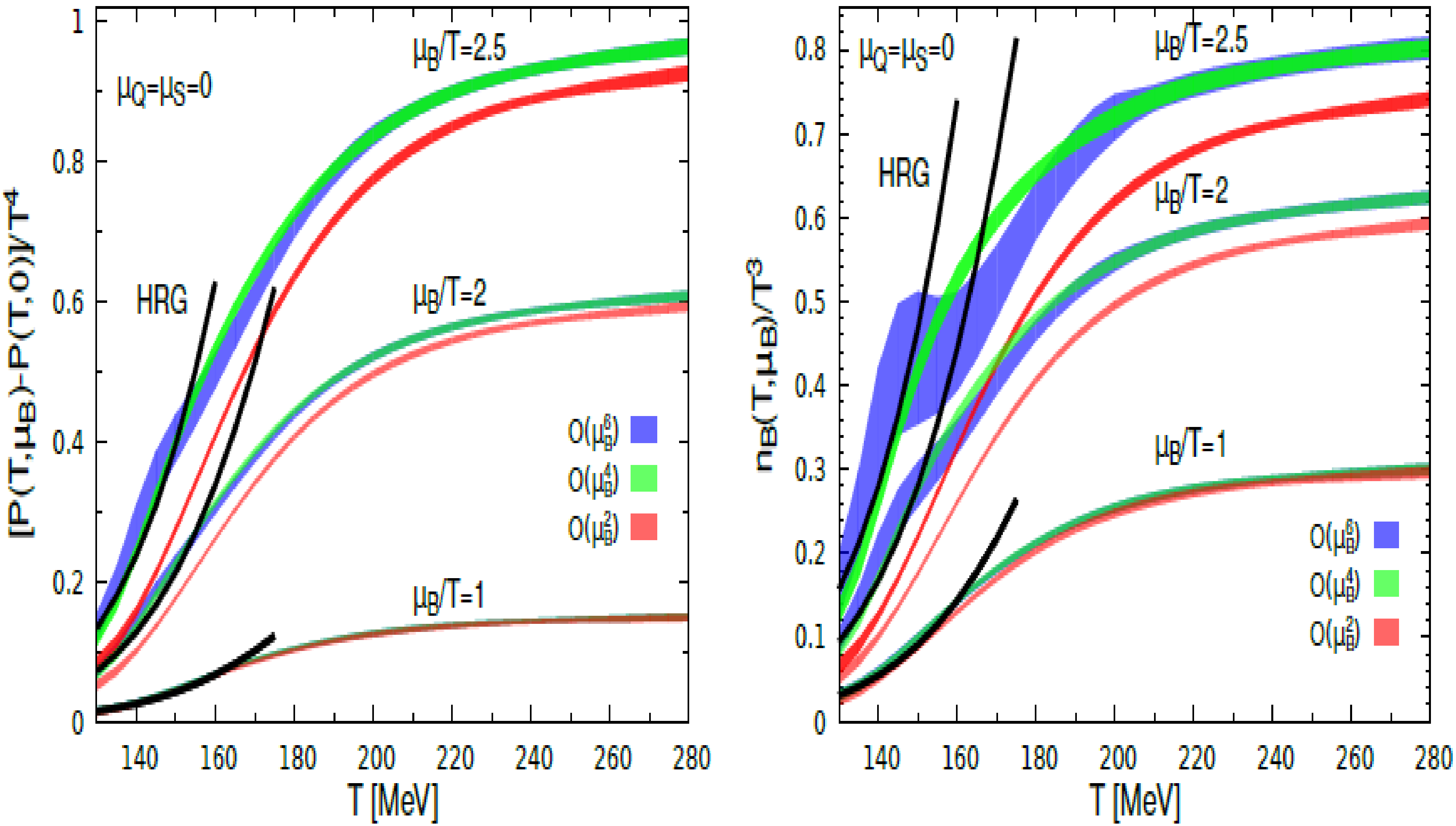
Hot QCD collaboration, Phys.Rev.D95, 054504 (2017)
'The QCD Equation of State to $O\left(\mu_B^6\right)$ From Lattice QCD'

$$N_f = 2 + 1$$

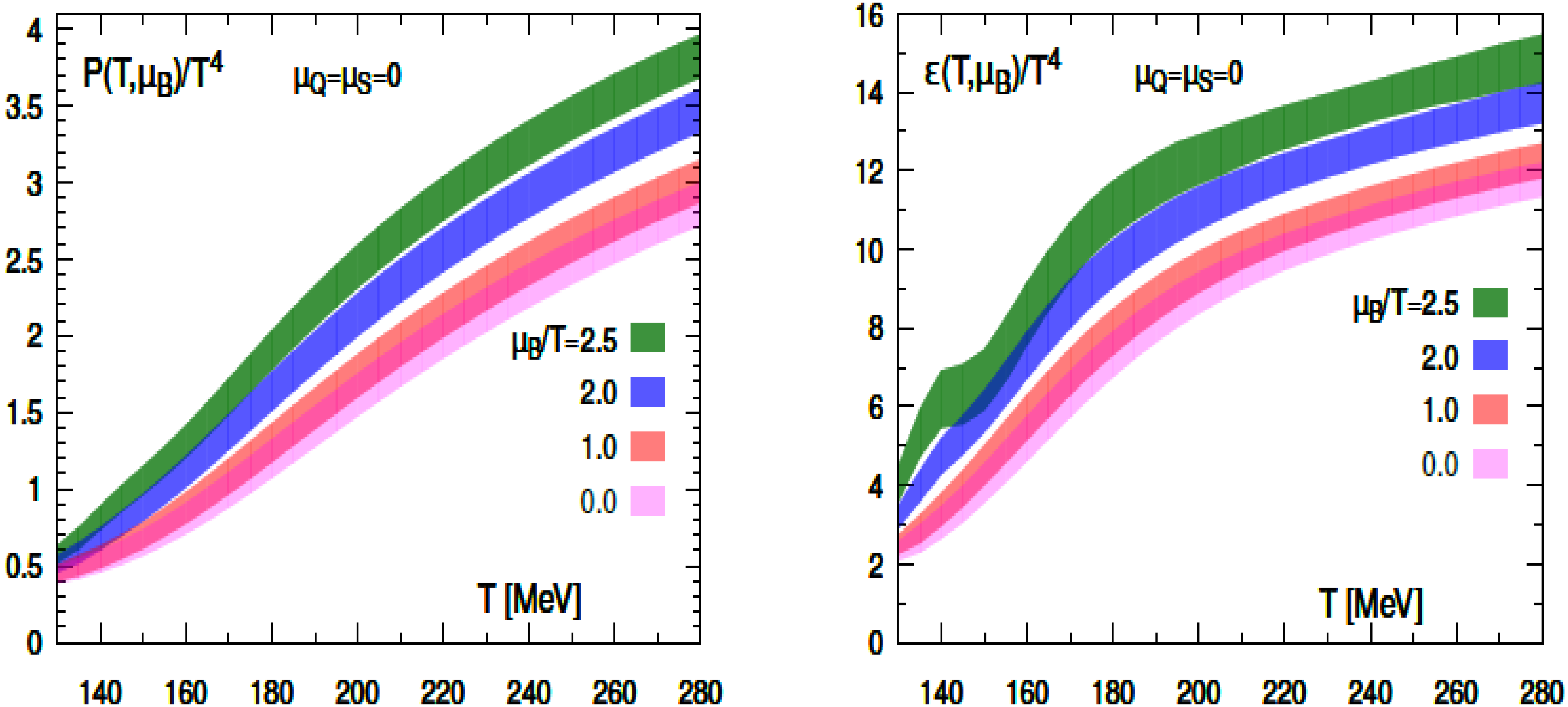
$$\frac{m_s}{m_l} = 27 \quad (m_\pi = 140\ \text{MeV})$$

$$N_t = 8,10,12,16$$

$$\frac{N_s}{N_t} = 4$$

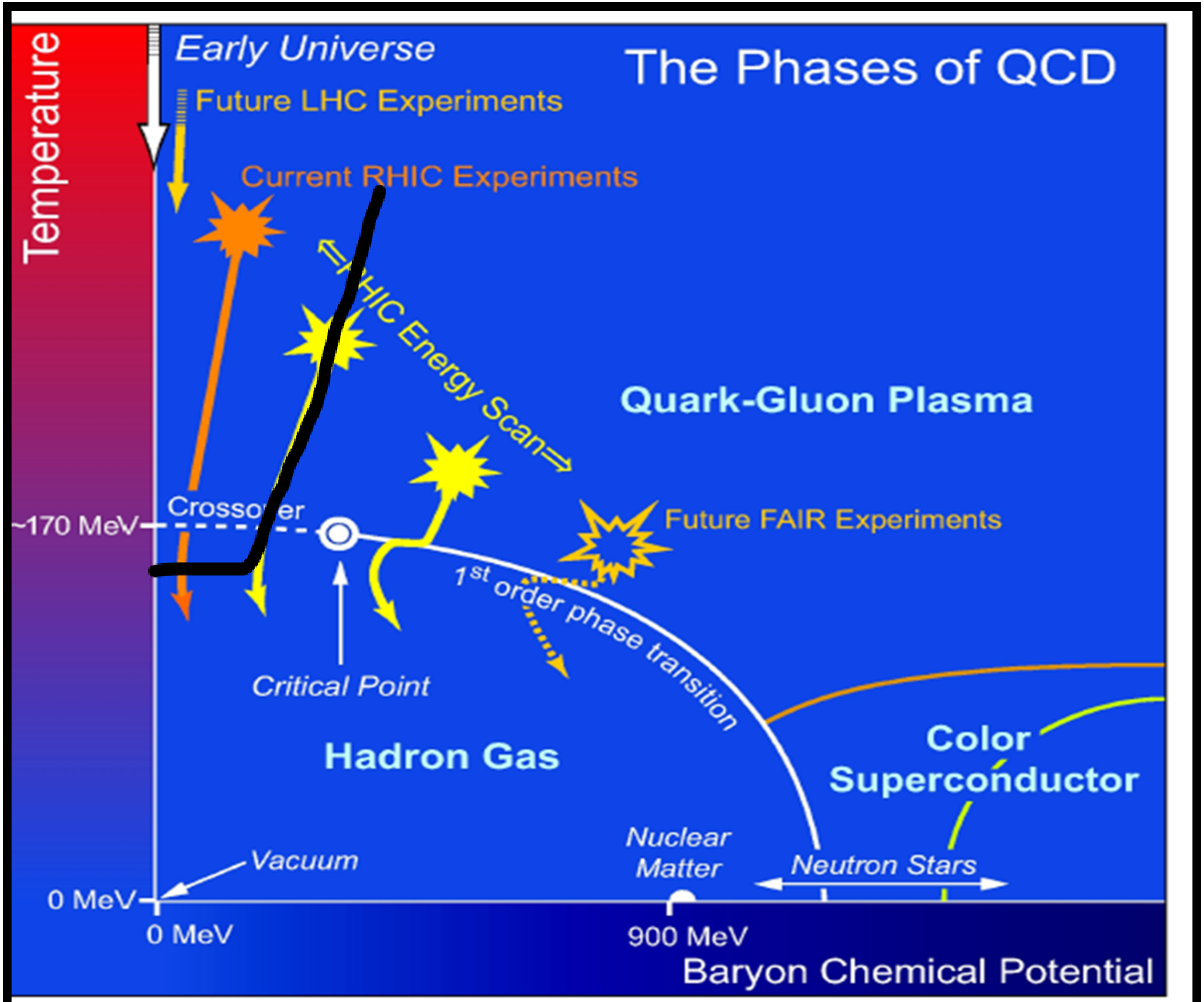


$\mu_Q=\mu_S=0$
$\mu_B/T=2.5$
$\mu_B/T=2$
$\mu_B/T=1$
HRG
$O(\mu_B^6)$
$O(\mu_B^4)$
$O(\mu_B^2)$
$[P(T,\mu_B)-P(T,0)]/T^4$
$n_B(T,\mu_B)/T^3$
T [MeV]



P(T,μB)/T4
μQ=μS=0
μB/T=2.5
2.0
1.0
0.0
T [MeV]
ε(T,μB)/T4
μQ=μS=0
μB/T=2.5
2.0
1.0
0.0
T [MeV]

Assuming that the current results obtained with expansion coefficients up to 6th order are indicative for the behavior of higher order expansion coefficients and taking into account the current errors on 6th order expansion coefficients we concluded that at temperatures T > 135 MeV the presence of a critical point in the QCD phase diagram for $\mu_B < 2T$ is unlikely.



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Baryon Chemical Potential

Imaginary μ_q

At imaginary chemical potential $\mu_q = i\mu_{qI}$ the sign problem is absent and standard Monte Carlo algorithms can be applied to simulate Lattice QCD. Can we use this?

Study of QCD at nonzero μ_{qI} can provide us with information about physical range of μ_q

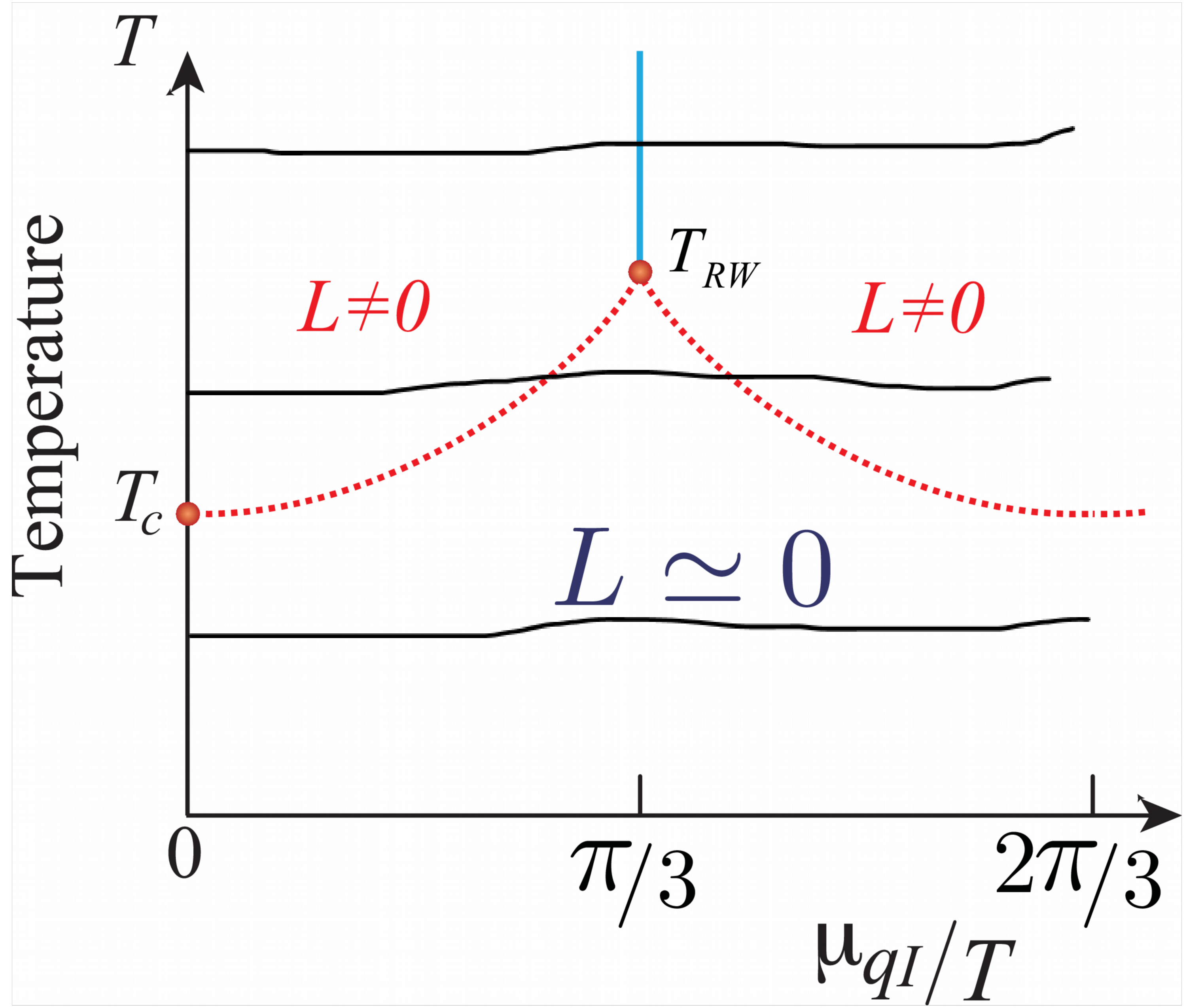
- extrapolation to $\mu_q = 0$ or analytical continuation to nonzero real μ_q

The QCD partition function Z is a periodic function of $\theta = \mu_{qI}/T$:

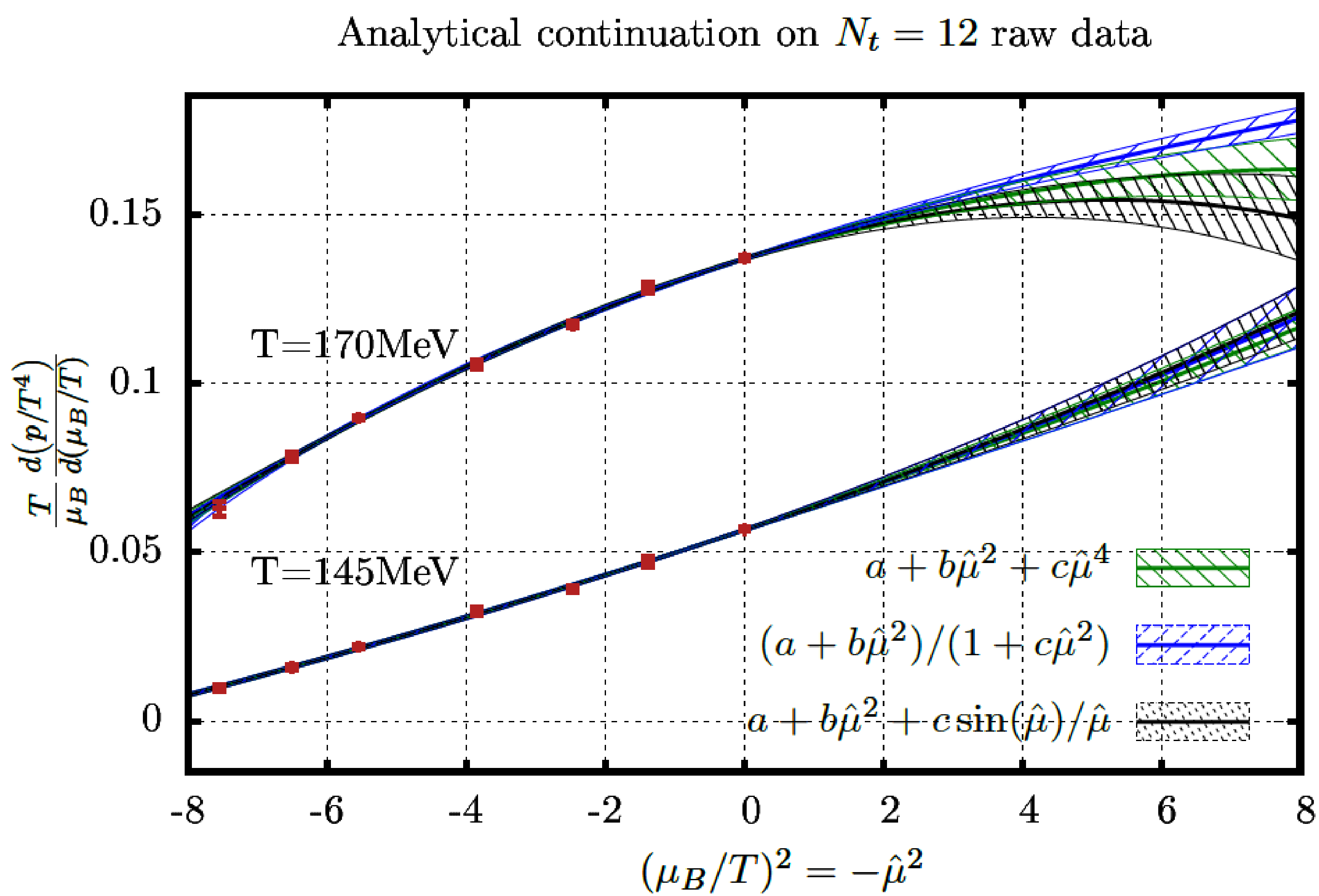
$$Z(\theta) = Z(\theta + 2\pi k/3)$$

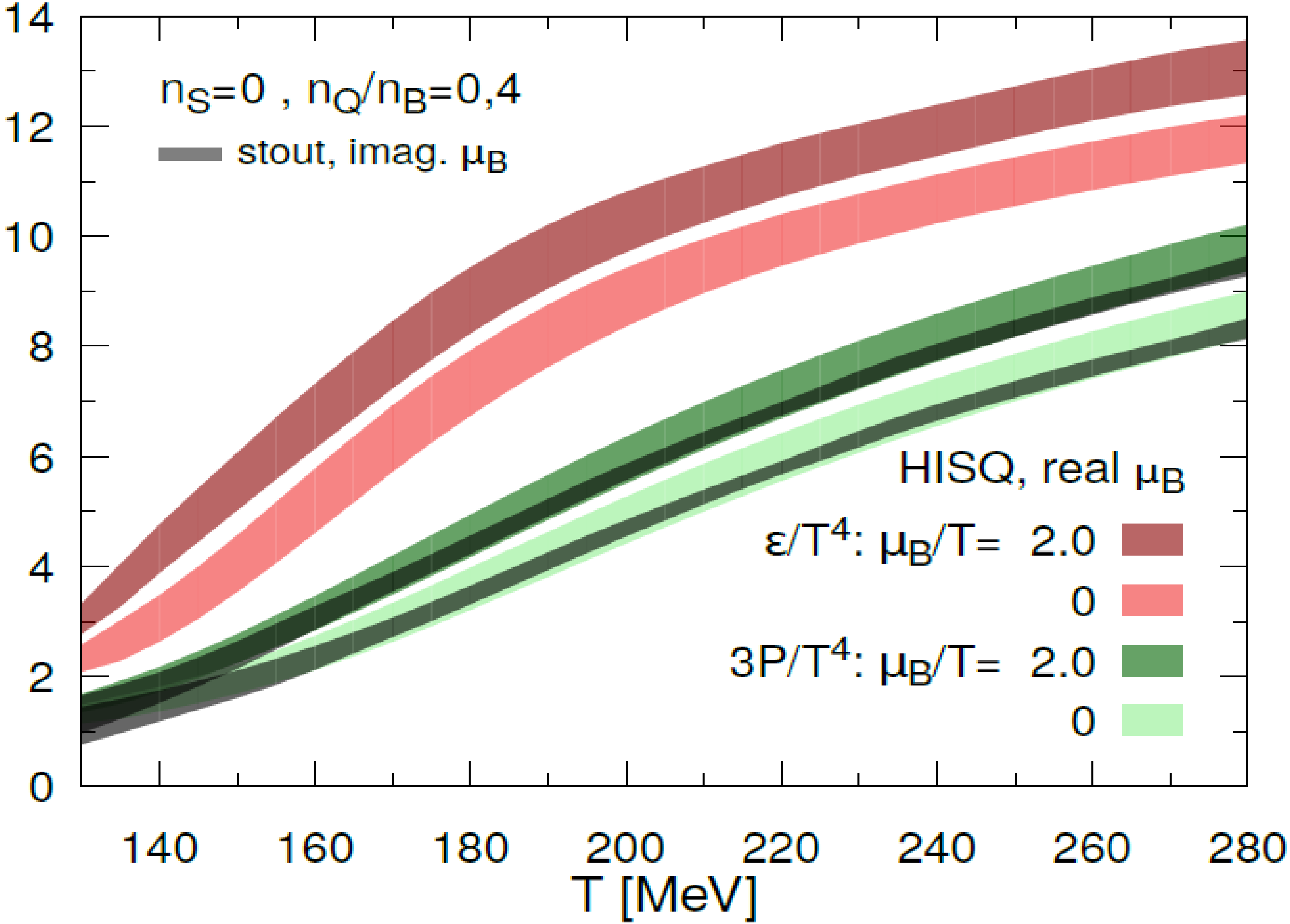
There are 1st order phase transitions at $\theta = (2k+1)\frac{\pi}{3}$

This symmetry is called Roberge-Weiss symmetry


T
Temperature
T_{RW}
$L \neq 0$
$L \neq 0$
T_c
$L \simeq 0$
0
$\pi/3$
$2\pi/3$
μ_{qI}/T

N_f=2+1, N_t=10,12,16, N_s=40,48,64
Fodor et al., 2016



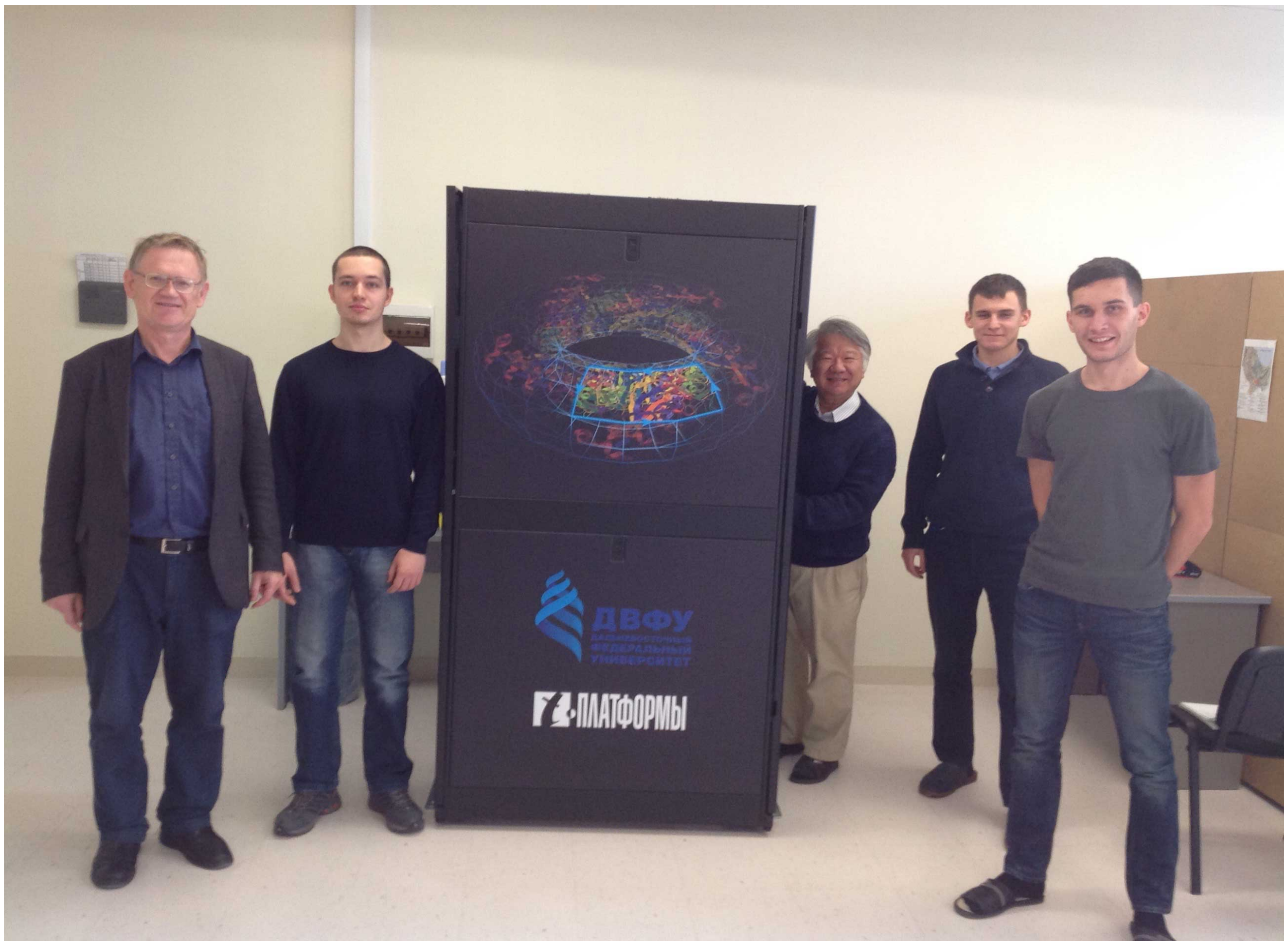


Pressure obtained by two different methods

FEFU lattice group

‘QCD at nonzero baryon density’

Nakamura, V.B., Molochkov, Boyda, Goy, Nikolaev, Zakharov, Iida, Wakayama



Canonical ensemble approach

$$Z_{GC}(\mu,T,V) = \mathrm{Tr}\left(e^{-\frac{\hat{H}-\mu\hat{N}}{T}}\right) = \sum_{n=-\infty}^{\infty} \langle n| e^{-\frac{\hat{H}}{T}} |n\rangle e^{\frac{\mu n}{T}}$$

$$= \sum_{n=-\infty}^{\infty} Z_C(n,T,V) e^{\frac{\mu n}{T}} = \sum_{n=-\infty}^{\infty} Z_n e^{\theta n} = \sum_{n=-\infty}^{\infty} Z_n \xi^n.$$

$$Z_n = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} Z_{GC}(\mu = i\mu_{Im}, T, V).$$

Using numerical data or fitted form of n_{ql} one can compute the grand canonical partition function by integration over imaginary chemical potential

$$LZ(\theta) \equiv \log \frac{Z_G(\theta)}{Z_G(0)} = \int_0^\theta d\tilde{\theta} n_{ql}(\tilde{\theta}) \tag{10}$$

Then

$$\frac{Z_G(\theta)}{Z_G(0)} = e^{LZ(\theta)} \tag{11}$$

Z_n can be also computed

$$\frac{Z_n}{Z_G(0)} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} e^{LZ(\theta)} \tag{12}$$

We obtained very promising results for canonical partition functions Z_n.

Simulation settings

We simulate $N_f = 2$ lattice QCD with clover improved Wilson fermions and Iwasaki improved gauge field action

To fix parameters (lattice spacing a, temperature T, quark mass m_q) we use $T = 0$ results of WHOT QCD collaboration

Currently quark mass is defined by ratio $m_\pi / m_\rho = 0.8$ ($m_\pi \approx 0.7$ GeV
Lattice size: $16^3 \times 4$
large lattice spacing: $a \approx 0.2$ fm
large volume: $L \approx 3.2$ fm
We simulate at imaginary chemical potential $\mu_q = i\mu_q^I$
At $T > T_c$ ($T/T_c = 1.08$; 1.35, 1.20)
at $T < T_c$ ($T/T_c = 0.84$; 0.93; 0.99)

In the deconfining phase we fit the data for n_{ql} to a polynomial of θ

$$n_{ql}(\theta) = \sum_{n=1}^{n_{max}} a_n \theta^{2n-1} \tag{10}$$

while in the confining phase (below T_c) we fit it to a Fourier expansion

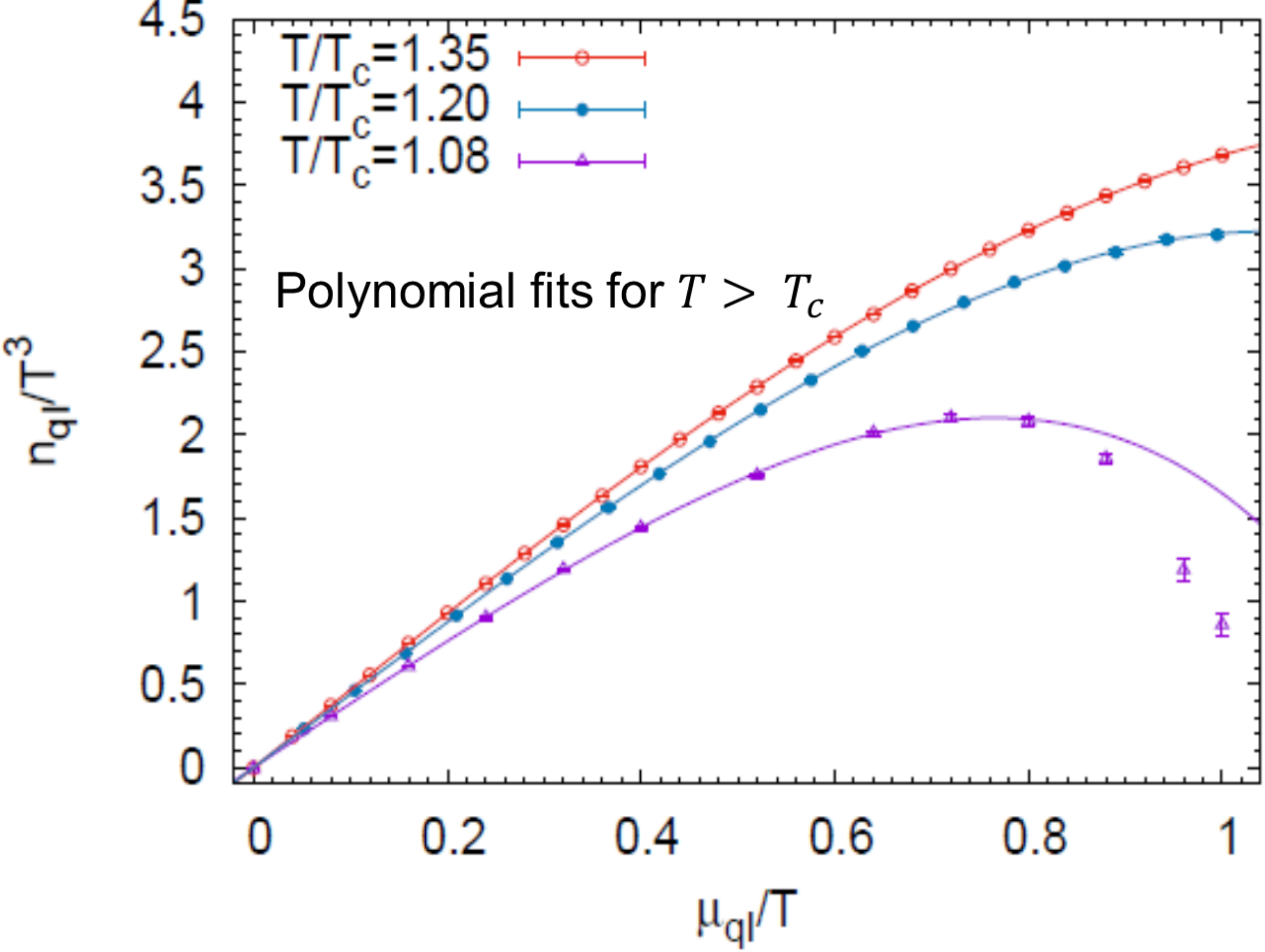
$$n_{ql}(\theta) = \sum_{n=1}^{n_{max}} f_{3n} sin(3n\theta) \tag{11}$$

Similar fits were used in Takahashi et al, 2014

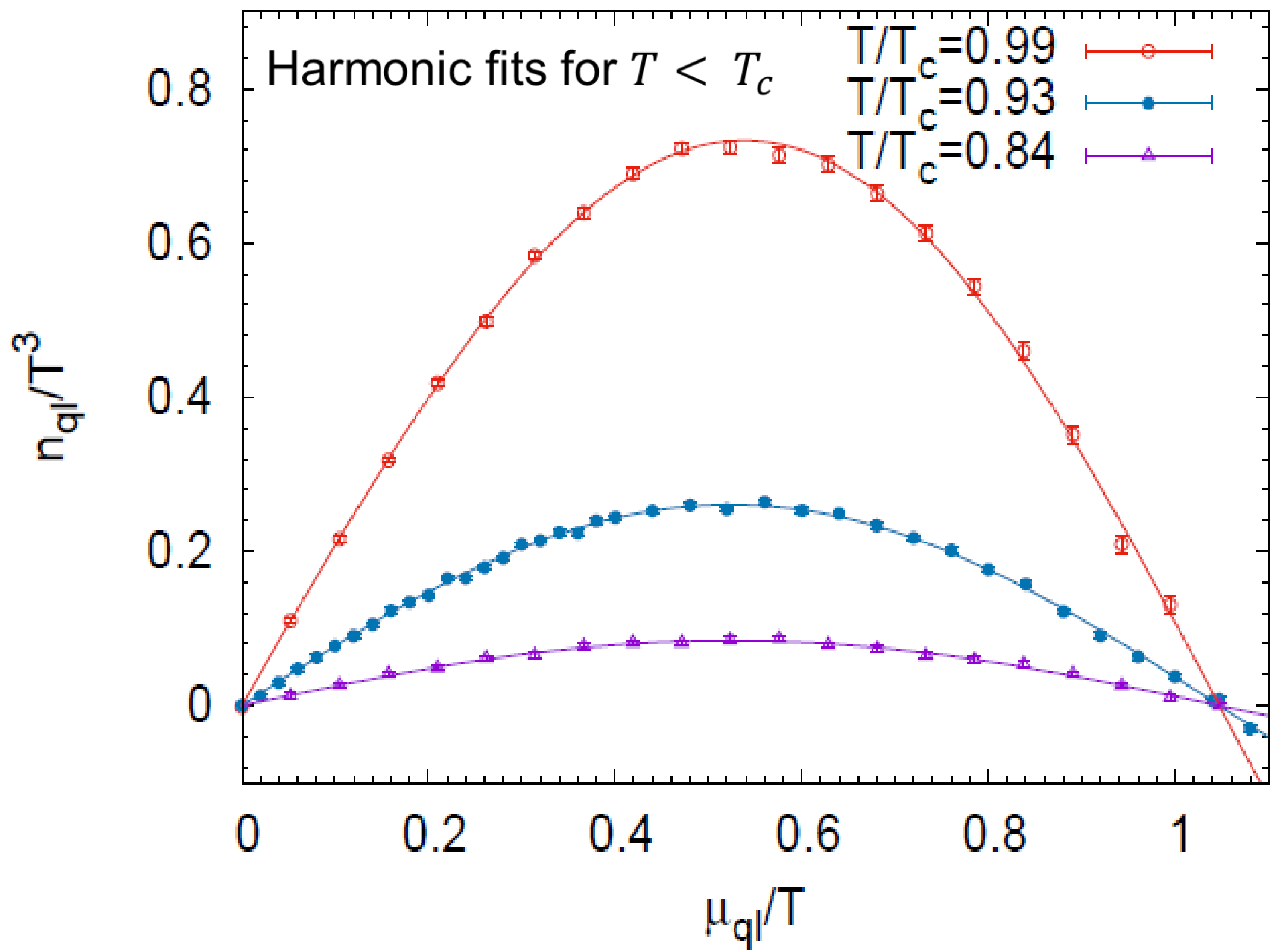
Gunther et al., 2016 arXiv:1607.02493

One more restriction on the fitting functions:

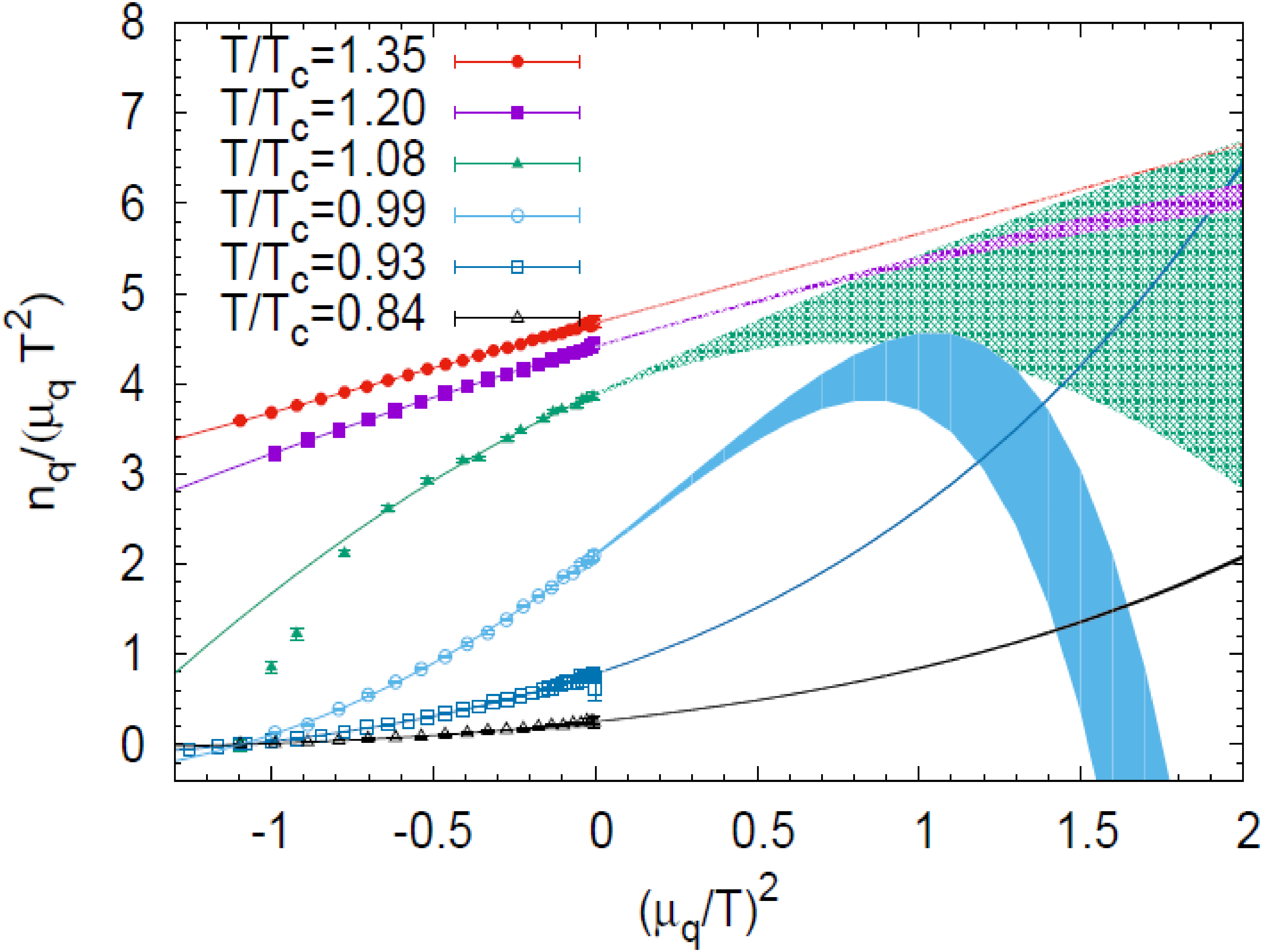
$$Z_n > 0$$



Imaginary number density at $T > T_{RW}$ and $T_c < T < T_{RW}$

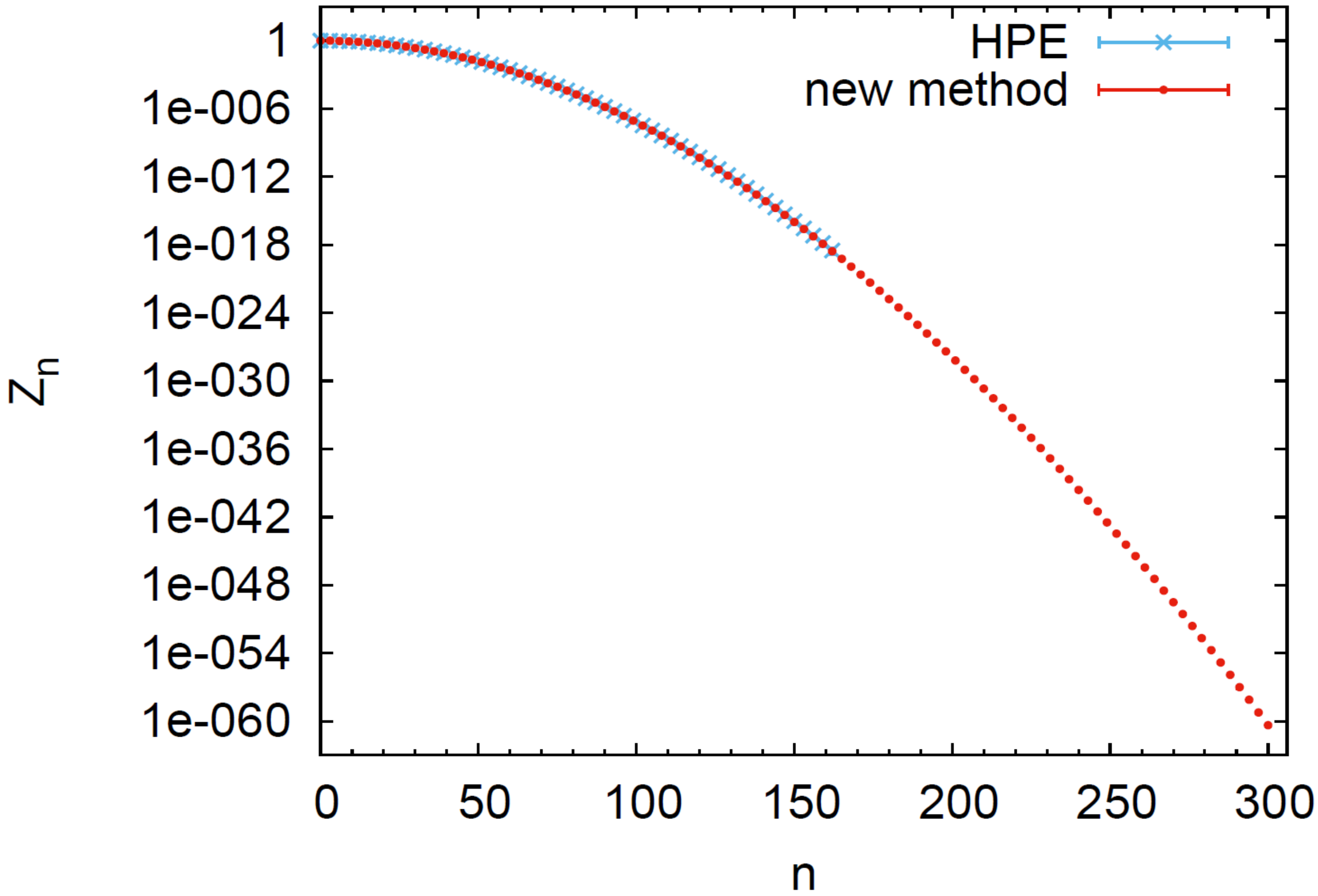


Harmonic fits for $T < T_c$
T/T_c=0.99
T/T_c=0.93
T/T_c=0.84
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
n_ql/T^3
μ_ql/T

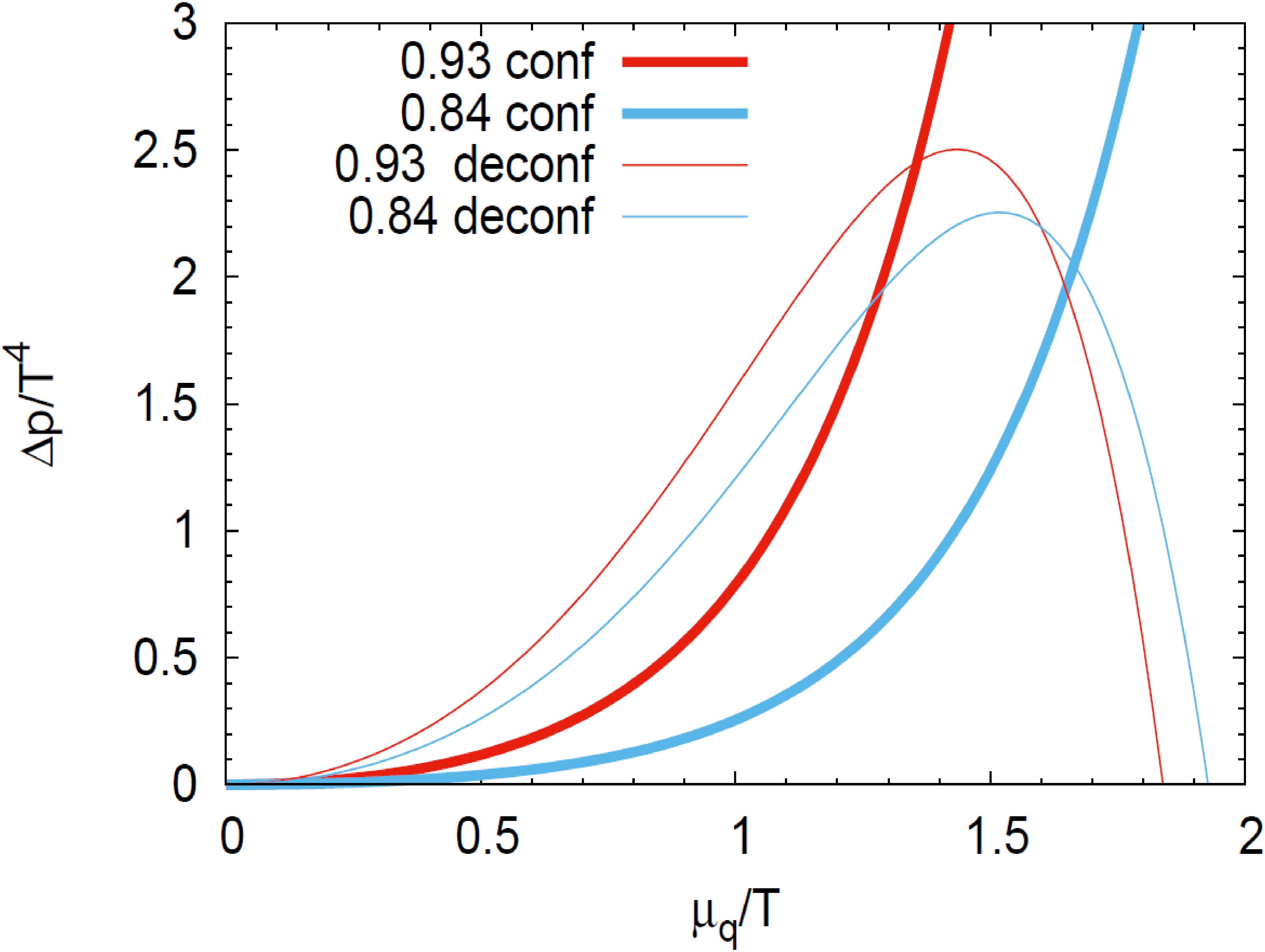


T/Tc=1.35
T/Tc=1.20
T/Tc=1.08
T/Tc=0.99
T/Tc=0.93
T/Tc=0.84
nq/(μq T2)
(μq/T)2
8
7
6
5
4
3
2
1
0
-1
-0.5
0
0.5
1
1.5
2

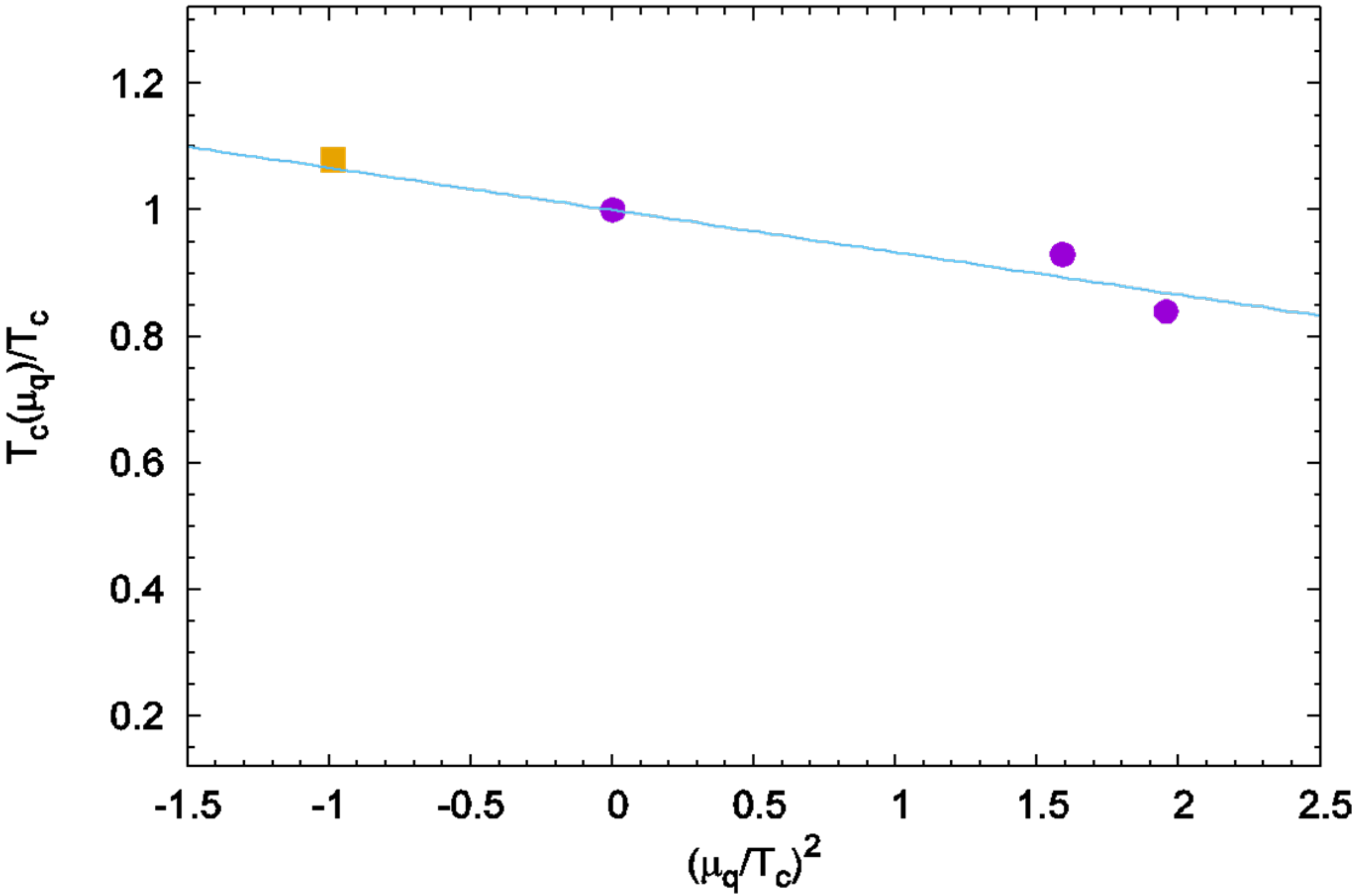
Deconfining phase:



To determine the transition line in the $\mu - T$ plane we extrapolate pressure from $T > T_c$ to $T < T_c$



0.93 conf
0.84 conf
0.93 deconf
0.84 deconf
Δp/T^4
μ_q/T
0
0.5
1
1.5
2
2.5
3



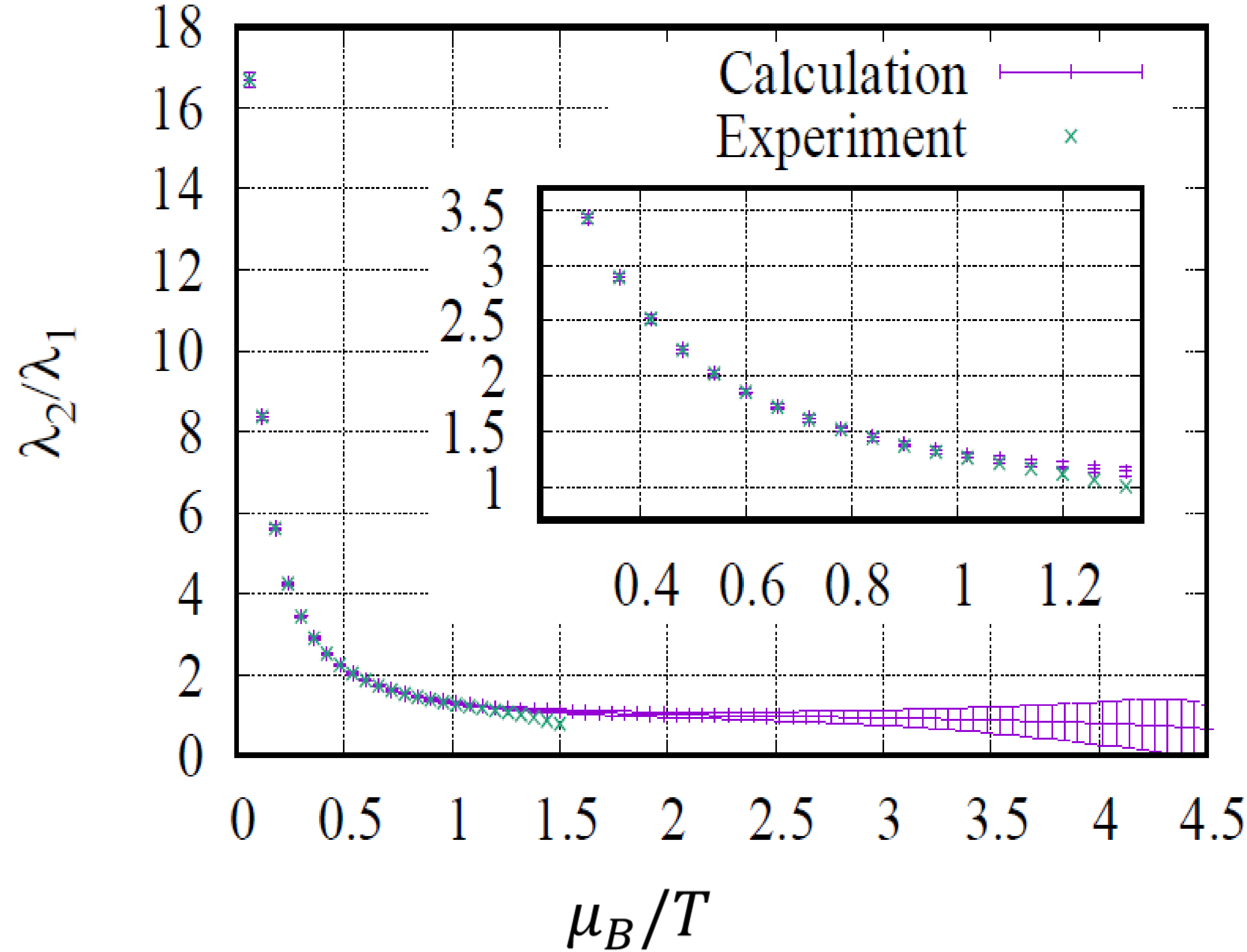
1.2
1
0.8
0.6
0.4
0.2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
$T_c(\mu_q)/T_c$
$(\mu_q/T_c)^2$

$$T_c(\mu_q)/T_c = 1 - C\left(\mu_q/T_c\right)^2$$

Our result: C=0.07 is in nice agreement with other results for N_f=2 lattice QCD:

C=0.051(3) De Forcrand, Philipsen, 2002

C=0.065(7) Wu, Luo, Chen, 2007



Calculation
Experiment
λ_2/λ_1
μ_B/T
0
2
4
6
8
10
12
14
16
18
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
1
1.5
2
2.5
3
3.5
0.4
0.6
0.8
1
1.2

Conclusions

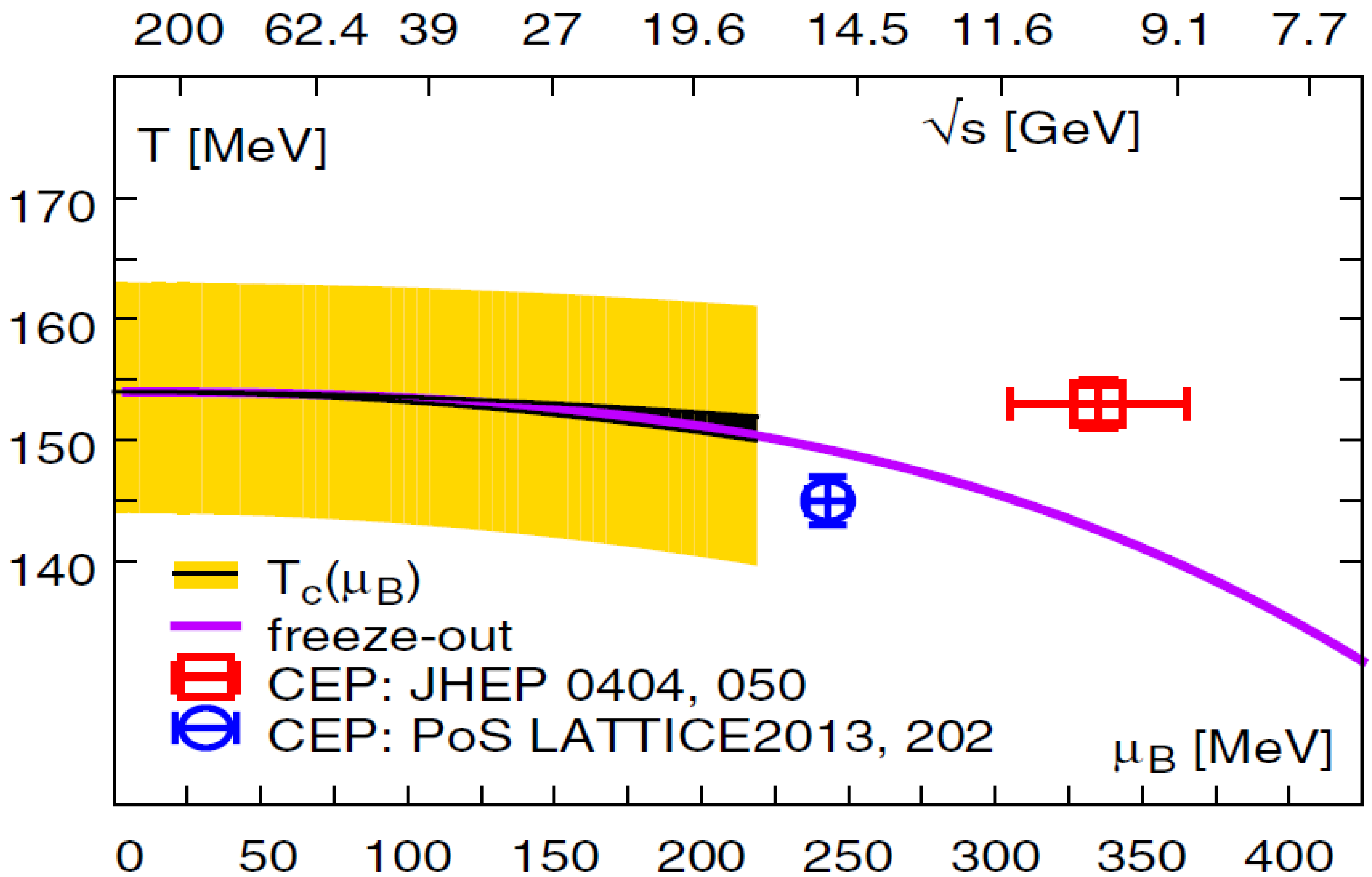
- Lattice QCD makes slow but steady progress in computations (EOS, cumulants, transition line) at $\mu_B > 0$

- Most successful so far are Taylor expansion and analytical continuation approaches

- Our group is developing a combination of the canonical ensemble and analytical continuation approaches. Results are encouraging.

- coefficients of Taylor expansion agree with direct computation, smaller errors
- in the deconfining phase our results are in nice agreement with hopping parameter expansion
- in the confining phase agreement is not so good but there is hope for improvement
- contrary to hopping parameter expansion - no limitations on quark mass
- observed agreement with the hopping parameter expansion means method works beyond Taylor expansion validity range.
- Pressure, number density and higher cumulants can be computed beyond Taylor expansion

$T_c(\mu)$



200
62.4
39
27
19.6
14.5
11.6
9.1
7.7
√s [GeV]
T [MeV]
170
160
150
140
Tc(μB)
freeze-out
CEP: JHEP 0404, 050
CEP: PoS LATTICE2013, 202
μB [MeV]
0
50
100
150
200
250
300
350
400

We use hopping parameter expansion (HPE) to evaluate the determinant ($\xi = e^{\mu_B a N_t} = e^{\mu_B/T}$):

$$Tr\left[ln\,\Delta\right] = Tr\left[ln\left(I - \kappa Q\right)\right] = -\sum_{n=1}^{\infty} \frac{\kappa^n}{n} Tr\left[Q^n\right] = \sum_{n=-\infty}^{\infty} W_n \left(e^{\mu_B a N_\tau}\right)^n$$

$$\det \Delta(U) = e^{Tr[ln\Delta]} = exp\left[\sum_{n=-N_{cut}}^{N_{cut}} W_n[U]\xi^n\right]$$

$W_n[U]$ may be calculated using stochastic estimators for $Tr\left[Q^n\right]$.

$$\det \Delta(U) = \sum_{n=-2N_x N_y N_z N_c}^{2N_x N_y N_z N_c} z_n[U]\xi^n$$