3d $\mathcal{N} = 2$ dualities with monopoles

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3d gauge theories admit Monopole operators, local disorder operators, creating a magnetic flux on a 2-sphere surrounding the insertion point.

What happen if we add monopole operators to the Lagrangian?

It's tricky, monopoles are not polynomial in the elementary fields.

Why do we care?

New non-trivial fixed point theories $\mathcal{T}_{\mathfrak{M}}$, new dualities.

How can we reach $\mathcal{T}_{\mathfrak{M}}$?

Conservative approach: go to the fixed point $\mathcal{T}_0$ of the theory without monopole deformations. If $\Delta[\mathfrak{M}] < 3$ (relevant def.) then we can turn it on and initiate a flow to $\mathcal{T}_{\mathfrak{M}}$. Other possibilities?

I will address some of these questions in the $\mathcal{N} = 2$ SQCD $U(N_c)$.

Consider the $U(N_c)$ SQCD with N_f flavors $Q, \tilde{Q}$ and $\mathcal{W} = 0$:

	$U(1)_R$	$U(1)_A$	$SU(N_f)_\ell$	$SU(N_f)_r$	$U(1)_T$
Q	R_Q	1	$\mathbf{N_f}$	$\mathbf{1}$	0
$\tilde{Q}$	R_Q	1	$\mathbf{1}$	$\overline{\mathbf{N_f}}$	0
M	$2R_Q$	2	$\mathbf{N_f}$	$\overline{\mathbf{N_f}}$	0
$\mathfrak{M}^+$	$(1-R_Q)N_f - N_c + 1$	$-N_f$	$\mathbf{1}$	$\mathbf{1}$	1
$\mathfrak{M}^-$	$(1-R_Q)N_f - N_c + 1$	$-N_f$	$\mathbf{1}$	$\mathbf{1}$	-1

Monopole charges under any Abelian symmetry are computed by

$$\delta Q(\mathfrak{M}) = -\frac{1}{2} \sum_{\text{fermions } \psi} Q(\psi) \left|\rho_\psi(\mathfrak{m})\right| ,$$

where the fermions ψ transform with ρ_ψ under the gauge group.

In particular the dimension $R[\mathfrak{M}^\pm]$ is given in terms of R_Q, in turned determined by F-maximization [Jafferis].

We want to consider the deformation

$$\mathcal{W}_{\text{mon}} = \mathfrak{M}^+ + \mathfrak{M}^-$$

which breaks both $U(1)_T$ and $U(1)_A$.

Conservative approach

$\mathcal{W}_{\text{mon}}$ is a good deformation in $\mathcal{T}_0$ if $\mathfrak{M}^{\pm}$ are relevant and above the unitarity bound:

$$1/2 < R[\mathfrak{M}^{\pm}]_{\mathcal{T}_0} < 2$$

this holds in a narrow window [Safdi-Klebanov-Lee],[Benini-Closset-Cremonesi]:

	$N_f = 1$	$N_f = 2$	$N_f = 3$	$N_f = 4$	$N_f = 5$	$N_f = 6$	$N_f = 7$
$N_c = 1$	$2/3$	1.18	1.69	2.19	2.69	3.20	3.70
$N_c = 2$		$1/2$	0.97	1.46	1.95	2.44	2.94
$N_c = 3$				0.78	1.24	1.72	2.20
$N_c = 4$					0.60	1.03	1.50
$N_c = 5$							0.84

For large N_c and N_f, if $N_f \lesssim 1.45\, N_c$, monopole operators become free decoupled fields ($R[\mathfrak{M}]_{\mathcal{T}_0} < 1/2$) and we cannot turn on $\mathcal{W}_{\text{mon}}$.

There alternative ways to reach $\mathcal{T}_{\mathfrak{M}}$. The first one is to from 4d, as a bonus we also get a dual description $\mathcal{T}'_{\mathfrak{M}}$.

$\mathcal{T}_{\mathfrak{M}}$ and its dual $\mathcal{T}'_{\mathfrak{M}}$ from 4d

The starting point is the 4d $\mathcal{N}=1$ Intriligator-Pouliot duality:

$$USP(2N_c),\ 2N_f \text{ flavors } Q_i \leftrightarrow USP(2N_f-2N_c-4),\ 2N_f \text{ flavors } q_i,\ \ \mathcal{W}=\sum_{a<b} M^{ab}q_a q_b$$

When this dual pair is compactified on $R^3 \times S^1$ KK monopoles contribute to the superpotential [Aharony-Hanany-Intriligator-Seiberg-Strassler], [Aharony-Razamat-Seiberg-Willett] and we obtain a 3d duality between:

$\mathcal{T}_1$: $USP(2N_c)$ with $2N_f$ flavors and $\mathcal{W}_1=\eta Y$

and

$\mathcal{T}_2$: $USP(2N_f-2N_c-4)$ with $2N_f$ flavors and $\mathcal{W}_2=\sum_{a<b} M^{ab}q_a q_b+\tilde{\eta}\tilde{Y}$

The localized S_b^3 partition functions with masses $(m_1,\cdots m_{2N_f})$ are equal:

$$Z_{\mathcal{T}_1}=Z_{\mathcal{T}_2}\,,$$

which follows from a non-trivial integral identity which holds when the masses satisfy: $\sum_{a=1}^{2N_f} m_a = iQ(N_f-N_c-1)$.

This is an effect of the ηY, $\tilde{\eta}\tilde{Y}$ terms which break the $U(1)_A$ symmetry.

$\mathcal{T}_{\mathfrak{M}}$ and its dual $\mathcal{T}'_{\mathfrak{M}}$ from 4d

The $\mathcal{T}_1 = \mathcal{T}_2$ duality reduces to the Aharony duality in a suitable real mass deformation [Aharony-Razamat-Seiberg-Willett].

Here take a different real mass deformation:

$$m_i \to m_i + s\,, \qquad m_{N_f+i} \equiv \tilde{m}_i \to \tilde{m}_i - s\,, \quad i = 1\,, \cdots, N_f\,, \quad s \to \infty$$

In the $s \to \infty$ limit the vec. mult. splits into a $U(N_c)$ part + extra massive parts, N_f flavors remain light and we find:

$\mathcal{T}_1$: $USP(2N_c)$, $2N_f$ flavors, $\mathcal{W}_1 = \eta Y \xrightarrow[s\to\infty]{} U(N_c)$, N_f flavors, $\mathcal{W} = \mathfrak{M}^+ + \mathfrak{M}^-$:$\mathcal{T}_{\mathfrak{M}}$

The monopole superpotential combines the original Y-monopole and the AHW potential for $USP(2N) \to U(N)$ and breaks $U(1)_T \times U(1)_A$.

Similarly on the dual side the limit yields:

$\mathcal{T}_2$: $USP(2N_f - 2N_c - 4)$, $2N_f$ flavors, $\mathcal{W}_2 = \sum_{a<b} M^{ab} q_a q_b + \tilde{\eta}\tilde{Y} \xrightarrow[s\to\infty]{}$
$U(N_f - N_c - 2)$, N_f flavors, $\mathcal{W} = \widehat{\mathfrak{M}}^+ + \widehat{\mathfrak{M}}^- + \sum_{a,b} M^{ab} q_a \tilde{q}_b$:$\mathcal{T}'_{\mathfrak{M}}$

At the level of the S_b^3 partition functions we find:

$$Z_{\mathcal{T}_1} \xrightarrow[s\to\infty]{} L_1 \times Z_{\mathcal{T}_{\mathfrak{M}}} = L_2 \times Z_{\mathcal{T}'_{\mathfrak{M}}} \xleftarrow[s\to\infty]{} Z_{\mathcal{T}_2}$$

$L_1 = L_2$ are the leading (divergent) contributions to electric and magnetic saddles that we are comparing.

The real mass parameters in $Z_{\mathcal{T}_{\mathfrak{M}}}$, $Z_{\mathcal{T}'_{\mathfrak{M}}}$ are consistent with $U(1)_T \times U(1)_A$ being broken: no FI parameters and constraint on the masses $\sum_{i,j=1}^{N_f}(m_i + \tilde{m}_j) = iQ(N_f - N_c - 1)$.

Hence we propose the following duality between:

$\mathcal{T}_{\mathfrak{M}}$: $U(N_c)$ with N_f flavors $Q, \tilde{Q}$ and $\mathcal{W} = \mathfrak{M}^+ + \mathfrak{M}^-$

and

$\mathcal{T}'_{\mathfrak{M}}$: $U(N_f - N_c - 2)$ with N_f flavors $q, \tilde{q}$ and $\mathcal{W} = \hat{\mathfrak{M}}^+ + \hat{\mathfrak{M}}^- + Mq\tilde{q}$

Electric theory: $U(N_c), N_f$ and $\mathcal{W} = \mathfrak{M}^+ + \mathfrak{M}^-$

Imposing that monopole operators have R-charge 2 we find

$$R[\mathfrak{M}^\pm] = N_f(1 - R_Q) - (N_c - 1) = 2 \to R_Q = 1 - \frac{N_c + 1}{N_f}.$$

The Coulomb branch is lifted, the Higgs branch contains N_f^2 mesons $Q_i \tilde{Q}_j$ of dimension $R[Q_i \tilde{Q}_j] = 2\left(1 - \frac{N_c+1}{N_f}\right)$ with $Rank[Q_i \tilde{Q}_j] \leq N_c$.

Mesons should be above their unitarity bound:

$$R[Q_i \tilde{Q}_j] = 2(1 - \frac{N_c + 1}{N_f}) > 1/2 \to N_f \geq \frac{4(N_c+1)}{3}.$$

Magnetic theory: $U(N_f - N_c - 2), N_f$ and $\mathcal{W} = \hat{\mathfrak{M}}^+ + \hat{\mathfrak{M}}^- + Mq\tilde{q}$

Imposing that monopole operators have charge 2 we find

$$R(\hat{\mathfrak{M}}^{\pm}) = 2 \leftrightarrow R_q = (N_c + 1)/N_f$$

There are N_f^2 singlets $R(M_{ij}) = 2(1 - R_q) = 2(1 - \frac{N_c+1}{N_f})$ which are mapped to mesons of the electric theory!

To see that $Rank[M_{ij}] \leq N_c$. Give rank r vev to M_{ij}, they give mass to r of the N_f flavors, so the theory flows to $U(N_f - N_c - 2)$ with $N_f - r$ flavors and

$$\mathcal{W} = M_{ij} Q_i \tilde{Q}_j + \mathfrak{M}^+ + \mathfrak{M}^-$$

Now take the Aharony dual which is $U(N_c + 2 - r)$ with $N_f - r$ flavors and

$$\mathcal{W} = O^+ \mathfrak{M}^+ + O^- \mathfrak{M}^- + O^+ + O^-$$

Now the F-terms of $O^{\pm}$ imply that both $\mathfrak{M}^+$ and $\mathfrak{M}^-$ must take a non-zero vev, which in a $U(k)$ theory is possible only if $k > 1$, so there are vacua only if $N_c + 2 - r > 1 \leftrightarrow r \leq N_c$.

Phase space

As we vary N_f and N_c we encounter various phases:

- $N_f \geq N_c + 3$, $\mathcal{T}_{\mathfrak{M}}$ is interacting with $\mathcal{T}'_{\mathfrak{M}}$ dual:
  - $N_f > \frac{4}{3}(N_c + 1)$, $\mathcal{T}_{\mathfrak{M}}$ is completely interacting.
  - $N_c + 3 \leq N_f \leq \frac{4}{3}(N_c + 1)$, $\mathcal{T}_{\mathfrak{M}}$ has an interacting SCFT and a decoupled free sector: mesons are below the unitarity bound and the cubic superpotential $M_{ab} q_a \tilde{q}_b$ is irrelevant so the singlets decouple.
- $N_f = N_c + 2$, $\mathcal{T}_{\mathfrak{M}}$ is a Wess-Zumino model with $\mathcal{W} = \det(M)$, irrelevant for $N_c \geq 3$.
- $N_f = N_c + 1$ deformed Higgs branch $\det(M) = 1$.
- $N_f \leq N_c$ there are no supersymmetric vacua.

Complex mass deformation: consistency check

Take $N_f \geq (N_c + 3)$. In the electric theory turn on $\mathcal{W}_m = m Q_{N_f} \tilde{Q}_{N_f}$.

Using the operator mp, on the magnetic side we have

$$\mathcal{W} = \widehat{\mathfrak{M}}^+ + \widehat{\mathfrak{M}}^- + \sum_{i,j}^{N_f} M_{i,j} q_i \tilde{q}_j + m M_{N_f,N_f}$$

the meson gets a vev:

$$\partial_{M_{N_f,N_f}} \mathcal{W} = q_{N_F} \tilde{q}_{N_F} + m = 0$$

and the theory is Higgsed down to $U(N_f - N_c - 3)$, $N_f - 1$ and

$$\mathcal{W} = \mathfrak{M}^+ + \mathfrak{M}^- + \sum_{i,j}^{N_F - 1} M_{ij} q_i \tilde{q}_j \,.$$

OK!

Complex mass deformations: the $N_f < N_c + 3$ region

Start with $N_f = N_c + 3$, a complex mass takes the electric theory to the line $U(N_c), N_f = N_c + 2$. On the magnetic side the $U(1)$ gauge group is completely Higgsed $\Rightarrow$ $(N_c + 2)^2$ singlets with $\mathcal{W} = \det(M)$.

Another complex mass takes the electric theory to $U(N_c), N_f = N_c + 1$. On the magnetic side we have $\mathcal{W} = \det(M) + mM_{N_c+2,N_c+2}$. From the F-terms we see that a dual is in terms of $\mathcal{W} = \lambda(\det M' - 1)$, which is a deformation of the classical Higgs branch.

Another complex mass takes the electric theory to $U(N_c), N_f = N_c$, on the magnetic side we have $\mathcal{W} = \lambda(\det M' - 1) + mM_{N_c+1,N_c+1}$. The F-terms lead to runaway behaviour.

Real mass deformation: consistency check

We start from our duality with N_f+2 and take the real mass limit $t \to \infty$

$$m_{N_f+1} \to m_{N_f+1} + t\,, \qquad \tilde{m}_{N_f+1} \to m_{N_f+1} - t$$
$$m_{N_f+2} \to m_{N_f+2} - t\,, \qquad \tilde{m}_{N_f+2} \to m_{N_f+2} + t$$

Define $\eta = 2m_{N_f+1} + 2m_{N_f+2}$ and $\xi = 2m_{N_f+1} - 2m_{N_f+2}$, then

$$\sum_{a,b=1}^{N_f} (m_a + \tilde{m}_b) + \eta = iQ(N_f - N_c + 1),$$

the constraint is lifted and the $U(1)_{top} \times U(1)_{axial}$ symmetry is restored. ξ becomes the FI parameter, the electric theory is $U(N_c), N_f$ and $\mathcal{W} = 0$.

On the magnetic side the singlet matrix reduces to a $N_f \times N_f$ block plus two extra singlets $S^{\pm}$ which couple linearly to the monopole operators. The magnetic theory indeed is $U(N_f - N_c)$ theory with N_f flavors and $\mathcal{W} = \sum_{a,b}^{N_f} M_{a,b} \tilde{q}_a q_b + \hat{\mathfrak{M}}_- S^+ + \hat{\mathfrak{M}}_+ S^-$.

Hence the $\mathcal{T}_{\mathfrak{M}} = \mathcal{T}'_{\mathfrak{M}}$ duality reduces to the Aharony duality!

Real mass deformation: SQCD with $\mathcal{W} = \mathfrak{M}^-$ and its dual

Start from $\mathcal{T}_{\mathfrak{M}} = \mathcal{T}'_{\mathfrak{M}}$ with $N_f + 1$ flavors and consider the limit:

$$m_{N_f+1} \to m_{N_f+1} + t\,, \qquad \tilde{m}_{N_f+1} \to m_{N_f+1} - t\,,$$

with $t \to \infty$. By defining $\eta = 2m_{N_f+1}$ the mass constraint becomes

$$\sum_{a,b=1}^{N_f} (m_a + \tilde{m}_b) + \eta = iQ(N_f - N_c)\ ,$$

and it is lifted since η is a free parameter. We obtain a duality between:

$\mathcal{T}_1$: $U(N_c)$ SQCD with N_f flavors , $\mathcal{W} = \mathfrak{M}^-$

and

$\mathcal{T}_2$: $U(N_f - N_c - 1)$ SQCD with N_f flavors, $N_f^2 + 1$ singlets ,
$\mathcal{W} = \widehat{\mathfrak{M}}^+ + \widehat{\mathfrak{M}}^- S^+ + \sum_{i,j=1}^{N_f} M^i{}_j \tilde{q}_i q^j$.

The superpotential $\mathfrak{M}^-$ breaks the $U(1)_T \times U(1)_A$ symmetry to the diagonal, indeed in the partition function the FI parameter and the axial mass are not independent.

Real mass deformations: Chern-Simons couplings

Giving a real mass to $2k$ charge -1 chirals, leads to the duality between:

$\mathcal{T}_A :\ U(N_c)_{\frac{k}{2}}$ with $(N_f, N_f - k)$ fund/antifund chirals , $\mathcal{W} = \mathfrak{M}^-$

and

$\mathcal{T}_B : U(N_f - N_c - 1)_{-\frac{k}{2}}$ with $(N_f, N_f - k)$ fund/antifund, $N_f(N_f - k)$ singlets,

$$\mathcal{W} = \widehat{\mathfrak{M}}^+ + \sum_i^{N_f} \sum_j^{N_f - k} M^i{}_j \tilde{q}_i q^j$$

which provide a generalization with monopole superpotential of the dualities discussed in [Benini-Closset-Cremonesi].

3D completion for $N_f \leq 3N_c + 2$

Start from $\mathcal{T}_0$, $N_f = 3N_c + 2$ plus N_f^2 chirals Φ_{ij} with $\mathcal{W} = \Phi_{ij}^3$.

Couple Φ_{ij} to $\mathcal{T}_0$ by $\mathcal{W} = \sum_{ij=1}^{N_f} \left(\Phi_{ij} Q_i \tilde{Q}_j + \Phi_{ij}^3 \right)$. Two possibilities:

- Flow-1: we reach $\mathcal{T}_{\Phi_{ij} Q_i \tilde{Q}_j + \Phi_{ij}^3}$ with $R[Q] = R[\Phi_{ij}] = 2/3$ and $R[\mathfrak{M}^{\pm}] = N_f \big(1 - R[Q]\big) - N_c + 1 = \frac{5}{3} < 2$.
- Flow-2: we reach $\mathcal{T}_{\Phi_{ij} Q_i \tilde{Q}_j}$ with $R[\Phi_{ij}] > 2/3$.

Flow-1 violates the F-theorem with few Φ_{ij} and theory follows Flow-2.

With enough chirals the theory follows Flow-1.

We add $\mathfrak{M}^+ + \mathfrak{M}^-$, this sets $R[Q]_{\mathcal{T}} = \frac{2N_c+1}{3N_c+2}$ and $R[\Phi_{ij}] = 2 - 2R[Q] > 2/3$, the N_f^2 terms Φ_{ij}^3 must be dropped.

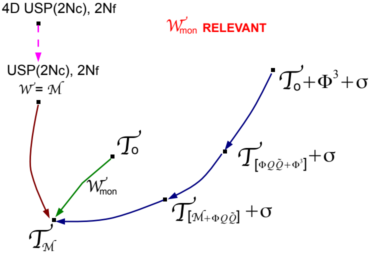
Finally we add N_f^2 extra chirals σ_{ij} and couple them linearly to the Φ_{ij}'s. Both Φ_{ij} and σ_{ij} become massive and integrating them out we get $\mathcal{T}_{\mathfrak{M}}$.

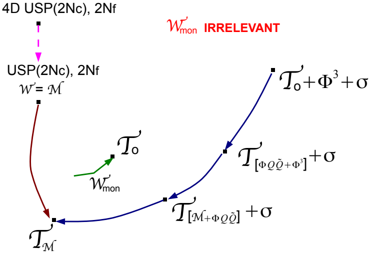
For $N_f = 3N_c + 2$ we can reach $\mathcal{T}_{\mathfrak{M}}$ by a chain of 3D RG flows:

$$\mathcal{T}_0 \oplus \mathcal{T}_{\Phi^3} \oplus \sigma \ \rightarrow\ \mathcal{T}_{\Phi q \tilde{q} + \Phi^3} \oplus \sigma \ \rightarrow\ \mathcal{T}_{\mathfrak{M} + \Phi q \tilde{q}} \oplus \sigma \ \rightarrow\ \mathcal{T}_{\mathfrak{M}}$$

$\rightarrow$Lower N_f via complex mass deformations.

Possible RG flows:





Summary and outlook

- A monopole deformed theory $\mathcal{T}_{\mathfrak{M}}$ can exist even if $\mathcal{W}_{mon}$ is irrelevant at $\mathcal{T}_0$ and admit a dual $\mathcal{T}'_{\mathfrak{M}}$.

  What about the non-susy case? 3D non-susy dualities with monopoles?

- For $U(N_c-1)$, $N_f=2N_c$, $\mathcal{T}_{\mathfrak{M}}$ is self dual and describes the duality wall for 4d $\mathcal{N}=2$ $SU(N)$ with N_f flavors (AGT dual of the Toda Braiding Kernel) [Le Floch].

- The $\mathcal{T}_{\mathfrak{M}}=\mathcal{T}'_{\mathfrak{M}}$ duality have been recently re-derived and generalized with a brane construction [Amariti-Orlando-Reffert].

- Duality $U(N_c)$ with N_f flavors and $\mathcal{W}=\mathfrak{M}^-$ crucial for the "sequential confinement" observed by [Benvenuti-Giacomelli].

- For $N_c=1$ and $N_f=3$, $\mathcal{T}_{\mathfrak{M}}$ and its dual are part of a web of dual abelian theories describing the low energy dynamics of D3 branes ending on pq-webs $\rightarrow$

Basic abelian mirror symmetry

The basic abelian $\mathcal{N}=2$ mirror duality relates

$$U(1)_{q,\tilde{q}}, \mathcal{W}=0 \leftrightarrow \{x,y,z\}, \mathcal{W}=xyz\,.$$

On the LHS the gauge invariant operators are the meson $M=q\tilde{q}$ and two monopoles $\mathfrak{M}^{\pm}$ which are mapped into the 3 singlets:

$$(M, \mathfrak{M}^{+}, \mathfrak{M}^{-}) \leftrightarrow (x,y,z)$$

The $\mathcal{N}=4$ version gives:

$$U(1)_{q,\tilde{q}}, \mathcal{W}=q\tilde{q}\Phi \leftrightarrow \{p,\tilde{p}\}, \mathcal{W}=0\,.$$

The map is now:

$$(\mathfrak{M}^{+}, \mathfrak{M}^{-}) \leftrightarrow (y,z)=(p,\tilde{p})\,.$$

Indeed $R[\mathfrak{M}^{\pm}]=1/2$ and are identified to a free twisted hyper [Borokhov-Kapustin-Wu'02].

Mirror symmetry as a Functional Fourier Transform

It was noticed [Kapustin-Strassler'99] that if we think of partition functions as functionals of background vector multiplets $\hat{V}$:

$$Z_{SQED}[\hat{V}] = \int DV \; DQ \; e^{\frac{S_V(V)}{g^2} + S_{BF}(V,\hat{V}) + S_H(V,Q)}\,,$$

$$Z_H(\hat{V}) = \int DQ \; e^{S_H(\hat{V},Q)}\,,$$

in the IR the statement of abelian $\mathcal{N} = 4$ mirror symmetry becomes:

$$Z_{SQED}[\hat{V}] = \int DV \; e^{S_{BF}(V,\hat{V})} \int DQ \; e^{S_H(V,Q)} = \int DQ \; e^{S_H(\hat{V},Q)} = Z_H(\hat{V})\,.$$

$\rightarrow$ The free hyper coincides with its Functional Fourier Transform (FFT):

$$\int DV \; e^{S_{BF}(V,\hat{V})} \, Z_H(V) = Z_H(\hat{V})$$

We can piecewise-generate $\mathcal{N} = 4$ and $\mathcal{N} = 2$ abelian mirror duals.

Mirror symmetry as an ordinary Fourier Transform

The localized S^3 partition functions become matrix integrals functions of real masses associated to background vector multiplets of the form $\hat{V} \sim m\theta\bar{\theta}$ [Kapustin-Willett-Yaakov'09].

The 1-loop contribution of an $\mathcal{N} = 4$ hypermultiplet is:

$$Z_H(\sigma) = \frac{1}{\cosh(\sigma)}.$$

And the the basic abelian $\mathcal{N} = 4$ mirror symmetry becomes

$$Z_{SQED}(\xi) = \int d\sigma \; e^{-2\pi i \sigma \xi} \frac{1}{\cosh(\sigma)} = \frac{1}{\cosh(\xi)} = Z_H(\xi)\,.$$

By dualizing all the fields with this trick one can generate mirror duals where the Coulomb and Higgs branches are swapped.

In the $\mathcal{N}=2$ case, with anomalous R-charges on the squashed S^3_b [Jafferis],[Hama-Hosomichi-Lee], the 1-loop contribution of a chiral with real mass m' for a $U(1)$ symmetry and R-charge r is:

$$s_b(\tfrac{Q}{2}(1-r)-m') \quad with \quad s_b(x)=\prod_{m,n\geq 0}\frac{mb+nb^{-1}+\frac{Q}{2}-ix}{mb+nb^{-1}+\frac{Q}{2}+ix}\,, \qquad Q=b+1/b\,.$$

We also introduce:

$$F_m(x)\equiv s_b(x+\frac{m}{2}+i\frac{Q}{4})s_b(-x+\frac{m}{2}+i\frac{Q}{4})$$

the contribution of an hyper with 'axial' mass m in which we absorbed r.

The basic abelian $\mathcal{N}=2$ mirror symmetry becomes the *pentagon* identity or 2-3 *Pachner* move:

$$Z_{SQED}=\int ds\; e^{-2\pi i s\xi}F_m(s)=s_b(m)F_{-m}(\xi)=Z_{xyz}\,.$$

A gauge theory triality

We now present a triality relating the following theories:

- $\mathcal{T}_A$: $U(1)^2$ quiver with 6 chiral fields $(A, \tilde{A}, Q, \tilde{Q}, P, \tilde{P})$ and

$$\mathcal{W}_{\mathcal{T}_A} = APQ + \tilde{A}\tilde{P}\tilde{Q}$$

- $\mathcal{T}_B$: $U(1)$ with 3 flavors $p_i, \tilde{p}_i$ with

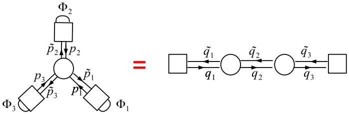
$$\mathcal{W}_{\mathcal{T}_B} = \sum_i^3 \Phi_i p_i \tilde{p}_i + \mathfrak{M}^+ + \mathfrak{M}^-$$

- $\mathcal{T}_C$: 6 chiral fields $(a, \tilde{a}, q, \tilde{q}, p, \tilde{p})$ with

$$\mathcal{W}_{\mathcal{T}_C} = apq + \tilde{a}\tilde{p}\tilde{q}$$

$\mathcal{T}_A = \mathcal{T}_B$

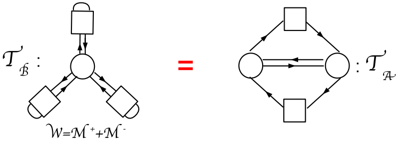
Consider the $U(1)$ theory with $N_f = 3$ flavors and $\mathcal{W} = \sum_{i=1}^{3} \Phi_i p_i \tilde{p}_i$ and its (piece-wise) mirror dual:



The fundamental monopoles of the electric theories are mapped into the long mesons:

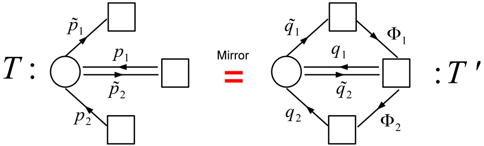
$$\mathfrak{M}^+ \leftrightarrow q_1 q_2 q_3 \,, \qquad \mathfrak{M}^- \leftrightarrow \tilde{q}_1 \tilde{q}_2 \tilde{q}_3 .$$

Now add $\mathcal{W} = q_1 q_2 q_3 + \tilde{q}_1 \tilde{q}_2 \tilde{q}_3$ on the RHS and $\mathcal{W} = \mathfrak{M}^+ + \mathfrak{M}^-$ and on the LHS. We find:



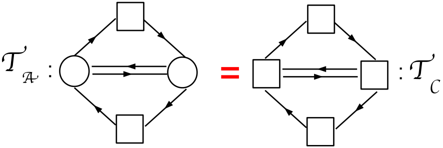
$\mathcal{T}_A = \mathcal{T}_C$

Consider the $U(1)$ theory with $N_f = 2$ flavors and $\mathcal{W} = 0$ and its (piece-wise) mirror dual:

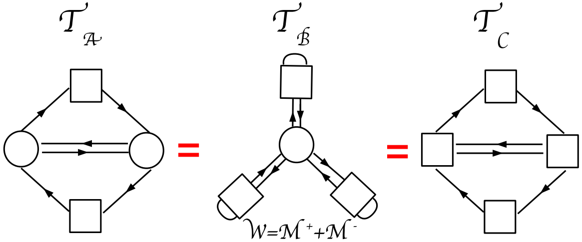


$$(p_1\tilde{p}_1, p_2\tilde{p}_2; p_1\tilde{p}_2, p_2\tilde{p}_1; \mathfrak{M}^+, \mathfrak{M}^-) \leftrightarrow (\Phi_1, \Phi_2; \mathfrak{M}^+, \mathfrak{M}^-; q_1\tilde{q}_2, q_2\tilde{q}_1)$$

- Gauge a symmetry Q acting on $(q_1, \tilde{q}_1, q_2, \tilde{q}_2, \Phi_1, \Phi_2)$ with charges $(1, 0, 0, -1, -1, 1)$ so that T' becomes $\mathcal{T}_A$.
- Q acts and on $(p_1, p_2, \tilde{p}_1, \tilde{p}_2)$ as $(0, 1, -1, 0)$. T becomes a $U(1)^2$ GLSM with charges $v_1 = (1, 1, -1, -1)$, $v_2 = (0, 1, -1, 0)$. Change basis to find two $U(1)_{q,\tilde{q}}, \mathcal{W} = 0$ or two XYZ models, that is $\mathcal{T}_C$.



Triality



Operator map at level $R = 2/3$:

$$\begin{pmatrix} & \mathfrak{M}^{1,0} & \mathfrak{M}^{1,1} \\ \mathfrak{M}^{-1,0} & & \mathfrak{M}^{0,1} \\ \mathfrak{M}^{-1,-1} & \mathfrak{M}^{0,-1} & \end{pmatrix} \leftrightarrow \begin{pmatrix} & p_1\tilde{p}_2 & p_1\tilde{p}_3 \\ p_2\tilde{p}_1 & & p_2\tilde{p}_3 \\ p_3\tilde{p}_1 & p_3\tilde{p}_2 & \end{pmatrix} \leftrightarrow \begin{pmatrix} & p & \tilde{a} \\ \tilde{p} & & q \\ a & \tilde{q} & \end{pmatrix}$$

Triality $Z_{\mathcal{T}_A} = Z_{\mathcal{T}_B} = Z_{\mathcal{T}_C}$

At the level of S_b^3 partition function the triality reads:

$$\begin{aligned} Z_{\mathcal{T}_A} =& \int dx_1 dx_2 \ e^{2\pi i(\xi_1 x_1 + \xi_2 x_2)} \ F_{2t}(x_2 - x_1) F_{2v-\phi}(x_1) F_{2v+\phi}(x_2) \\ &= s_b(2t) s_b(2v \pm \phi) \int ds \ F_{-2t}(s) F_{-2v+\phi}(s + \xi_1) F_{-2v-\phi}(\xi_2 - s) = Z_{\mathcal{T}_B} \\ &= F_{2t}(\xi_1 + \xi_2) F_{2v+\phi}(\xi_1) F_{2v-\phi}(\xi_2) = Z_{\mathcal{T}_C} \,. \end{aligned}$$

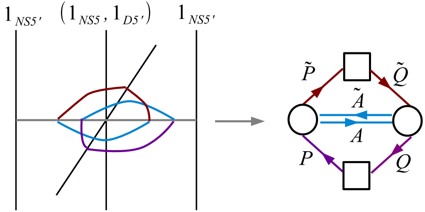
- $Z_{\mathcal{T}_A} = Z_{\mathcal{T}_B}$ follows from piecewise dualisation, that is from repeated use of the *pentagon* identity.
- $Z_{\mathcal{T}_B} = Z_{\mathcal{T}_C}$ is known as *new pentagon* (or *ultimate integral identity*).
- $Z_{\mathcal{T}_A} = Z_{\mathcal{T}_C}$ is a new interesting identity!

pq-webs

Consider a configurations of 5-branes and 3-branes in Type IIB [Brunner-Hanany-Karch-Lust], [Cremonesi].

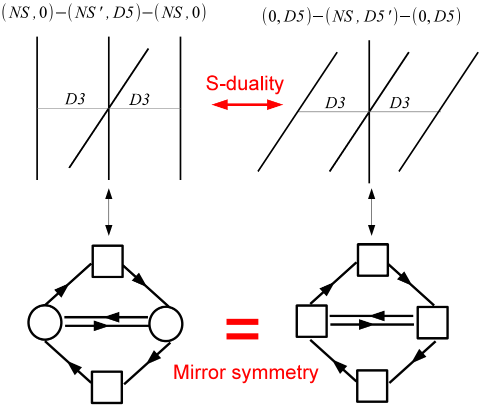
- NSs along 012345
- D5s along 012457
- D3s along 0126

The D5 branes are split by the NS'. We take D3 branes ending on each side of the pq-web and terminate them on two spectator NSs.
The low energy theory on the N D3 branes is a 3d $\mathcal{N}=2$ $U(N)^2$ theory:



If the spectator branes are $D5'$ the nodes are un-gauged.

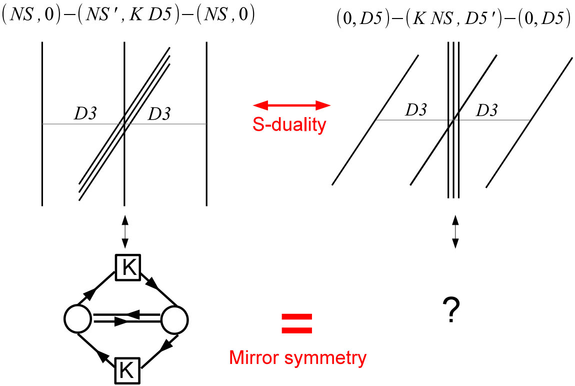
$\mathcal{T}_A = \mathcal{T}_C$ as pq-web S-duality



$$\int dx_1 dx_2 e^{2\pi i(\xi_1 x_1+\xi_2 x_2)} F_{2t}(x_2-x_1)F_{2v-\phi}(x_1)F_{2v+\phi}(x_2) = F_{2t}(\xi_1+\xi_2)F_{2v+\phi}(\xi_1)F_{2v-\phi}(\xi_2)\,.$$

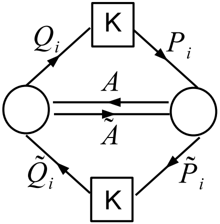
This observation suggests that the Fourier transform implements the rotation of spectator branes via S-transform [Gaiotto-Kim].

pq-webs $(NS', KD5)$ and $(KNS, D5')$



$\mathcal{T}_{A,K}$

The theory associated to the $(NS', KD5)$-web is $\mathcal{T}_{A,K}$:



$$R[Q_i] = R[\tilde{Q}_i] = R[P_i] = R[\tilde{P}_i] = r$$

$$R[A] = R[\tilde{A}] = 2 - 2r$$

$$W = \sum_i^K \left(Q_i A P_i + \tilde{Q}_i \tilde{A} \tilde{P}_i\right)$$

$$GLOBAL\ SYMM : SU(K)^2 \times U(1)^2 \times U(1)^2_{top}$$

The chiral ring of $\mathcal{T}_{A,K}$ is generated by

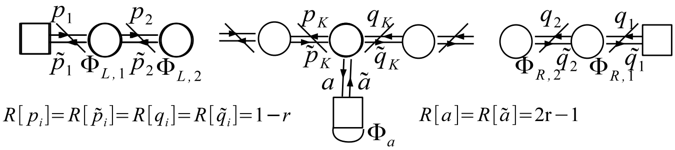
- $K^2 + K^2$ mesons $R[P_i \tilde{P}_j] = R[Q_i \tilde{Q}_j] = 2r$
- 1 meson $R[A\tilde{A}] = 2(2 - 2r)$
- 4 monopoles $R[\mathfrak{M}^{\pm 1,0}] = R[\mathfrak{M}^{0,\pm 1}] = K(1 - r) + 2r - 1$
- 2 monopoles $R[\mathfrak{M}^{\pm(1,1)}] = 2K(1 - r)$

Piece-wise dualize $\mathcal{T}_{A,K} \to \mathcal{T}_{B,K}$

We use the basic abelian duality $2K+1$ times:

$$\{P_i, \tilde{P}_i\}, \mathcal{W} = 0 \to U(1)_{p_i,\tilde{p}_i}, \mathcal{W} = p_i \tilde{p}_i \Phi_{L,i}\,,$$
$$\{Q_i, \tilde{Q}_i\}, \mathcal{W} = 0 \to U(1)_{q_i,\tilde{q}_i}, \mathcal{W} = q_i \tilde{q}_i \Phi_{R,i}\,,$$
$$\{A, \tilde{A}\}, \mathcal{W} = 0 \to U(1)_{a,\tilde{a}}, \mathcal{W} = a \tilde{a} \Phi_a\,.$$

Functional integration over the two original gauge $U(1)$'s yields functional delta. We can present the result as the linear quiver $\mathcal{T}_{B,K}$.

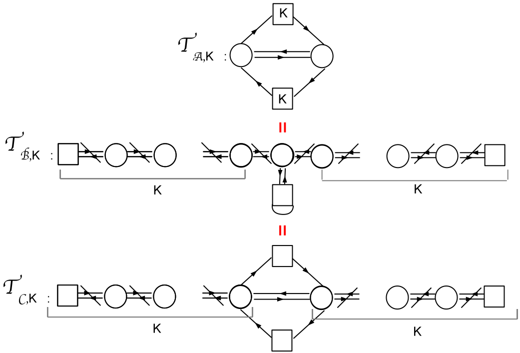


$$R[\mathfrak{M}^{\pm(0,\ldots,0,1,\ldots,1|1|1,\ldots,1,0,\ldots,0)}] = 2(1-(1-r)) + 1 - (2r-1) = 2\,,$$

these $2K$ terms contribute to the superpotential!

We can also replace $\mathcal{T}_B = \mathcal{T}_C$ in $\mathcal{T}_{B,K}$ and obtain a new thoery $\mathcal{T}_{C,K}$.

Summarising we have the following triality:



with

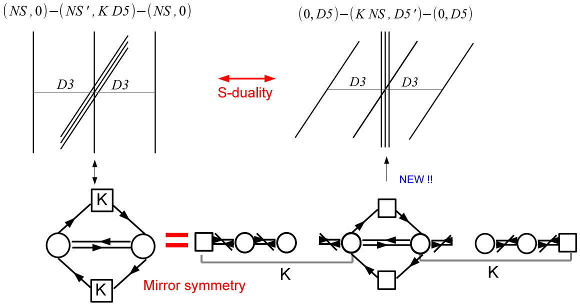
$$\mathcal{W}_{\mathcal{T}_{A,K}} = \sum_i^K (AP_iQ_i + \tilde{A}\tilde{P}_i\tilde{Q}_i)$$

$$\mathcal{W}_{\mathcal{T}_{B,K}} = \sum_i^K (q_i\tilde{q}_i\Phi_{i,L} + p_i\tilde{p}_i\Phi_{i,R}) + a\tilde{a}\Phi_a + \mathfrak{M}^{\pm(0,\dots,0|1|0,\dots,0)} + \mathfrak{M}^{\pm(0,\dots,1|1|1,\dots,0)} + \dots$$

$$\mathcal{W}_{\mathcal{T}_{C,K}} = \sum_i^{K-1} (q_i\tilde{q}_i\Phi_{i,L} + p_i\tilde{p}_i\Phi_{i,R}) + ABC + \tilde{A}\tilde{B}\tilde{C} + \mathfrak{M}^{\pm(0,\dots,1|1,\dots,0)} + \mathfrak{M}^{\pm(0,\dots,1,1|1,1,\dots,0)} + \dots$$

Of course $Z_{\mathcal{T}_{A,K}} = Z_{\mathcal{T}_{B,K}} = Z_{\mathcal{T}_{C,K}}$ and the operators match.

So we have found the low energy description of the S-dual pq-web:

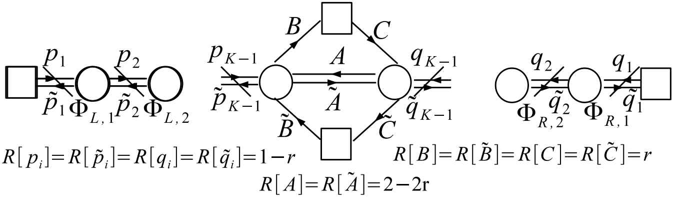


More general pq-webs can be obtain by suitable real mass deformations.

- Generalisation to the $(HNS', KD5)$-web case ?
- Add more D3 branes, non-abelian case?

Piece-wise mirror symmetry tells us that monopole superpotentials are necessary to describe these brane set-ups!

$\mathcal{T}_{C,K}$ and operator map



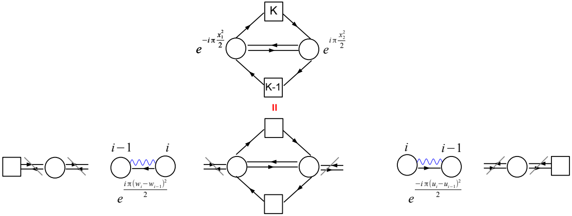
There are $2K-2$ monopoles with $R=2$ contributing to $\mathcal{W}$:

$$\mathcal{W}_{\mathcal{T}_{C,K}} = \mathcal{W}_{flip} + ABC + \tilde{A}\tilde{B}\tilde{C} + \mathfrak{M}^{\pm(0,\ldots,1|1,\ldots,0)} + \mathfrak{M}^{\pm(0,\ldots,1,1|1,1,\ldots,0)} + \cdots$$

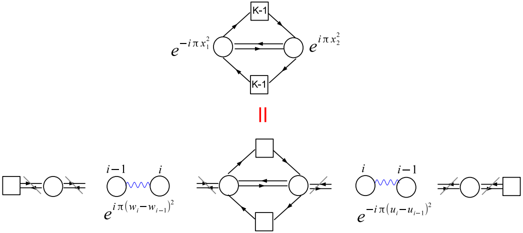
- LHS: $K-1$ singlets $\Phi_{i,L}$, meson $B\tilde{B}$, $K(K-1)$ monopoles $\mathfrak{M}^{(1,\ldots,0|0,\ldots,0)}, \mathfrak{M}^{(1,1\ldots,0|0,\ldots,0)}, \cdots$ and similarly on the RHS, all with $R=2r \leftrightarrow K^2+K^2$ mesons in $\mathcal{T}_{A,K}$.
- 4 mesons $R[\prod_i^{K-1} p_i B] = R[\prod_i^{K-1} \tilde{p}_i \tilde{B}] = R[C\prod_i^{K-1} q_i] = R[\tilde{C}\prod_i^{K-1} \tilde{q}_i] = (K-1)(1-r)+r \leftrightarrow$ monopoles $\mathfrak{M}^{\pm 1,0}, \mathfrak{M}^{0,\pm 1}$ in $\mathcal{T}_{A,K}$.
- 2 mesons $R[\tilde{A}\prod_i^{K-1} p_i\tilde{q}_i] = R[A\prod_i^{K-1} \tilde{p}_i q_i] = 2(K-1)(1-r)+2-2r \leftrightarrow$ monopoles $\mathfrak{M}^{\pm 1,1}$ in $\mathcal{T}_{A,K}$.
- 1 meson $R[A\tilde{A}] = 2(2-2r) \leftrightarrow$ meson in $\mathcal{T}_{A,K}$.

Real mass deformation

- Remove half D5 on one side:



- Remove half D5 on each side (in opposite directions):



Iterate and get D3 branes ending on more general pq-webs.