Cardy Formula for SUSY Theories & Localization

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CERN - Geometry of String and Gauge Theories

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Introduction

- $\mathcal{N} = 1, d = 4$ theories with an unbroken $R$-symmetry

- Interesting observable: generalization of Witten index defined by quantization on compact space-like slice
Introduction

SUSY theory on a curved space by coupling to background fields in the “new minimal” sugra multiplet

\((g_{\mu\nu}, A^R_\mu, V_\mu)\)

Festuccia, Seiberg
Introduction

In particular: $\mathcal{M}_3 \times \mathbb{R}$

$\mathcal{M}_3$ Seifert manifold: two Killing spinors $\zeta, \tilde{\zeta}$, $K_\mu = \zeta \sigma_\mu \tilde{\zeta}$

SUSY algebra: $\{\delta_\zeta, \delta_{\tilde{\zeta}}\} = i \delta K$

$[H, \delta_\zeta] = [H, \delta_{\tilde{\zeta}}] = 0 \quad \delta_\zeta^2 = \delta_{\tilde{\zeta}}^2 = 0$

Klare, Tomasiello, Zaffaroni; Dumitrescu, Festuccia, Seiberg; Closset, Dumitrescu, Festuccia, Komargodski
Generalization of Witten index: \( \text{Tr}_{\mathcal{H}(M_3)}((-1)^F e^{-\beta H}) \)
counting of short states

Path integral on \( M_3 \times S^1_{\beta} \)
with periodic fermions

\[ Z_{M_3 \times S^1(\beta)} \]
Introduction

- Independent on continuous coupling constant
- RG invariant
- Application: Dualities
Introduction

- Superconformal case

\[ \mathcal{H}(S^3) \equiv \{ \mathcal{O} \} \]

\[ Z_{S^3 \times S^1} = \sum_{\mathcal{O}} (-1)^F e^{-\beta(\Delta + 1/2R)} \]

- Superconformal index: counts BPS operators

Kinney, Maldacena, Minwalla, Raju; Romelsberger
Introduction

• Our aim is to study the Cardy limit $\beta \to 0$ of these partition functions.

• It captures the density of short states with high energy/dimension.

• We can derive simple universal formulas.
Introduction

- $Z_{\mathcal{M}_3 \times S^1}(\beta)$ for a gauge theory can be computed using localization
- One-loop det depends on holonomies around $S^1$
- Additional parameters depending on $\mathcal{M}_3$
Explicit formulae for 1-loop determinants on $\mathcal{M}_3 = S^3/\mathbb{Z}_N, S^1 \times \Sigma_g$

Benini, Nishioka, Yamazaki; Razamat, Willet; Closset, Shamir; Assel, Cassani, Martelli; Nishioka, Yaakov; Honda, Yoshida; Benini, Zaffaroni

Picking correct integration contour can be subtle

Benini, Eager, Hori, Tachikawa
Introduction

- Effective action for the holonomies in the limit $\beta \to 0$

- It can be derived on general $\mathcal{M}_3$, depends on it in a simple way
Result

In the limit $\beta \to 0$

$$Z_{S^1 \times M_3} \xrightarrow{\beta \to 0} e^{-\frac{\pi^2 \text{Tr}(R) L M_3}{12\beta}} \int d^r a \, e^{-V_{\text{eff}}_{M_3}(a)} + \ldots$$

"potential" for the holonomies
Result

\[ V_{M_3}^{\text{eff}}(a) = - \sum_f \sum_{\rho_f \in \mathcal{R}_f} \left[ \frac{\pi^3 i A_{M_3}}{6 \beta^2} \kappa(\rho_f \cdot a) \right. \]
\[
+ \frac{\pi^2 (R_f L_{M_3} - \rho_f \cdot l_{M_3})}{2 \beta} \vartheta(\rho_f \cdot a) \left. \right],
\]

\[ \kappa(x) \equiv \{x\}(1 - \{x\})(1 - 2\{x\}), \]
\[ \vartheta(x) \equiv \{x\}(1 - \{x\}). \]

\(A_{M_3}, L_{M_3}, l^i_{M_3}:\) integrals of local densities in the background/gauge fields
Result

- It was derived by Ardehali for $\mathcal{M}_3 = S^3$ using expression of 1-loop determinants.

- Alternative approach (effective 3d theory) that is easy to generalize. No need to know full determinant in advance.
Example

SU(2) with fundamental flavors
Example

ISS model: SU(2) + quadruplet

Intriligator, Seiberg, Shenker
Example

SU(3) SQCD (from Ardehali)
Result

Limit of the integral?

When $V_{\mathcal{M}_3}^{\text{eff}}(a_{\text{min}}) = 0$

$$Z_{S^1 \times \mathcal{M}_3} \xrightarrow{\beta \to 0} e^{-\frac{\pi^2 \text{Tr}(R)L_{\mathcal{M}_3}}{12\beta}} + \ldots \times Z_{\mathcal{M}_3}$$

Asymptotics fixed by gravitational anomalies

LD, Komargodski
Result

What just said applies if

\[ V_{M_3}^{\text{eff}}(a_{\text{min}}) = 0 \]

Additional contribution for cases with \( V_{M_3}^{\text{eff}}(a_{\text{min}}) < 0 \)

Examples known so far: ‘misleading anomaly matching’, all have \( a - c > 0 \)
Result

- Theories with $V_{\mathcal{M}_3}^{\text{eff}}(a_{\text{min}}) < 0$

- In this case the leading asymptotics of the partition function rather than $Z_{S^1 \times \mathcal{M}_3} \rightarrow e^{-\frac{\pi^2 \text{Tr}(R)L}{12\beta} \mathcal{M}_3} + \ldots \times Z_{\mathcal{M}_3}$ is

$$Z_{S^1 \times \mathcal{M}_3} \rightarrow e^{-\frac{\pi^2 \text{Tr}(R)L}{12\beta} \mathcal{M}_3} + V_{\mathcal{M}_3}^{\text{eff}}(a_{\text{min}}) + \ldots \times Y_{\mathcal{M}_3}$$

Physical interpretation?
Results

- $V_{\mathcal{M}_3}^{\text{eff}}(a_{\text{min}}) < 0$: one-loop determinant of $Z_{\mathcal{M}_3}$ grows exp at infinity

- This falsifies an assumption in LD, Komargodski

- We lack a more physical characterization of theories with this behavior
Derivation

- Couple the theory to bkgd metric and $G$ gauge field

\[ g_{MN} dX^M dX^N = (dX^4 + c_\mu dx^\mu)^2 + h_{\mu\nu} dx^\mu dx^\nu \]

\[ A_M dX^M = (dX^4 + c_\mu dx^\mu) A_4 + A_\mu dx^\mu \]

They are part of supersymmetric multiplets
Derivation

In a weakly coupled point:

KK fermions on $M_3$

$$M_n = \frac{2\pi}{\beta} (n + \rho_f \cdot a)$$

$$a = \frac{\beta A_4}{2\pi}$$

We take $\beta \to 0$ with $a$ finite
Derivation

Formula in three steps:

1) Integrate out
2) Supersymmetrize
3) Evaluate on SUSY configuration
1) Integrate out:

Mixed Chern-Simons terms between gauge, R-symmetry, KK.

Coefficient = infinite sum over KK tower, zeta-function regularization

\[ S_k(s, \rho_f \cdot a) = \sum_{n \in \mathbb{Z}} \text{sgn}(n + \rho_f \cdot a)n^{k-1}|n + \rho_f \cdot a|^{-s} \]
Derivation

2) Supersymmetrize:

Chern-Simons terms completed to full supersymmetric Lagrangians.

Key: these Lagrangian contains term that are activated on the background.
Derivation

For instance, R-KK:

\[ i \epsilon^{\mu \nu \rho} A^{(R) \mu} \partial_\nu c_\rho - \frac{1}{2} f^2_{\mu \nu} - \frac{1}{2} (A^{(R)}_4)^2 + \frac{1}{4} R \]

Integrated over \( \mathcal{M}_3 \)
defines the length \( L_{\mathcal{M}_3} \).
Derivation

3) Evaluate on the SUSY configuration of gauge fields:

\[(A^i_{\mu}, \sigma^i, D^i) \text{ with } \begin{cases} 
\sigma & \equiv A_4 \\
A_\mu & \equiv A_\mu - A_4 c_\mu \\
D & \equiv D - A_4 A_4^{(R)} \end{cases} \]

\[ A^i_4 = \frac{2\pi}{\beta} a^i \]

\[ A^i_\mu, D^i \text{ depend on } M_3. \]

They enter the density \[ l^i_{M_3}. \]
Derivation

For instance, plugging in the gauge-R:

\[ \frac{2\pi}{\beta} (\rho f \cdot a) \times \]

\[ (i\epsilon^{\mu\nu\rho} A^{(R)}_\mu \partial_\nu c_\rho - \frac{1}{2} f_{\mu\nu} - \frac{1}{2}(A^{(R)}_4)^2 + \frac{1}{4} R) \]
Derivation

The \( a \)-dependent coefficients combine to give

\[
V_{\mathcal{M}_3}^{\text{eff}}(a) = - \sum_f \sum_{\rho_f \in \mathcal{K}_f} \left[ \frac{\pi^3 i A_{\mathcal{M}_3}}{6 \beta^2} \kappa(\rho_f \cdot a) \right. \\
+ \frac{\pi^2 (R_f L_{\mathcal{M}_3} - \rho_f \cdot l_{\mathcal{M}_3})}{2 \beta} \vartheta(\rho_f \cdot a) \left. \right],
\]

\[
\kappa(x) \equiv \{x\}(1 - \{x\})(1 - 2\{x\}),
\]

\[
\vartheta(x) \equiv \{x\}(1 - \{x\}).
\]
Derivation

Integrate/sum over the conf of the dynamical gauge fields

Finally we get

\[
Z_{S^1 \times \mathcal{M}_3} \xrightarrow{\beta \to 0} e^{-\frac{\pi^2 \text{Tr}(R) L \mathcal{M}_3}{12 \beta}} \int d^r a \ e^{-V_{\mathcal{M}_3}^{\text{eff}}(a) + \ldots}
\]

\(\alpha^i\)-independent prefactor
Applications

- It matches with the limit of the 1-loop determinants on $M_3 = S^3/\mathbb{Z}_N, S^1 \times \Sigma_g$

- More general backgrounds: $M_3$ can also be a non-trivial $S^1$ fibration over a Riemann surface with $g > 1$

Closset, Kim, Willett
Applications

Superconformal case

\[ \text{Tr} R \sim a - c \]

\[ \sum_{\mathcal{O}} (-1)^F e^{-\beta(\Delta + \frac{R}{2})} \sim e^{-\frac{16\pi^2}{3}(a-c)\frac{r_3}{\beta}} \]

Similar to Cardy formula in 2d CFT

\[ Z_{T^2} = \sum_{\mathcal{O}} e^{-\beta\Delta} \sim e^{\frac{\pi^2}{3}c\frac{r_1}{\beta}} \]
Applications

Usually: \((a, c)\) via correlators of the stress-energy tensor/anomalies

Formula for \(a - c\) via the sum over BPS operators

Ardehali, Liu, Szepietowski
Beccaria, Tseytlin
Applications

Theories with extended SUSY $N=2$ in 4d $e (2,0)$ in 6d

With one condition on the fugacities, more SUSY: Schur Index

Only 1 parameter left

Related to the character of 2d chiral algebras

Beem, Lemos, Liendo, Peelaers, Rastelli
\[ I(p, q, y) = \text{Tr} \left[ (-1)^F p^{j_2-j_1 + \frac{R}{2}} q^{j_2+j_1 + \frac{R}{2}} y^f \right] \]

\[ R = \frac{4}{3} \left( I_3 - \frac{r}{2} \right), \quad f = I_3 + r \]

- 4d Schur Index \( p = (pq)^{\frac{2}{3}} y \equiv e^{-\beta\omega} \)

- Dependence on \( q \) drops

- We obtain \( \log I^{\text{Schur}} \sim -\frac{8\pi^2}{\beta\omega} (a - c) \)

Agrees with the character of the chiral algebra. Different info than central charge: \( c^{2d} = -c^{4d} \)

Ardehali; Rastelli

Thursday, July 13, 17
\[ I(p, q, t, y) = \text{Tr} \left[ (-1)^F p^{R + h_1} q^{R + h_2} t^{R - h_3} y f \right] \]
\[ R = \frac{R + r}{2}, \quad f = R - r \]

- **6d Schur Index** \( y \sqrt{\frac{q t}{p}} = 1, \quad p \equiv e^{-\beta \omega} \)

- Dependence on \( q, t \) drops

- We obtain \( \log I^{\text{Schur}} \approx \frac{\pi r_g}{12 \omega} \left( \frac{2\pi}{\beta} \right) \)

\( \beta^{-3} \) cancels. Agrees with the proposed correspondence to \( \mathcal{W}_g \) chiral algebra.
Outlook

• Characterize theories with $V_{\mathcal{M}_3}^{\text{eff}}(a_{\text{min}}) < 0$, sign of $a - c$?

• Physical interpretation of Cardy behavior for $V_{\mathcal{M}_3}^{\text{eff}}(a_{\text{min}}) < 0$

• Relation to the bound on the gap for holographic theories?

$$\Delta_\text{gap}^2 \leq \left| \frac{c}{c - a} \right|$$

Camanho, Edelstein, Maldacena, Zhiboedov
Thank you!