Cardy Formula for SUSY Theories & Localization

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CERN - Geometry of String and Gauge Theories

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Introduction

- $\mathcal{N}=1, d=4$ theories with an unbroken R-symmetry
- Interesting observable: generalization of Witten index defined by quantization on compact space-like slice

Introduction

- SUSY theory on a curved space by coupling to background fields in the "new minimal" sugra multiplet

$$(g_{\mu\nu}, A_\mu^R, V_\mu)$$

Festuccia, Seiberg

Introduction

- In particular: $\mathcal{M}_3 \times \mathbb{R}$
- $\mathcal{M}_3$ Seifert manifold: two Killing spinors $\zeta, \tilde{\zeta}$ $\quad K_\mu = \zeta \sigma_\mu \tilde{\zeta}$
- SUSY algebra: $\{\delta_\zeta, \delta_{\tilde{\zeta}}\} = i\delta_K$

$$[H, \delta_\zeta] = [H, \delta_{\tilde{\zeta}}] = 0 \quad \delta_\zeta^2 = \delta_{\tilde{\zeta}}^2 = 0$$

Klare, Tomasiello, Zaffaroni; Dumitrescu, Festuccia, Seiberg; Closset, Dumitrescu, Festuccia, Komargodski

Introduction

- Generalization of Witten index: $\mathrm{Tr}_{\mathcal{H}(\mathcal{M}_3)}((-1)^F e^{-\beta H})$ counting of short states
- Path integral on $\mathcal{M}_3 \times S^1_\beta$ with periodic fermions

$$Z_{\mathcal{M}_3 \times S^1}(\beta)$$

Introduction

- Independent on continuous coupling constant
- RG invariant
- Application: Dualities

Introduction

- Superconformal case

$$\mathcal{H}(S^3) \equiv \{\mathcal{O}\}$$

$$Z_{S^3\times S^1} = \sum_{\mathcal{O}} (-1)^F e^{-\beta(\Delta+1/2R)}$$

- Superconformal index: counts BPS operators

Kinney, Maldacena, Minwalla, Raju; Romelsberger

Introduction

- Our aim is to study the Cardy limit $\beta \to 0$ of these partition functions
- It captures the density of short states with high energy/dimension
- We can derive simple universal formulas

Introduction

- $Z_{\mathcal{M}_3 \times S^1}(\beta)$ for a gauge theory can be computed using localization
- One-loop det depends on holonomies around S^1
- Additional parameters depending on $\mathcal{M}_3$

Introduction

- Explicit formulae for 1-loop determinants on $\mathcal{M}_3 = S^3/\mathbb{Z}_N, S^1 \times \Sigma_g$

Benini, Nishioka, Yamazaki; Razamat, Willet; Closset, Shamir; Assel, Cassani, Martelli; Nishioka, Yaakov; Honda, Yoshida; Benini, Zaffaroni

- Picking correct integration contour can be subtle

Benini, Eager, Hori, Tachikawa

Introduction

- Effective action for the holonomies in the limit $\beta \to 0$
- It can be derived on general $\mathcal{M}_3$, depends on it in a simple way

Result

- In the limit $\beta \to 0$

$$Z_{S^1\times\mathcal{M}_3} \xrightarrow[\beta\to 0]{} e^{-\frac{\pi^2 \mathrm{Tr}(R) L_{\mathcal{M}_3}}{12\beta}} \int d^r a \, e^{-V^{\mathrm{eff}}_{\mathcal{M}_3}(a)+\dots}$$

"potential" for the holonomies

Result

$$V^{\text{eff}}_{\mathcal{M}_3}(a) = -\sum_f \sum_{\rho_f \in \mathfrak{R}_f} \left[\frac{\pi^3 i A_{\mathcal{M}_3}}{6\beta^2} \kappa(\rho_f \cdot a) + \frac{\pi^2 (R_f L_{\mathcal{M}_3} - \rho_f \cdot l_{\mathcal{M}_3})}{2\beta} \vartheta(\rho_f \cdot a) \right] ,$$

$$\kappa(x) \equiv \{x\}(1-\{x\})(1-2\{x\}) ,$$

$$\vartheta(x) \equiv \{x\}(1-\{x\}) .$$

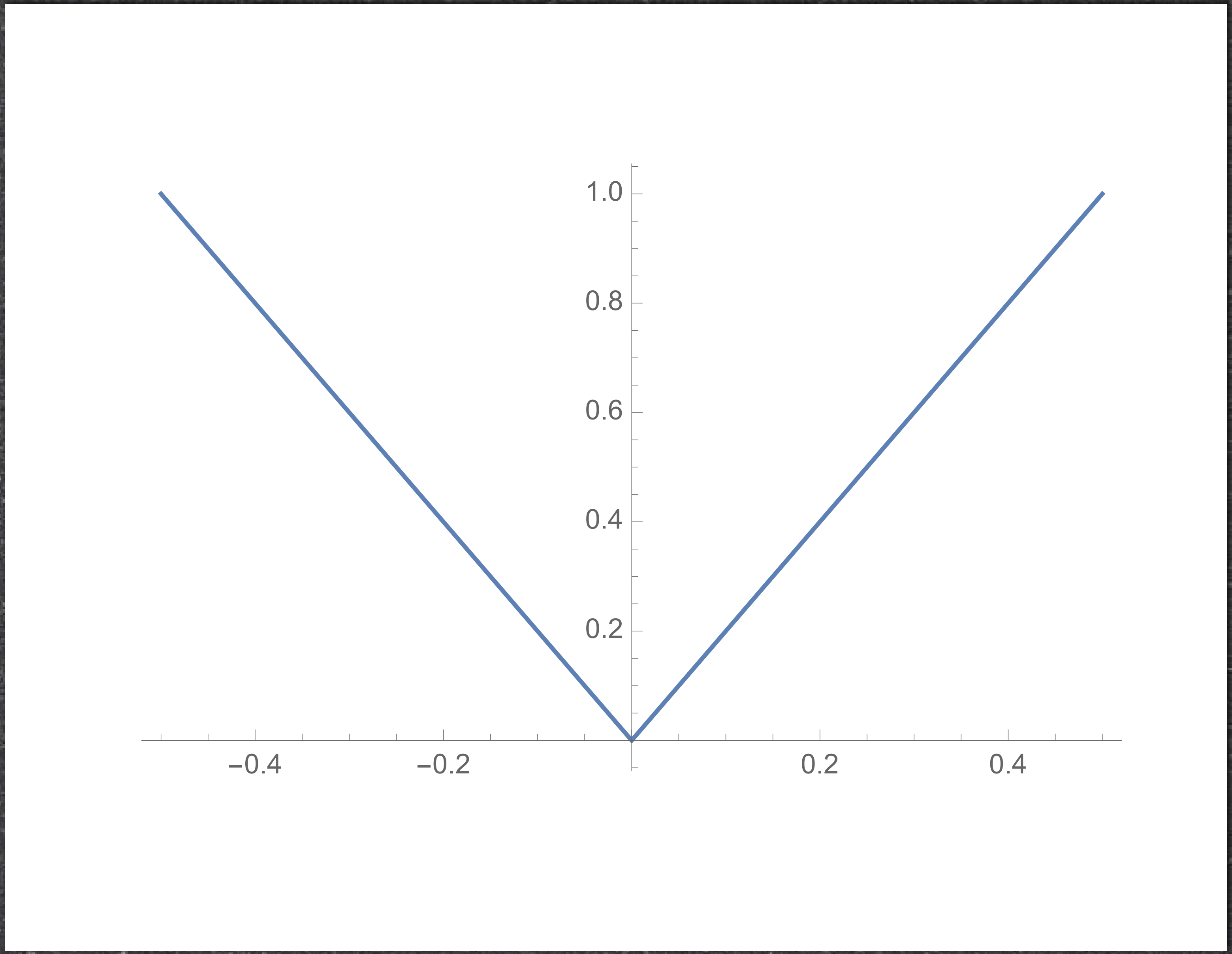
$A_{\mathcal{M}_3}, L_{\mathcal{M}_3},\ l^i_{\mathcal{M}_3}$: integrals of local densities in the background/gauge fields

Result

- It was derived by Ardehali for $\mathcal{M}_3 = S^3$ using expression of 1-loop determinants

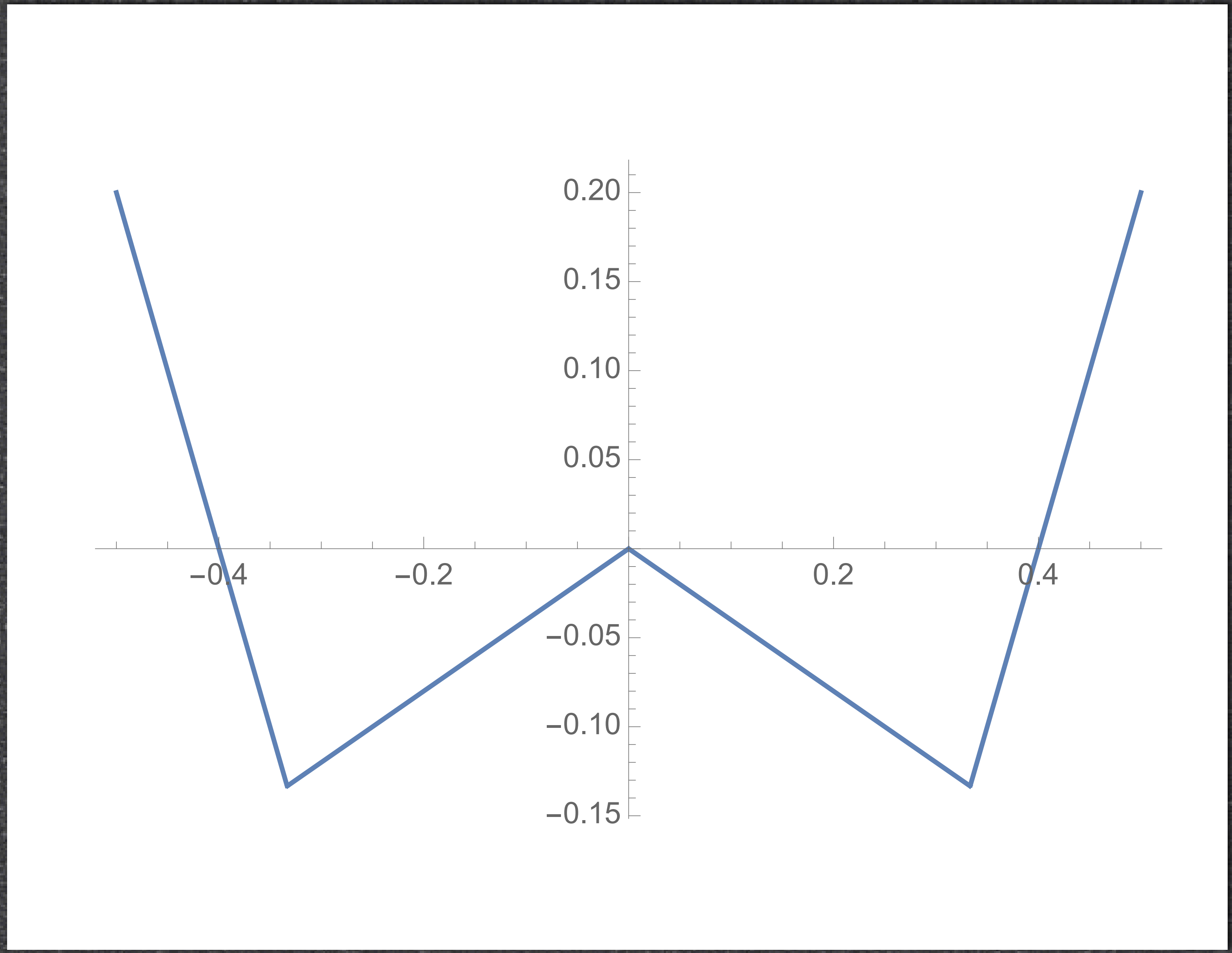
- Alternative approach (effective 3d theory) that is easy to generalize. No need to know full determinant in advance

Example



SU(2) with fundamental flavors

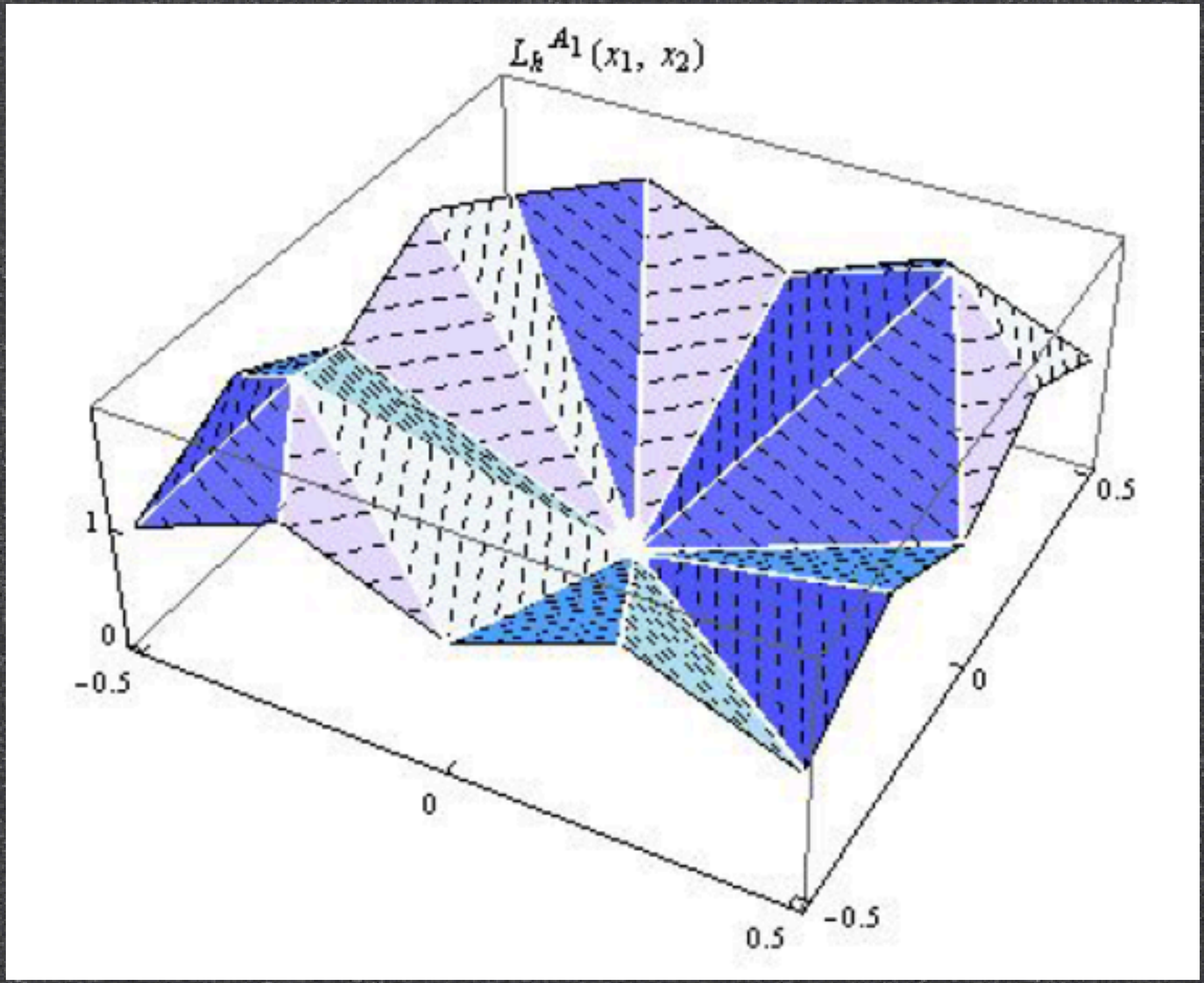
Example



ISS model: SU(2) + quadruplet

Intriligator, Seiberg, Shenker

Example



SU(3) SQCD (from Ardehali)

Result

- Limit of the integral?
- When $V^{\mathrm{eff}}_{\mathcal{M}_3}(a_{\min}) = 0$

$$Z_{S^1\times\mathcal{M}_3} \underset{\beta\to 0}{\longrightarrow} e^{-\frac{\pi^2\mathrm{Tr}(R)L_{\mathcal{M}_3}}{12\beta}+\cdots} \times Z_{\mathcal{M}_3}$$

Asymptotics fixed by gravitational anomalies

LD, Komargodski

Result

- What just said applies if $V^{\text{eff}}_{\mathcal{M}_3}(a_{\min}) = 0$
- Additional contribution for cases with $V^{\text{eff}}_{\mathcal{M}_3}(a_{\min}) < 0$
- Examples known so far: ‘misleading anomaly matching’, all have $a - c > 0$

Result

- Theories with $V^{\text{eff}}_{\mathcal{M}_3}(a_{\min}) < 0$
- In this case the leading asymptotics of the partition function rather than

$$Z_{S^1\times\mathcal{M}_3} \underset{\beta\to 0}{\longrightarrow} e^{-\frac{\pi^2\mathrm{Tr}(R)L_{\mathcal{M}_3}}{12\beta}+\cdots} \times Z_{\mathcal{M}_3}$$

is

$$Z_{S^1\times\mathcal{M}_3} \underset{\beta\to 0}{\longrightarrow} e^{-\frac{\pi^2\mathrm{Tr}(R)L_{\mathcal{M}_3}}{12\beta}+V^{\text{eff}}_{\mathcal{M}_3}(a_{\min})+\cdots} \times Y_{\mathcal{M}_3}$$

Physical interpretation?

Results

- $V^{\text{eff}}_{\mathcal{M}_3}(a_{\min}) < 0$: one-loop determinant of $Z_{\mathcal{M}_3}$ grows exp at infinity
- This falsifies an assumption in LD, Komargodski
- We lack a more physical characterization of theories with this behavior

Derivation

- Couple the theory to bkgd metric and G gauge field

$$g_{MN}dX^M dX^N = (dX^4 + c_\mu dx^\mu)^2 + h_{\mu\nu}dx^\mu dx^\nu$$
$$A_M dX^M = (dX^4 + c_\mu dx^\mu)A_4 + \mathcal{A}_\mu dx^\mu$$

They are part of supersymmetric multiplets

Derivation

- In a weakly coupled point: KK fermions on $\mathcal{M}_3$

$$M_n = \frac{2\pi}{\beta}(n + \rho_f \cdot a) \qquad\qquad a = \frac{\beta A_4}{2\pi}$$

G

KK

We take $\beta \to 0$ with a finite

Derivation

- Formula in three steps:

  1) Integrate out
  2) Supersymmetrize
  3) Evaluate on SUSY configuration

Derivation

- 1) Integrate out:

  Mixed Chern-Simons terms between gauge, R-symmetry, KK.

  Coefficient = infinite sum over KK tower, zeta-function regularization

$$S_k(s, \rho_f \cdot a) = \sum_{n \in \mathbb{Z}} \mathrm{sgn}(n + \rho_f \cdot a) n^{k-1} |n + \rho_f \cdot a|^{-s}$$

Derivation

- 2) Supersymmetrize:

Chern-Simons terms completed to full supersymmetric Lagrangians.

Key: these Lagrangian contains term that are activated on the background.

Derivation

- For instance, R-KK:

$$i\epsilon^{\mu\nu\rho}\mathcal{A}^{(R)\mu}\partial_\nu c_\rho - \frac{1}{2}f_{\mu\nu}^2 - \frac{1}{2}(A_4^{(R)})^2 + \frac{1}{4}R$$

Integrated over $\mathcal{M}_3$ defines the length $L_{\mathcal{M}_3}$.

Derivation

- 3) Evaluate on the SUSY configuration of gauge fields:

$$(\mathcal{A}^i_\mu, \sigma^i, \mathcal{D}^i) \quad \text{with} \quad \begin{cases} \sigma & \equiv A_4 \\ \mathcal{A}_\mu & \equiv A_\mu - A_4 c_\mu \\ \mathcal{D} & \equiv D - A_4 A_4^{(R)} \end{cases}$$

$$A^i_4 = \frac{2\pi}{\beta} a^i$$

A^i_μ, D^i depend on $\mathcal{M}_3$.

They enter the density $l^i_{\mathcal{M}_3}$.

Derivation

- For instance, plugging in the gauge-R:

$$\longrightarrow \frac{2\pi}{\beta}(\rho_f \cdot a)\times$$

$$(i\epsilon^{\mu\nu\rho}\mathcal{A}^{(R)\mu}\partial_\nu c_\rho - \frac{1}{2}f_{\mu\nu}^2 - \frac{1}{2}(A_4^{(R)})^2 + \frac{1}{4}R)$$

Derivation

- The a-dependent coefficients combine to give

$$V^{\text{eff}}_{\mathcal{M}_3}(a) = -\sum_f \sum_{\rho_f \in \mathfrak{R}_f} \left[\frac{\pi^3 i A_{\mathcal{M}_3}}{6\beta^2} \kappa(\rho_f \cdot a) + \frac{\pi^2 (R_f L_{\mathcal{M}_3} - \rho_f \cdot l_{\mathcal{M}_3})}{2\beta} \vartheta(\rho_f \cdot a) \right] ,$$

$$\kappa(x) \equiv \{x\}(1-\{x\})(1-2\{x\}) ,$$

$$\vartheta(x) \equiv \{x\}(1-\{x\}) .$$

Derivation

- Integrate/sum over the conf of the dynamical gauge fields

- Finally we get

$$Z_{S^1\times\mathcal{M}_3} \xrightarrow[\beta\to 0]{} e^{-\frac{\pi^2 \mathrm{Tr}(R) L_{\mathcal{M}_3}}{12\beta}} \int d^r a \, e^{-V^{\mathrm{eff}}_{\mathcal{M}_3}(a)+\ldots}$$

a^i-independent prefactor

Applications

- It matches with the limit of the 1-loop determinants on $\mathcal{M}_3 = S^3/\mathbb{Z}_N, S^1 \times \Sigma_g$

- More general backgrounds: $\mathcal{M}_3$ can also be a non-trivial S^1 fibration over a Riemann surface with $g > 1$

Closset, Kim, Willett

Applications

- Superconformal case

$$\mathrm{Tr}R \sim a - c$$

$$\sum_{\mathcal{O}} (-1)^F e^{-\beta(\Delta + \frac{R}{2})} \underset{\beta \to 0}{\simeq} e^{-\frac{16\pi^2}{3}(a-c)\frac{r_3}{\beta}}$$

Similar to Cardy formula in 2d CFT

$$Z_{T^2} = \sum_{\mathcal{O}} e^{-\beta\Delta} \underset{\beta \to 0}{\simeq} e^{\frac{\pi^2}{3} c \frac{r_1}{\beta}}$$

Applications

- Usually: (a, c) via correlators of the stress-energy tensor/anomalies
- Formula for $a - c$ via the sum over BPS operators

Ardehali, Liu, Szepietowski
Beccaria, Tseytlin

Applications

- Theories with extended SUSY N=2 in 4d e (2,0) in 6d
- With one condition on the fugacities, more SUSY: Schur Index
- Only 1 parameter left
- Related to the character of 2d chiral algebras

Beem, Lemos, Liendo, Peelaers, Rastelli

$$\mathcal{I}(p,q,y) = \mathrm{Tr}\left[(-1)^F p^{j_2-j_1+\frac{R}{2}} q^{j_2+j_1+\frac{R}{2}} y^f\right]$$

$$R = \frac{4}{3}\left(I_3 - \frac{r}{2}\right), f = I_3 + r$$

- 4d Schur Index $p = (pq)^{\frac{2}{3}} y \equiv e^{-\beta\omega}$
- Dependence on q drops
- We obtain $\log \mathcal{I}^{\mathrm{Schur}} \underset{\beta\to 0}{\sim} -\frac{8\pi^2}{\beta\omega}(a-c)$

Agrees with the character of the chiral algebra. Different info than central charge: $c^{2d} = -c^{4d}$

Ardehali; Rastelli

$$\mathcal{I}(p,q,t,y) = \mathrm{Tr}\left[(-1)^F p^{\mathcal{R}+h_1} q^{\mathcal{R}+h_2} t^{\mathcal{R}-h_3} y^f\right]$$

$$\mathcal{R} = \frac{R+r}{2}\,, \quad f = R-r$$

- 6d Schur Index $y\sqrt{\frac{qt}{p}} = 1\,,\ p \equiv e^{-\beta\omega}$
- Dependence on q, t drops
- We obtain $\log \mathcal{I}^{\mathrm{Schur}} \underset{\beta\to 0}{\sim} \frac{\pi\, \boldsymbol{r}_{\mathfrak{g}}}{12\,\omega}\left(\frac{2\pi}{\beta}\right)$

β^{-3} cancels. Agrees with the proposed correspondence to $\mathcal{W}_{\mathfrak{g}}$ chiral algebra.

Outlook

- Characterize theories with $V^{\text{eff}}_{\mathcal{M}_3}(a_{\min}) < 0$, sign of $a - c$?
- Physical interpretation of Cardy behavior for $V^{\text{eff}}_{\mathcal{M}_3}(a_{\min}) < 0$
- Relation to the bound on the gap for holographic theories?

$$\Delta^2_{\text{gap}} \leq \left| \frac{c}{c-a} \right|$$

Camanho, Edelstein, Maldacena, Zhiboedov

Thank you!