THE INFRARED PHYSICS OF BAD THEORIES

Stefano Cremonesi
(Durham University)

Geometry of String and Gauge Theories, CERN, 17/7/2017

Joint work with B. Assel (CERN) 1707.03403

3d YANG-MILLS / MAXWELL GAUGE THEORIES

The gauge coupling is dimensionful in 3d:

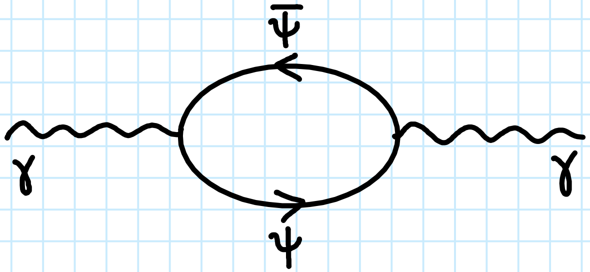
$[g^2] = M$. $\longrightarrow$ Dimensionless coupling $\frac{g^2}{\mu}$.

- <u>UV</u> ($\mu \to \infty$): superrenormalizable and weakly coupled. $\left(\frac{g^2}{\mu} \xrightarrow[\mu\to\infty]{} 0 \right)$
  - $\rightarrow$ Abelian gauge theories are well defined
  - $\rightarrow$ No upper bound on the number of matter fields

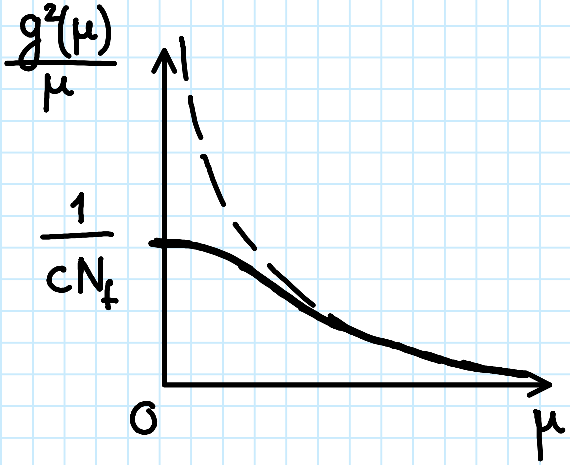
- <u>IR</u> ($\mu \to 0$): interesting dynamics, e.g.
  - $\rightarrow$ Interacting CFT (if enough matter)
  - $\rightarrow$ Mass gap / Confinement / SSB (not enough matter)

Non-perturbative corrections due to magnetic monopoles (instantons in 3d). [Polyakov '77]

EXAMPLE: 3d QED

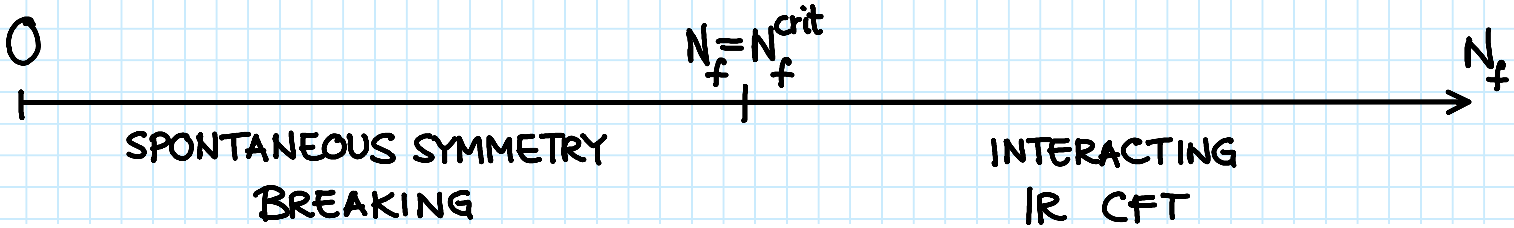


$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\infty)} + c \cdot \frac{N_f}{\mu} \qquad (c>0)$$



$$\frac{\mu}{g^2(\mu)} = \frac{\mu}{g^2(\infty)} + c N_f \xrightarrow[\mu \to 0]{} c N_f$$

$\underline{N_f \gg 1}$: $\underline{\text{Free}}$ UV fixed point $\xrightarrow[\text{flow}]{\text{RG}}$ $\underline{\text{Weakly coupled}}$ IR fixed point



[Pisarski '84, Appelquist et al '88]

3d $\mathcal{N}=4$ GAUGE THEORIES

- Lower dim cousins of 4d $\mathcal{N}=2$ gauge theories (& 6d $\mathcal{N}=(1,0)$)

  8 supercharges

  GAUGE GROUP G $\longrightarrow$ VECTOR MULTIPLET

  MATTER REPRESENTATION $\mathcal{R}$ $\longrightarrow$ HYPER MULTIPLET

- Rich moduli spaces of SUSY vacua:

  — HIGGS BRANCH $(\mathcal{H})$: classically exact (HYPERKÄHLER QUOTIENT)

  $\circlearrowright$ $SU(2)_H$

  — COULOMB BRANCH $(\mathcal{C})$: quantum corrected (HYPERKÄHLER)

  $\circlearrowright$ $SU(2)_C$

  (1-loop + non-pert.)

  [Seiberg, Witten '96; Dorey et al '97, '98]

  — MIXED BRANCHES

3d $\mathcal{N}=4$ GAUGE THEORIES

- Mirror symmetry (particle/vortex duality) [Intriligator, Seiberg '96]

$$\begin{tikzcd} \text{UV} & A_{UV} \arrow[dr] & & B_{UV} \arrow[dl] \\ \text{IR} & & A_{IR}=B_{IR} & \end{tikzcd}$$

$$\mathcal{H}^{A_{IR}} = \mathcal{C}^{B_{IR}}$$
$$\mathcal{C}^{A_{IR}} = \mathcal{H}^{B_{IR}}$$

- Recent exact results:

  1) Supersymmetric localization [Kapustin, Willett, Yaakov '09; ...]

  2) Algebraic description of the Coulomb branch

  [SC, Hanany, Zaffaroni '13; Nakajima '15; Bullimore, Dimofte, Gaiotto '15]

3d $\mathcal{N}=4$ GAUGE THEORIES: THE GOOD, THE BAD AND THE UGLY

[Gaiotto, Witten '08]

- Naive IR limit $\quad g^2(\mu) \longrightarrow \infty$: $\qquad \mathcal{N}=4$ SCFT ?
- Unitarity bound : $\qquad \Delta = R^{sc} \geq \frac{1}{2} \qquad$ for chiral primary operators of $\mathcal{N}=2$ SCA.

Assume $\quad R^{sc}_{IR} = R^{sc}_{UV} := R_C + R_H$. $\qquad$ Do monopole operators V_m obey the unitarity bound?

<u>"GOOD" THEORIES</u> : $\qquad R^{sc}_{UV}[V_m] > \frac{1}{2} \;\; \forall m$.

$\xrightarrow[RG]{}$ Interacting SCFT with $R^{sc}_{IR} = R^{sc}_{UV}$.

<u>"UGLY" THEORIES</u> : $\qquad R^{sc}_{UV}[V_m] \geq \frac{1}{2} \;\; \forall m$, $\quad = \frac{1}{2}$ for some m .

$\xrightarrow[RG]{}$ Interacting SCFT with $R^{sc}_{IR} = R^{sc}_{UV}$ $\quad$ + free fields

<u>"BAD" THEORIES</u> : $\qquad R^{sc}_{UV}[V_m] < \frac{1}{2} \quad (\leq 0) \qquad$ for some m .

$\xrightarrow[RG]{}$?

3d $\mathcal{N}=4$ "U(N), N_f" SQCD: THE GOOD, THE BAD AND THE UGLY

- Gauge group: $U(N)$. Matter: N_f hypers in fundamental rep.
- Monopole operators V_m. Magnetic charge $m=(m_1,\dots,m_N)\in\mathbb{Z}^N/S_N$.

$$\boxed{R^{sc}_{UV}[V_m]=\frac{1}{2}\Big(\underbrace{N_f\sum_a |m_a|}_{\text{HYPERS}}-\underbrace{\sum_{a\neq b}|m_a-m_b|}_{\text{W-BOSONS}}\Big)}$$

[Borokhov, Kapustin, Wu '02]
[Benna, Klebanov, Klose '09]
[Bashkirov, Kapustin '10] ...

$N_f\geq 2N$: "GOOD" $U(N),\ N_f\geq 2N \xrightarrow[RG]{}$ "$T_{U(N),N_f}$" interacting SCFT

$N_f=2N-1$: "UGLY" $U(N),\ N_f=2N-1 \xrightarrow[RG]{}$ Interacting SCFT + free fields $V_{(\pm1,0,\dots,0)}$

$N_f\leq 2N-2$: "BAD" $U(N),\ N_f\leq 2N-2 \xrightarrow[RG]{}$?

HINTS FROM SUPERSYMMETRIC LOCALIZATION

- SUSY partition function on S^3 : $Z_{U(N),N_f}$. [Kapustin, Willett, Yaakov '09]
  - GOOD/UGLY : $Z_{U(N),N_f}$ converges.
  - BAD : $Z_{U(N),N_f}$ diverges.

1. $$\boxed{\underbrace{Z_{U(N),2N-1}}_{\text{UGLY}} = \underbrace{Z_{U(N-1),2N-1}}_{\text{GOOD}} \cdot \underbrace{Z_{2 \text{ free chirals}}}_{\text{FREE}}}$$

$$U(N),\ 2N-1 \xrightarrow[\text{RG}]{} T_{U(N-1),2N-1} + (1 \text{ free twisted hyper}).$$ ✓

2. $$\boxed{\underbrace{Z_{U(N),\,N\leq N_f\leq 2N-2}}_{\text{BAD}}\Big|_{\text{reg.}} = \underbrace{Z_{U(N_f-N),\,N_f}}_{\text{GOOD}} \cdot \underbrace{Z_{2(2N-N_f) \text{ free chirals}}}_{\text{FREE}}}$$

SEIBERG DUALITY CONJECTURE [Yaakov '13] :

$$U(N),\ N\leq N_f\leq 2N-2 \xrightarrow[\text{RG}]{} T_{U(N_f-N),N_f} + (2N-N_f \text{ free twisted hypers})\ ?$$

We will see that:

① is an exact IR equivalence.

② is not an exact IR equivalence: RHS is a local IR effective description of LHS near a certain point of its moduli space.

$\boxed{\text{MAIN THEME: IR EFFECTIVE PHYSICS from LOCAL GEOMETRY OF MODULI SPACE.}}$

PLAN:

1) Moduli space of vacua of $U(N)$, N_f SQCD
2) Singular loci of CB, roots of HB's, and IR SCFT's
3) Remarks on "Seiberg duality" conjecture

MODULI SPACE OF VACUA OF $U(N), N_f$ SQCD

CLASSICAL MODULI SPACE OF VACUA

- HYPERS $\supset$ $H = \begin{pmatrix} Q \\ \tilde{Q}^\dagger \end{pmatrix}$ $SU(2)_H$ doublet

$$Q^a{}_\alpha \in \mathbb{C}^{N \times N_f}$$
$$\tilde{Q}^\alpha{}_a \in \mathbb{C}^{N_f \times N}$$

- VECTORS $\supset$ A_μ, $\underbrace{(\phi^1, \phi^2, \phi^3)}_{SU(2)_c \text{ triplet}}$

$$\phi^i \in \mathfrak{u}(N)$$

- SCALAR POTENTIAL:

$$V = \frac{g^2}{2} \operatorname{Tr} \sum_{\hat{i}=1}^{3} \left(\mu_{\hat{i}}(H, H^\dagger)\right)^2 + \frac{1}{2g^2} \operatorname{Tr} \sum_{j<k} \left(i[\phi^j, \phi^k]\right)^2 + \sum_{i=1}^{3} \left\| (\phi^i \otimes \mathbb{1}_2) H \right\|^2$$

- VACUUM EQUATIONS:

$$\mu_{\hat{i}} := \operatorname{tr}_2 (H H^\dagger \sigma_{\hat{i}}) = 0 \qquad [\phi^j, \phi^k] = 0 \qquad (\phi^i \otimes \mathbb{1}_2) H = 0$$
$$\hat{i} = 1,2,3 \qquad j,k = 1,2,3 \qquad i = 1,2,3$$

CLASSICAL MODULI SPACE OF VACUA

- VACUUM EQUATIONS:

$$\mu_{\hat{i}} := \mathrm{tr}_2\left(HH^\dagger \sigma_{\hat{i}}\right) = 0 \qquad [\phi^j, \phi^k] = 0 \qquad (\phi^i \otimes \mathbb{1}_2) H = 0$$

$\hat{i} = 1,2,3$ $\qquad$ $j,k = 1,2,3$ $\qquad$ $i = 1,2,3$

- CLASSICAL MODULI SPACE:

$$\boxed{\mathcal{M} = \bigcup_{\ell} \left(C_{N-\ell} \times \mathcal{H}_\ell \right)}$$

Union of MIXED BRANCHES

$$\langle \phi^i \rangle = \mathrm{diag}\left(\phi^i_1, \dots, \phi^i_{N-\ell}, 0, \dots, 0\right)$$

"Coulomb" component

$$\langle Q \rangle = \begin{pmatrix} O_{(N-\ell)\times N_f} \\ \hline Q_{\ell \times N_f} \end{pmatrix}, \quad \langle \tilde{Q}^\dagger \rangle = \begin{pmatrix} O_{(N-\ell)\times N_f} \\ \hline \tilde{Q}^\dagger_{\ell \times N_f} \end{pmatrix}$$

"Higgs" component

HIGGS BRANCH

- $\langle$Vector multiplet scalars$\rangle = 0$.
- Classically exact. [Intriligator, Seiberg '96]

- $\underline{\mathcal{N}=4 \text{ language}}$: HYPERKÄHLER QUOTIENT [Hitchin, Karlhede, Lindström, Roček '87]

$$\mathcal{H}[U(N), N_f] = \mathbb{H}^{N\times N_f} //// \, U(N) = \vec{\mu}^{-1}(\vec{0}) / U(N)$$

- $\underline{\mathcal{N}=2 \text{ language}}$: cplx algebraic variety

$$\mathcal{H}[U(N), N_f] = \left\{ Q \in \mathbb{C}^{N\times N_f}, \, \tilde{Q} \in \mathbb{C}^{N_f \times N} \,\middle|\, Q\tilde{Q} = 0 \,\wedge\, QQ^\dagger - \tilde{Q}^\dagger \tilde{Q} = 0 \right\} / U(N)$$

$$\cong \left\{ Q \in \mathbb{C}^{N\times N_f}, \, \tilde{Q} \in \mathbb{C}^{N_f \times N} \,\middle|\, Q\tilde{Q} = 0 \right\} / GL(N, \mathbb{C})$$

$$\cong \left\{ M := \tilde{Q} Q \in \mathbb{C}^{N_f \times N_f} \,\middle|\, M^2 = 0 \,, \, \mathrm{rk}(M) \leq N \right\}$$

TO BE IMPROVED

$$\mathcal{H}[U(N),N_f] \cong \left\{ M := \tilde{Q}Q \in \mathbb{C}^{N_f\times N_f} \;\middle|\; M^2=0\,,\ \mathrm{rk}(M) \le \underset{\underline{\ell_{max}(N,N_f)}}{\underset{\shortparallel}{\boxed{\min\left(N,\lfloor N_f/2\rfloor\right)}}} \right\}$$

$$\equiv \bigcup_{\ell=0}^{\ell_{max}} \mathcal{H}_\ell \,,$$

where $\quad \mathcal{H}_\ell \cong \left\{ M := \tilde{Q}Q \in \mathbb{C}^{N_f\times N_f} \;\middle|\; M^2=0\,,\ \mathrm{rk}(M) = \ell \right\}$. $\quad$ On $\mathcal{H}_\ell$:

$$Q = \begin{pmatrix} 0 & k_1 & & & & & & \\ & & 0 & k_2 & & & & \\ & & & & \ddots & & & 0 \\ & & & & & 0 & k_\ell & \\ & & & 0 & & & & 0 \end{pmatrix} \qquad \tilde{Q}^\dagger = \begin{array}{cc} \begin{pmatrix} k_1 & 0 & & & & & & \\ & & k_2 & 0 & & & & \\ & & & & \ddots & & & 0 \\ & & & & & k_\ell & 0 & \\ & & & 0 & & & & 0 \end{pmatrix} & N \\ \longleftarrow N_f \longrightarrow & \end{array}$$

- GAUGE: $U(N) \longrightarrow U(N-\ell)$
- FLAVOUR: $SU(N_f) \longrightarrow SU(N_f-2\ell)$

$$\Rightarrow \quad M = \begin{pmatrix} 0 & k_1^2 & & & & & & & \\ 0 & 0 & & & & & & & \\ & & 0 & k_2^2 & & & & & \\ & & 0 & 0 & & & & & 0 \\ & & & & \ddots & & & & \\ & & & & & 0 & k_\ell^2 & & \\ & & & & & 0 & 0 & & \\ & & & 0 & & & & & 0 \end{pmatrix}$$

Nilpotent orbit $(2^\ell, 1^{N_f-2\ell})$ of $SL(N_f,\mathbb{C})$

$\dim_{\mathbb{H}} \mathcal{H}_\ell = \ell(N_f-\ell)$ $\quad$ massless hypers

CLASSICAL COULOMB BRANCH

- $\langle$ Hyper multiplet scalars $\rangle = 0$.

- **$\mathcal{N}=4$ language:** $\langle \phi^i \rangle = \mathrm{diag}(\phi^i_1, \ldots, \phi^i_N)$ GAUGE: $U(N) \longrightarrow U(1)^N$

$U(1)^N$ photons A_a $\xleftrightarrow[\frac{2\pi}{g^2} * dA_a = d\gamma_a]{}$ Periodic scalars $\gamma_a \sim \gamma_a + 2\pi$ ("Dual photons")

$$\Rightarrow \mathcal{C}[U(N), N_f] \approx (\underbrace{\mathbb{R}^3}_{\phi^i_a} \times \underbrace{S^1}_{\gamma_a})^N / S_N$$

- **$\mathcal{N}=2$ language:** cplx algebraic variety

$$\mathcal{C}[U(N), N_f] \approx (\mathbb{C} \times \mathbb{C}^*)^N / S_N$$

$\varphi_a = \phi^1_a + i\phi^2_a$ $\qquad u^\pm_a = e^{\pm(\phi^3_a/g^2 + i\gamma_a)}$ s.t. $\boxed{u^+_a u^-_a = 1}$.

(Abelian monopole operators)

CLASSICAL MODULI SPACE : SUMMARY

The classical moduli space of vacua of 3d $\mathcal{N}=4$ $U(N), N_f$ SQCD is a union of mixed branches

$$\boxed{\mathcal{M}[U(N), N_f] = \bigcup_{\ell=0}^{\ell_{max}(N,N_f)} \left(\mathcal{C}_{N-\ell} \times \mathcal{H}_{\ell} \right)} \quad , \quad \ell_{max}(N, N_f) = \min\left(N, \lfloor N_f/2 \rfloor\right)$$

on which the gauge symmetry is broken $U(N) \longrightarrow U(1)^{N-\ell}$

flavour symmetry " $SU(N_f) \longrightarrow SU(N_f - 2\ell)$.

- NOTE on $\ell = \ell_{max}$:
  - GOOD THEORIES: $\ell_{max} = N$.

    Gauge group COMPLETELY HIGGSED on $\mathcal{C}_0 \times \mathcal{H}_N$ (PURE HIGGS BRANCH).
  - UGLY / BAD THEORIES: $\ell_{max} = \lfloor N_f/2 \rfloor < N$.

    Gauge group PARTIALLY HIGGSED on $\mathcal{C}_{N-\ell_{max}} \times \mathcal{H}_{\ell_{max}}$ (MIXED BRANCH).

QUANTUM CORRECTED COULOMB BRANCH

- $\mathcal{C}_{\text{class}}[U(N), N_f] \approx (\mathbb{C} \times \mathbb{C}^*)^N / S_N$ is inadequate if matter hypers / W-bosons are light.

- HK metric on $\mathcal{C}$: 1-loop corrections + possibly non-pert. corrections (BAD theories).

- [Bullimore, Dimofte, Gaiotto '15]:

The cplx algebraic description of $\mathcal{C}$ ($\mathcal{N}=2$ language) has no non-pert. corrections. $\mathcal{C}$ is parametrized by Weyl invariant polynomials of $(\varphi_a, u_a^\pm)$. Relations follow from

QUANTUM CORRECTED ABELIANIZED RELATIONS

$$\boxed{u_a^+ u_a^- = \frac{\prod_{\alpha=1}^{N_f} (\varphi_a - m_\alpha)}{\prod_{b \neq a} (\varphi_b - \varphi_a)^2}} \quad \overset{N}{\underset{a=1}{\forall}}$$

$\leftarrow$ HYPERS

$\leftarrow$ W-BOSONS

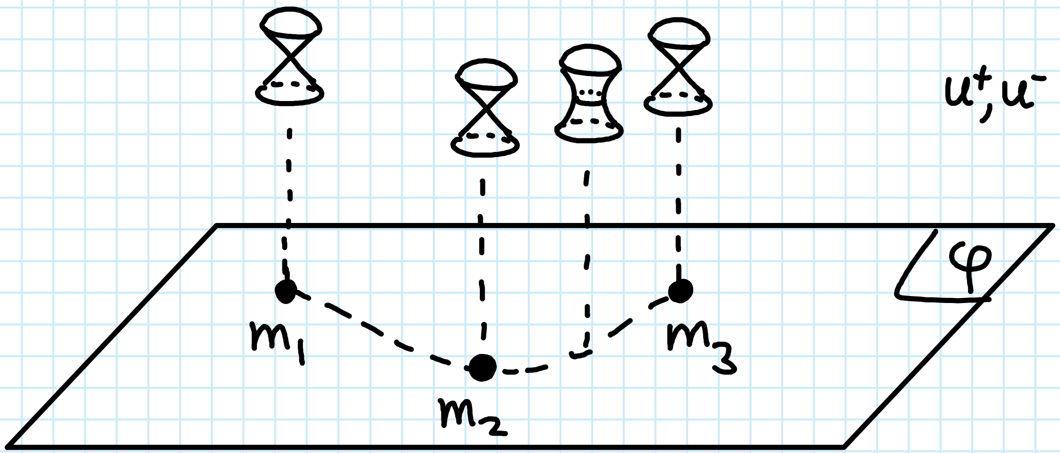
which are valid outside the discriminant locus where W-bosons have zero cplx mass.

EXAMPLE: 3d $\mathcal{N}=4$ SQED

- 1-loop exact metric [Intriligator, Seiberg '96]: multi-Taub-NUT. [Gibbons, Hawking '78]

- Algebraically, $\mathcal{C}[U(1), N_f] = \widehat{\mathbb{C}^2/\mathbb{Z}_{N_f}}$

$$\boxed{u^+u^- = \prod_{\alpha=1}^{N_f}(\varphi - m_\alpha)}$$



- Dual photon γ shrinks if there are massless hypers. [Dorey, Tong '99]

- Singularity on Coulomb branch (if $m_\alpha = m_\beta$) $\longleftrightarrow$ Root of Higgs branch.

COULOMB BRANCH : GAUGE INVARIANT DESCRIPTION

Rewrite abelianized relations so that they extend to the discriminant locus:

$$\left[u_a^+ \prod_{b\neq a}(\varphi_a-\varphi_b)\right]\cdot\left[u_a^- \prod_{b\neq a}(\varphi_a-\varphi_b)\right] - \left[\prod_{\alpha=1}^{N_f}(\varphi_a-m_\alpha)\right] = 0 \qquad \forall_{a=1}^{N} \; .$$

$$\Rightarrow \underbrace{\left[\sum_{a=1}^{N} u_a^+ \prod_{b\neq a}(z-\varphi_b)\right]}_{=:\,U^+_{N-1}(z)}\cdot\underbrace{\left[\sum_{a=1}^{N} u_a^- \prod_{b\neq a}(z-\varphi_b)\right]}_{=:\,U^-_{N-1}(z)} - \underbrace{\left[\prod_{\alpha=1}^{N_f}(z-m_\alpha)\right]}_{=:\,P_{N_f}(z)} = -\underbrace{\left[\prod_{a=1}^{N}(z-\varphi_a)\right]}_{=:\,Q_N(z)}\cdot\, \tilde{Q}_{\tilde{N}}(z)$$

$$\uparrow$$

$$\underline{\tilde{N} = \max(N_f-N,\, N-2)}$$

GENERATING POLYNOMIAL OF CHIRAL RING RELATIONS:

$$\boxed{R_{N+\tilde{N}}(z) := Q_N(z)\,\tilde{Q}_{\tilde{N}}(z) + U^+_{N-1}(z)\,U^-_{N-1}(z) - P_{N_f}(z) = 0} \quad \forall z\in\mathbb{C}$$

GENERATING POLYNOMIAL OF CHIRAL RING RELATIONS:

$$\boxed{\mathcal{R}_{N+\tilde N}(z) := Q_N(z)\,\tilde Q_{\tilde N}(z) + U^+_{N-1}(z)\,U^-_{N-1}(z) - P_{N_f}(z) = 0} \quad \forall z$$

- $Q_N(z) = \sum_{n=0}^{N} (-1)^n\, \phi_n\, z^{N-n} \qquad (\phi_0 = 1)$ $\qquad U^\pm_{N-1}(z) = \sum_{n=0}^{N-1} (-1)^n\, V^\pm_n\, z^{N-1-n}$

  $\tilde Q_{\tilde N}(z) = \sum_{n=0}^{\tilde N} (-1)^n\, \tilde\phi_n\, z^{\tilde N-n}$ $\qquad P_{N_f}(z) = \sum_{n=0}^{N_f} (-1)^n\, M_n\, z^{N_f-n} = z^{N_f}$ today $(m_\alpha = 0)$

$$\{\mathcal{O}_i\} = \{\, \phi_{1\le n\le N},\ V^\pm_{0\le n\le N-1},\ \tilde\phi_{0\le n\le \tilde N} \,\}$$

<u>GENERATORS</u> (GAUGE INVARIANT OP's)

- $\mathcal{R}_{N+\tilde N}(z) = \sum_{n=0}^{N+\tilde N} (-1)^n\, R_n\, z^{N+\tilde N-n} = 0 \ \forall z \longrightarrow N+\tilde N+1$ <u>RELATIONS</u> $\boxed{R_n(\mathcal{O}) = 0}$:

$$\begin{cases} R_0, \dots, R_{\tilde N} = 0 & \text{determine } \{\tilde\phi_n\}. \\ R_{\tilde N+1}, \dots, R_{\tilde N+N} = 0 & : \ N \text{ relations among } 3N \text{ generators } \{\phi_n, V^\pm_n\}. \end{cases}$$

[SC, Hanany, Zaffaroni '13]

E.g. for BAD THEORIES: $\quad R_k = \sum_{n_1+n_2=k} \left(\phi_{n_1}\tilde\phi_{n_2} + V^+_{n_1} V^-_{n_2}\right) - (-1)^{N_f} M_{N_f-2N+2+k} = 0$.

SINGULAR LOCI OF CB,

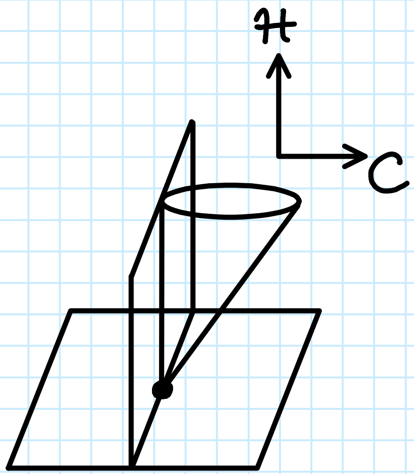
ROOTS OF HB,

AND INFRARED SCFT's

SINGULARITIES OF MODULI SPACE AND IR SCFT's

- What is the IR fate of 3d $\mathcal{N}=4$ $U(N), N_f$ SQCD theories?
  It depends... on N, N_f and the vacuum.

- Interacting SCFT's at singularities of moduli space:
  - Extra massless d.o.f.
  - Locally conical geometry



- Look for singularities of quantum corrected COULOMB BRANCH.
  They should include the ROOTS OF HIGGS BRANCHES. Where?

- Identify IR SCFT from local geometry of moduli space near the singularity:

~[Argyres, Plesser, Seiberg '96]

LOCAL GEOMETRY NEAR SINGULARITY = MODULI SPACE OF SCFT

SINGULAR LOCI OF THE COULOMB BRANCH

- COULOMB BRANCH $\mathcal{C}$ defined by:
  - GENERATORS: $\{\mathcal{O}_i\} = \{\phi_{1\le n\le N},\ V^{\pm}_{0\le n\le N-1},\ \tilde{\Phi}_{0\le n\le \tilde{N}}\}$
  - RELATIONS: $R_n(\mathcal{O}_i) = 0 \qquad 0\le n\le N+\tilde{N}$

- JACOBIAN MATRIX: $J = \left(J_n{}^i\right) = \left(\frac{\partial R_n}{\partial \mathcal{O}_i}\right) \quad : \quad \mathrm{rk}(J) \le N+\tilde{N}+1$

- SINGULAR LOCUS: rk(J) not maximal.

$$\mathcal{C}_{\mathrm{sing}} = \{\mathcal{O} \mid \mathrm{rk}(J) < N+\tilde{N}+1\} \cap \mathcal{C} = \bigcup_{\ell=1}^{\ell_{\max}} \mathcal{C}^{(\ell)}_{\mathrm{sing}},$$

where $\mathcal{C}^{(\ell)}_{\mathrm{sing}} = \{\mathcal{O} \mid \mathrm{rk}(J) = N+\tilde{N}+1-\ell\} \cap \mathcal{C}$. $\longleftarrow$ (Quaternionic) codim. ℓ singular locus

Nested sequence: $\underset{\text{Most singular}}{\mathcal{C}^* = \mathcal{C}^{(\ell_{\max})}_{\mathrm{sing}}} \subset \dots \subset \overline{\mathcal{C}^{(2)}_{\mathrm{sing}}} \subset \overline{\mathcal{C}^{(1)}_{\mathrm{sing}}} = \mathcal{C}_{\mathrm{sing}} \subset \underset{\text{Full CB}}{\overline{\mathcal{C}^{(0)}_{\mathrm{sing}}} = \mathcal{C}}$.

SINGULAR LOCI $=$ ROOTS OF HIGGS BRANCHES

- We find that the <u>CODIM. ℓ SINGULAR LOCUS</u> is given by:

$$\overline{C^{(\ell)}_{sing}} = \left\{ \Phi_{N-i} = \tilde{\Phi}_{\tilde{N}-i} = V^{\pm}_{N-1-i} = 0 \quad \overset{\ell-1}{\underset{i=0}{\forall}} \right\} \cap C$$

$$\cong C\left[U(N-\ell), N_f - 2\ell\right]$$

obtained by setting to zero ℓ triples $(\varphi_a, u^+_a, u^-_a)$.

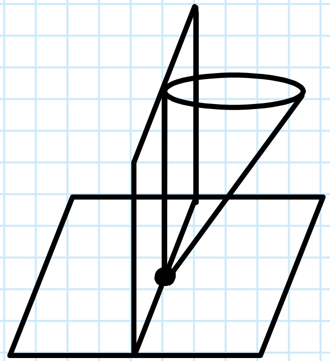
- Exactly the right structure to be the root $C_{N-\ell}$ of the HB $\mathcal{H}_\ell$:

$U(N-\ell)$ unbroken gauge symmetry
$SU(N_f-2\ell)$ unbroken flavour symmetry
on $\mathcal{H}_\ell$

SINGULAR LOCUS $C^{(\ell)}_{sing}$ ON CB $\longleftrightarrow$ ROOT $C_{N-\ell}$ OF HB $\mathcal{H}_\ell$

IR PHYSICS FROM LOCAL GEOMETRY

- Zoom near a generic point of $\overline{C^{(\ell)}_{\text{sing}}}$ (i.e. away from $C^{(\ell+1)}_{\text{sing}}$).



LOCAL CB GEOMETRY

$$\boxed{\mathcal{U}[C^{(\ell)}_{\text{sing}}] = \underbrace{\mathbb{C}^{2(N-\ell)}}_{\text{PARALLEL}} \times \underbrace{C[U(\ell), N_f]}_{\text{TRANSVERSE}}}$$

PARALLEL TRANSVERSE to $C^{(\ell)}_{\text{sing}}$

Including transverse HB directions: $\mathcal{H}[U(\ell), N_f]$ (and mixed branches).

$$\updownarrow$$

- INFRARED SCFT

$$\boxed{T_{IR}[C^{(\ell)}_{\text{sing}}] = \begin{pmatrix} N-\ell \text{ FREE} \\ \text{TWISTED HYPERS} \end{pmatrix} + T_{U(\ell), N_f}}$$

IR SCFT at ORIGIN of moduli space of GOOD $U(\ell), N_f$

$\mathbb{C}^* \;\shortparallel$

<u>CLAIM</u>: $U(N), N_f$ at a generic point of the codim. ℓ singular locus of its CB flows to $T_{IR}[C^{(\ell)}_{\text{sing}}] = T_{U(\ell), N_f} + (N-\ell$ free twisted hypers$)$.

REMARKS ON SEIBERG DUALITY CONJECTURE

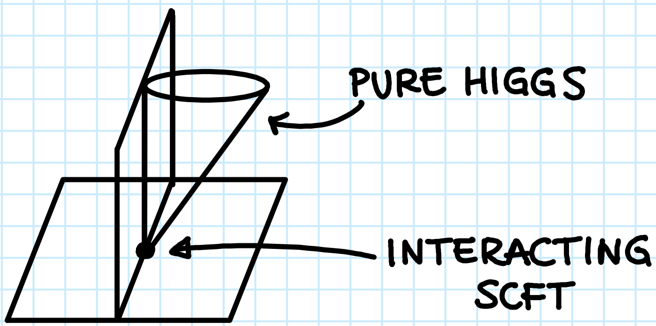
IR EFFECTIVE DESCRIPTION vs IR DUALITY

<u>CLAIM</u>: $U(N), N_f$ at a generic point of the codim. ℓ singular locus of its CB flows to $T_{IR}[\mathcal{C}^{(\ell)}_{sing}] = T_{U(\ell),N_f} + (N-\ell$ free twisted hypers$)$.

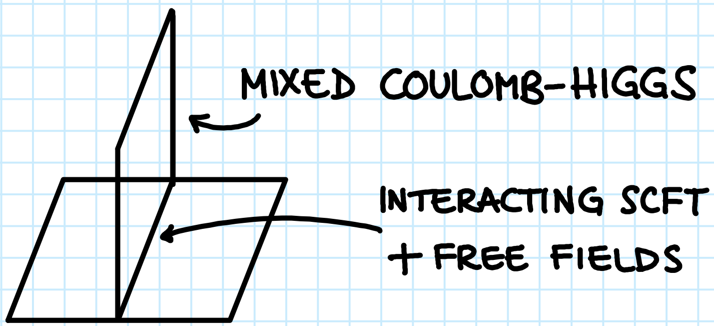
- <u>LOCAL IR EFFECTIVE DESCRIPTION</u> near (any) given point of CB $\mathcal{C}$.

It holds for all N, N_f. GOOD/BAD/UGLY theories only differ in value of ℓ_{max}:

<u>GOOD</u>: $\ell_{max}(N_f, N) = N$



<u>UGLY/BAD</u>: $\ell_{max}(N, N_f) = \left\lfloor \frac{N_f}{2} \right\rfloor < N$



- $\neq$ <u>EXACT IR EQUIVALENCE (DUALITY)</u>:

moduli space and deformations are equal <u>GLOBALLY</u>.

UGLY $U(N)$, $N_f = 2N-1$

EXACT IR EQUIVALENCE:

UGLY: $U(N), 2N-1$ $\longrightarrow$ $T_{U(N-1),2N-1} + 1$ f.t.h. $\longleftarrow$ $U(N-1), 2N-1 + 1$ f.t.h.: GOOD + FREE

- **SUSY PARTITION FUNCTIONS** match on the nose.

- **HIGGS BRANCH** matches: $\ell_{max} = N-1$ for both UGLY and GOOD.

- **COULOMB BRANCH** matches:

UGLY $\quad Q_N \widetilde{Q}_{N-1} + U^+_{N-1} U^-_{N-1} = P_{2N-1}$ $\quad \Longleftrightarrow \quad$ $Q^D_{N-1} \widetilde{Q}^D_N - U^{D+}_{N-2} U^{D-}_{N-2} = P_{2N-1}$ $\quad$ GOOD + FREE

$V^\pm_0$ decoupled

$$Q^D = \widetilde{Q}$$
$$U^{D\pm} = U^\pm - V^\pm_0 Q$$
$$\widetilde{Q}^D = Q + 3 \text{ terms}$$

BAD $U(N)$, $N \leq N_f \leq 2N-2$

SEIBERG DUALITY CONJECTURE [Yaakov '13] :

$$U(N),\ N \leq N_f \leq 2N-2 \xrightarrow[RG]{} T_{U(N_f-N),N_f} + (2N-N_f \text{ free twisted hypers}) \ ?$$

$\underset{\text{BAD}}{U(N), N_f}$ and $\underset{\text{GOOD}}{U(N_f-N), N_f}$ + $\underset{\text{FREE}}{(2N-N_f \text{ f.t.h.})}$ are <u>not</u> IR equivalent!

- <u>SUSY PARTITION FUNCTIONS</u> only match upon regularization of Z^{bad}.
- <u>HIGGS BRANCH</u> differs : $\ell^{\text{bad}}_{\text{max}} = \left\lfloor \frac{N_f}{2} \right\rfloor > \ell^{\text{good}}_{\text{max}} = N_f - N$.
- <u>COULOMB BRANCH</u> differs : no free factor $\mathbb{C}^{2(2N-N_f)}$ for bad theory.

BAD $U(N)$, $N \leq N_f \leq 2N-2$

- Is the equality of regularized partition functions just a coincidence?
- Putative "dual" $\equiv$ IR effective description of the bad theory near $C^{(N_f-N)}_{sing}$. Why?

- <u>SYMMETRIC VACUUM</u> $\mathcal{P}$: unique vacuum preserving all the global symmetry

$$U(1)_J \times SU(N_f) \times U(1)_c \times U(1)_H$$

lies on $C^{(N_f-N)}_{sing}$ of the bad theory (and $\nexists$ if $N_f < N$).

- Locally near $\mathcal{P}$ the moduli spaces agree (and all deformations are available).

- Regularizing SUSY p.f. requires Fayet-Iliopoulos parameter $\neq 0$:
  - $\longrightarrow$ C is lifted: only $\mathcal{P}$ survives.
  - $\longrightarrow$ $\mathcal{H}$ is partially lifted: $\underset{\text{BAD}}{T^*Gr(N, N_f)} \cong \underset{\text{GOOD}}{T^*Gr(N_f-N, N_f)}$

CONCLUSIONS

- MODULI SPACE OF VACUA of 3d $\mathcal{N}=4$ $U(N), N_f$ SQCD:

  Classically exact Higgs branches are glued to quantum corrected Coulomb branch at its singular loci.

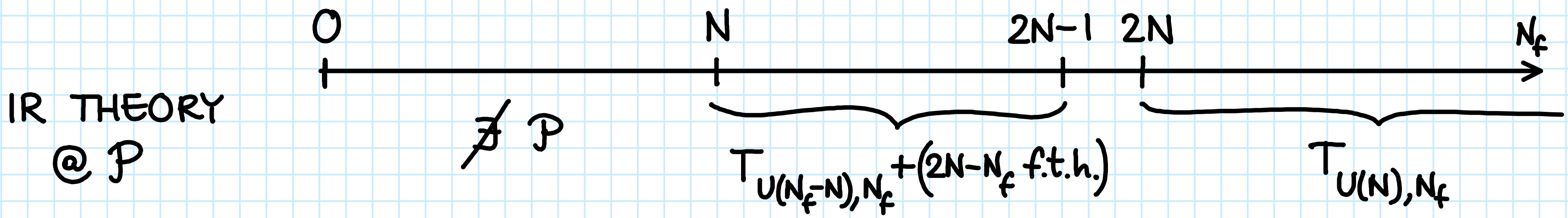
- The gauge theory flows to an IR SCFT which depends on the choice of vacuum.

  The IR SCFT is identified from the local geometry of the moduli space near the vacuum.

- UGLY = GOOD + FREE — Full-fledged IR duality.

- BAD ≠ GOOD + FREE — Moduli spaces are different.

  RHS is local IR effective description of LHS near the SYMMETRIC VACUUM $\mathcal{P}$.



SOME OPEN QUESTIONS:

- What is special about $\mathcal{P}$ when we put the SQFT in curved space?
  - Only at $\mathcal{P}$ are all deformations available.
  - Are other Coulomb vacua lifted (for $FI=0$)?
  - Or they aren't, but Goldstone bosons make Z ill-defined?

- Derive $\mathcal{N}=2$ dualities from $\mathcal{N}=4$ IR effective theory at $\mathcal{P}$ + mass deformation?

  ~ [Argyres, Plesser, Seiberg '96]

- Circular quivers have holographic cascading RG flows. [Aharony, Hashimoto, Hirano, Ouyang '09;...]

  → Cascade explained by IR physics of symmetric vacuum $\mathcal{P}$? ~ [Benini, Bertolini, Closset, SC '08]