Witten Index for Non-compact Dynamics

based on 1602.03530 and 1702.01749

with Piljin Yi (KIAS)

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Twisted Partition Function

Noncompact Dynamics and Localization

Non-compact Chirals

Free Chiral; U(1); ADHM

Pure Yang-Mills

D0 mechanics

Quivers

Rational invariants and nonprimitive dynamics

Summary and Outlook

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Witten Index and Deformation

- D-brane bound state problems via Witten index of $\mathcal{N}$=4 QMs

$$\mathcal{I}(\mathbf{y}) = \lim_{\beta\to\infty} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2R+2J_3}e^{-\beta H}\right]$$

- As an integral quantity, it is insensitive to small deformations but ***only to small deformations***.

- E.g. Wall-crossing phenomena

Twisted Partition Function

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- Non-compact dynamics is of the former type.
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The Formula for Ω

Localization of d=1 $\mathcal{N}$=4 GLSM [Hori, Kim, Yi `14]

cf. [Benini, Eager, Hori, Tachikawa `13]

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$$g_{\mathrm{v}}(u) = \frac{1}{2\sinh^r(\frac{z}{2})}\prod_{\alpha\in\Delta_G}\frac{\sinh(-\frac{\alpha\cdot u}{2})}{\sinh(\frac{\alpha\cdot u - z}{2})}; \quad g_{\mathrm{m}}(u) = \prod_{a=1}^{A}\prod_{\rho\in\mathcal{R}_a}\frac{\sinh(-\frac{\rho\cdot u + (\frac{R_a}{2}-1)z + F_a\cdot\mu}{2})}{\sinh(\frac{\rho\cdot u + \frac{R_a}{2}z + F_a\cdot\mu}{2})}$$

where $e^{z/2} = \mathbf{y}$ and $e^\mu = x$

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  - **JK-Res** is the sum of "Jeffrey-Kirwan residues" over singularities

$$\underset{u=0}{\text{JK-Res}}_{\,\eta:\mathbf{Q}_0} \frac{\mathrm{d}^r u}{\prod_{p=1}^{r} Q_{i_p} \cdot u} \equiv \begin{cases} \frac{1}{|\det(Q_{i_1}, \cdots Q_{i_r})|}, & \text{if } \eta \in \text{Cone}(Q_{i_1}, \cdots, Q_{i_r}) \\ 0, & \text{otherwise} \end{cases}$$

$\mathcal{I}$ via Ω ?

- The two objects agree for a compact dynamics, but they do not for a non-compact dynamics.

- **Question #1**: are these two related at all, and if so, can we extract one from the other?

- **Question #2**: what if there remain asymptotic directions that cannot be controlled by the flavor symmetry G_F?

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- **Question #2**: what if there remain asymptotic directions that cannot be controlled by the flavor symmetry G_F?

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- **<u>Question #1</u>**: are these two related at all, and if so, can we extract one from the other?

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- R-charges disagree. Further, the true count must be zero.

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- The singlet sectors do not agree, yet again.

U(1) GLSM

- The L^2 cohomology for asymptotically conical geometry
  [Hausel, Hunsicker, Mazzero `02]

$$H^n_{L^2}(M) = \begin{cases} H^n(M, \partial M) & \text{if } n < d \, (= \frac{1}{2}\dim_{\mathbb{R}} M) \\ \mathrm{Im}(H^n(M, \partial M) \to H^n(M)) & \text{if } n = d \\ H^n(M) & \text{if } n > d \end{cases}$$

leading to

$$\mathcal{I}^{\zeta>0}_{N,K}(\mathbf{y}) = \begin{cases} (-1)^{N+K-1}(\mathbf{y}^{-N+K+1} + \cdots \mathbf{y}^{N-K-1}) & \text{if } N > K \\ 0 & \text{otherwise} \end{cases}$$

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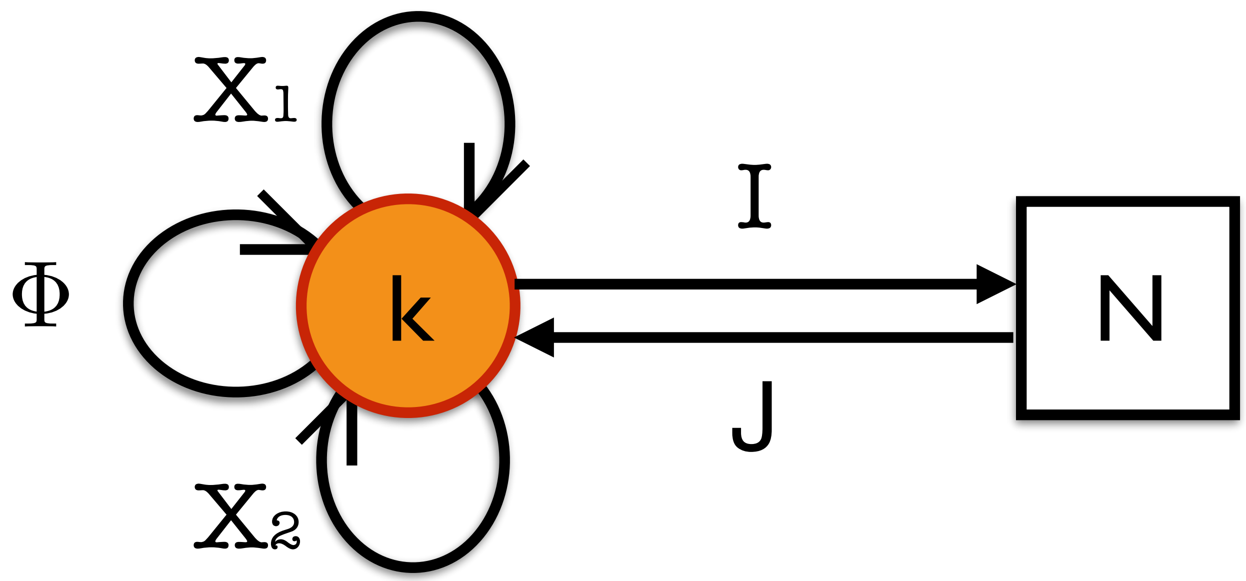
Would this work for other theories?

Would things get better with more supersymmetries?

ADHM

- k D0's and N D4's



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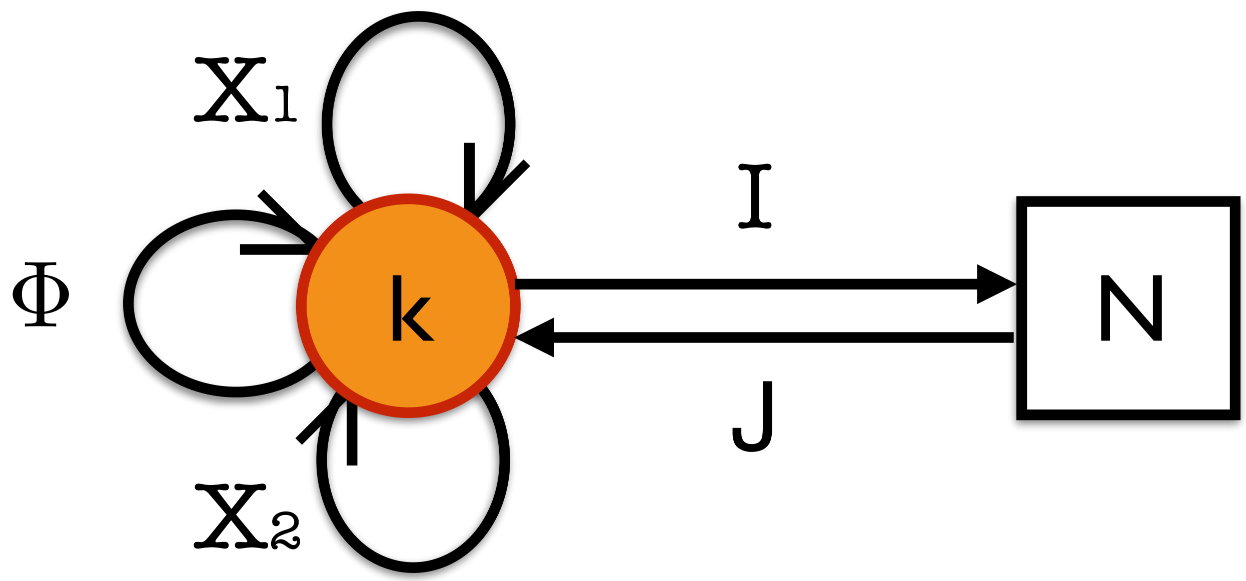

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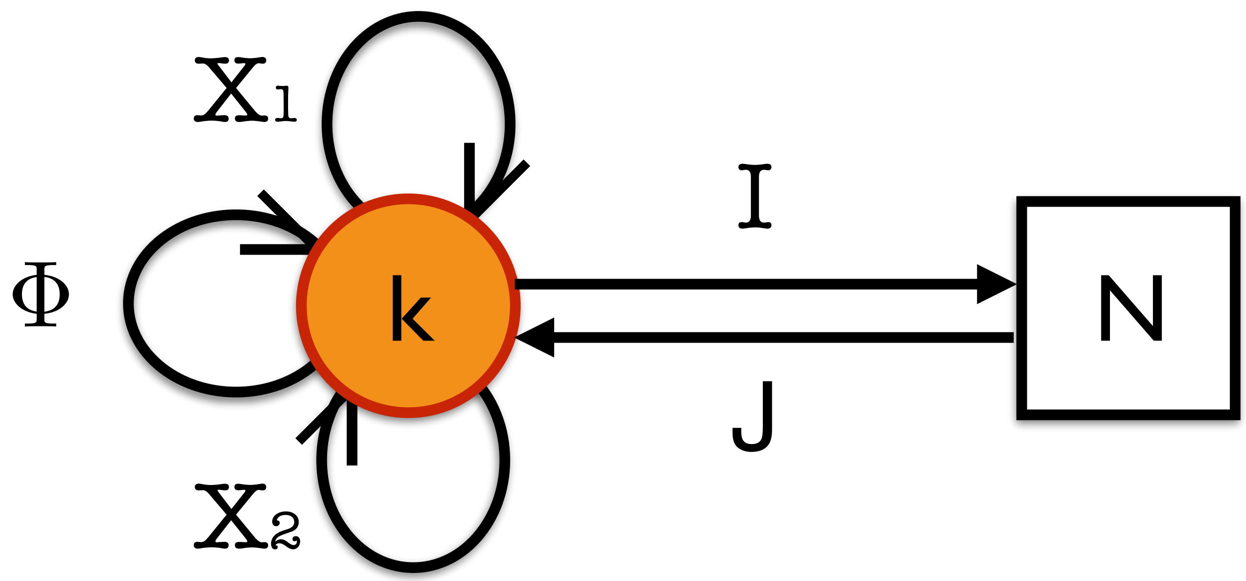

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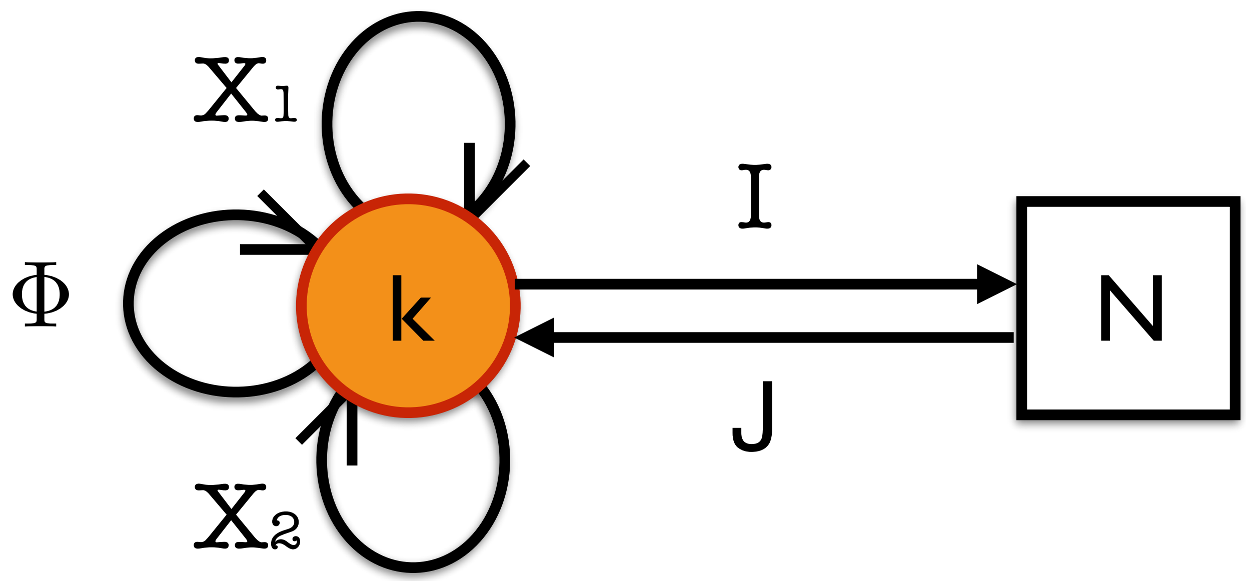

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- The only interesting part is the $U(1)_{\tilde{R}}$

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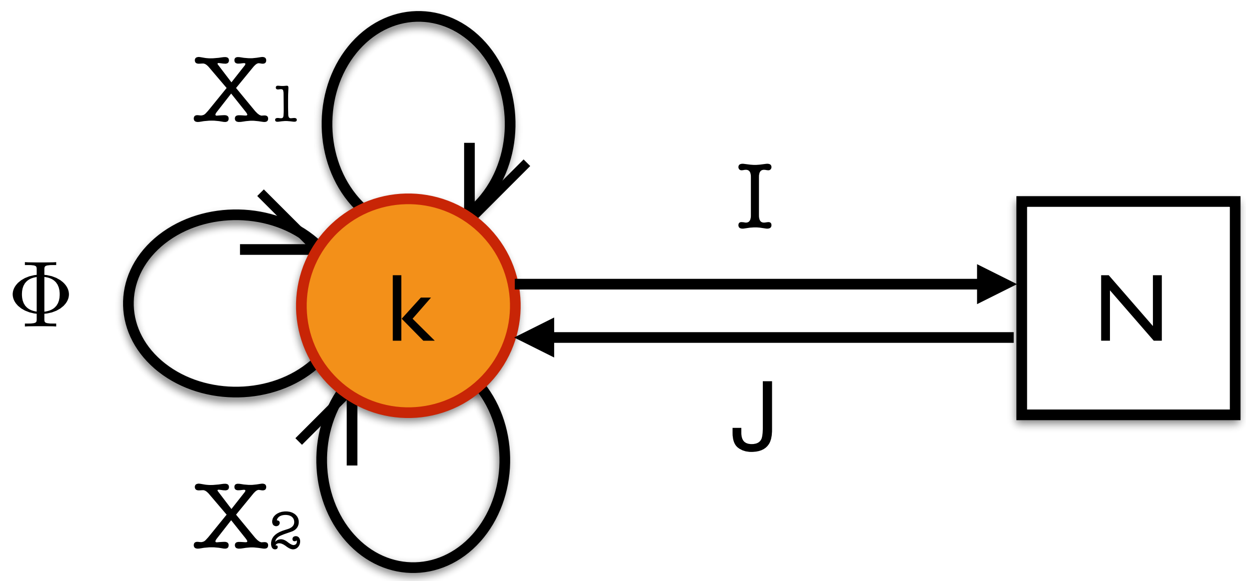

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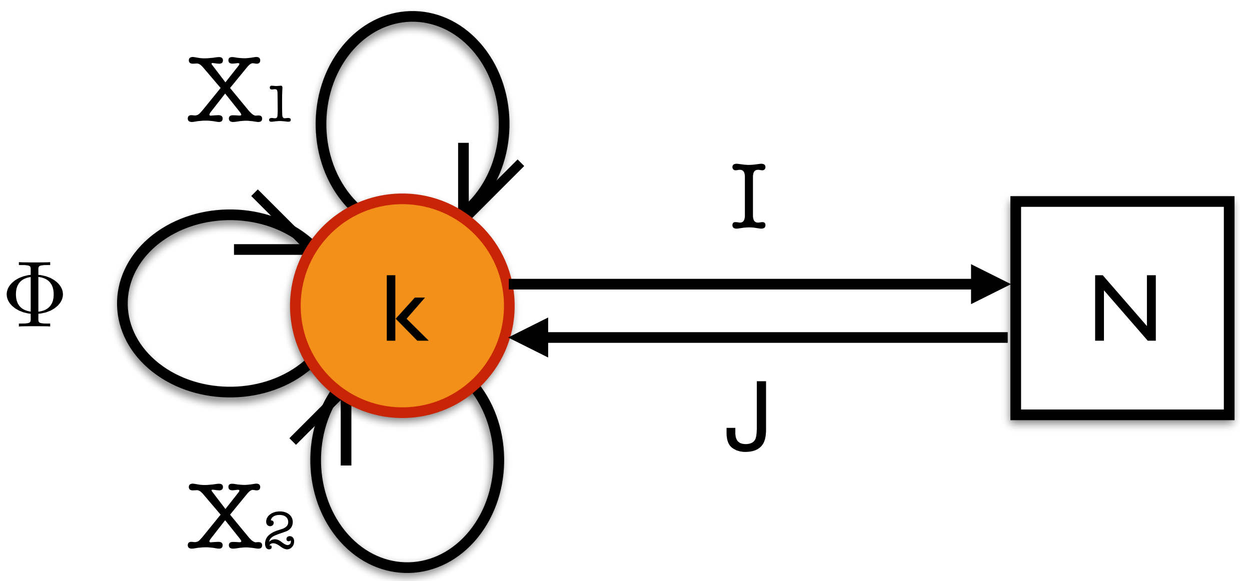

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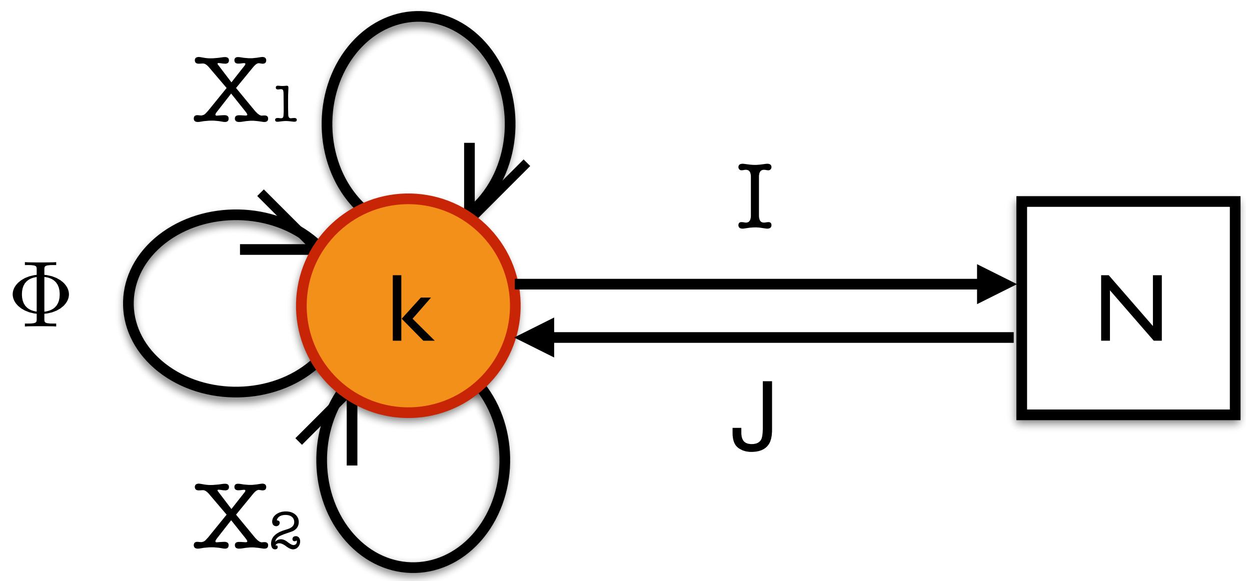

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- The true L^2 cohomology is computed to be 1.

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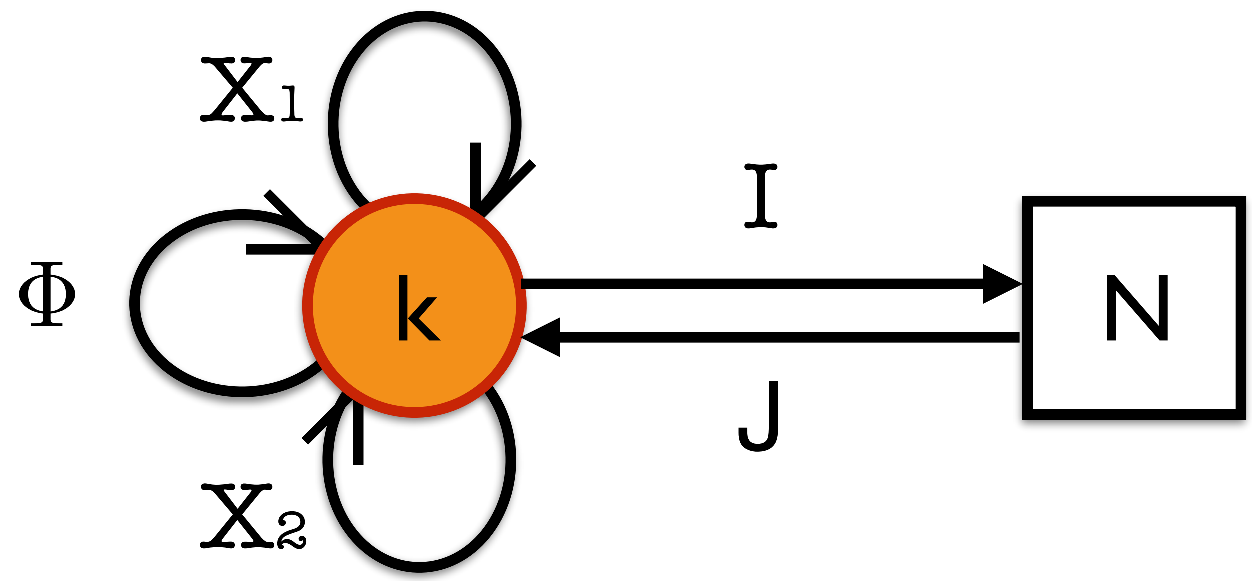

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- With k>1, however, more complicated structures arise and the mantra does not work as nicely…

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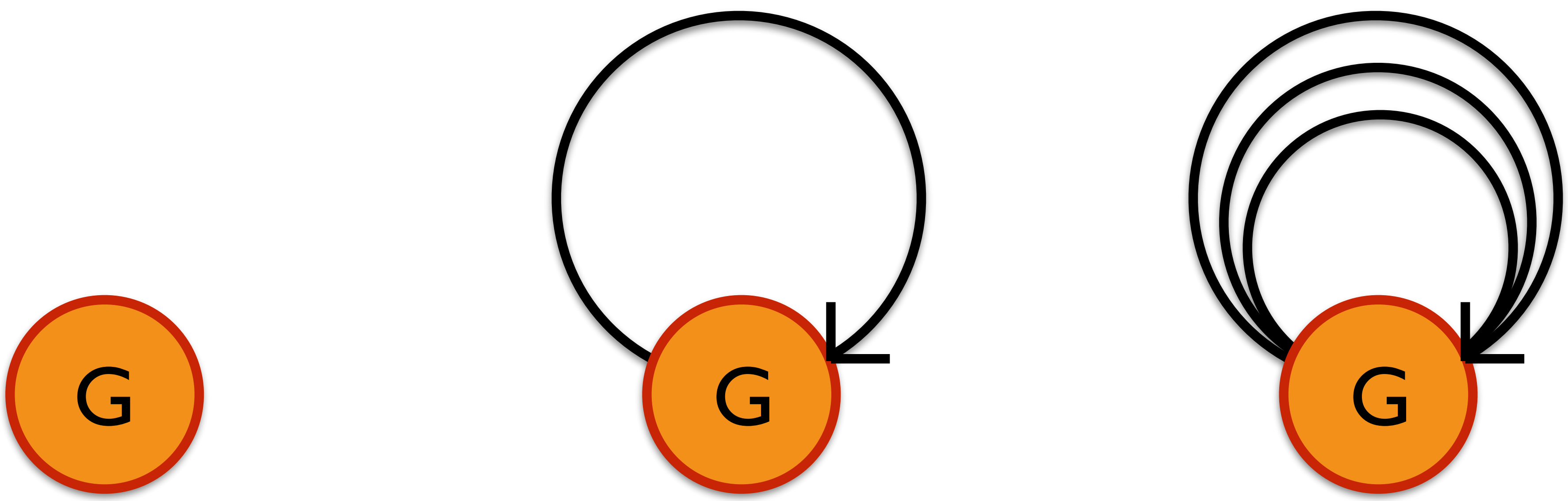
Pure Yang-Mills

- $\mathcal{N}$=4,8, and 16 Pure SYM QMs with G=ABCDEFG



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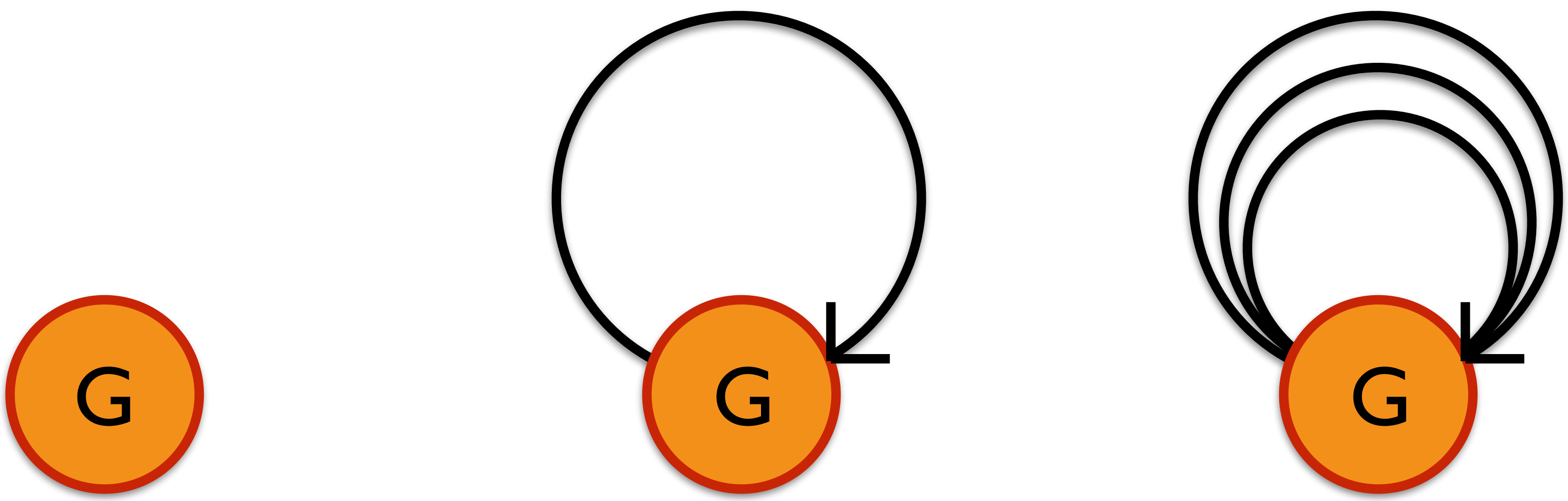

- IIA/M duality: D0 bound state problem for the case of G=A
  [Yi `97] [Sethi, Stern `97] [Green, Gutperle `97] [Moore, Nekrasov, Shatashvili `98] $\cdots$

Non-compact Vector

- Again, the object in question is the twisted partition function.
- With gapless directions from a chiral:

  Chemical potential could lift all of them.
- With gapless directions from a vector:

  Further subtleties arise as the flat directions cannot be lifted completely.

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  Chemical potential could lift all of them.
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  Further subtleties arise as the flat directions cannot be lifted completely.
- What does the residue formula mean then?

Pure Yang-Mills
Bulk and Boundary

- If non-compact vectors cannot be controled by any means, one must expect β-dependence in Ω
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- Recall that the Witten index can split as

$$\mathcal{I} = \mathcal{I}_{\rm bulk} + \delta\mathcal{I}$$

with $\mathcal{I}_{\rm bulk} = \lim_{\beta\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2R+2J_3}x^{G_F}e^{-\beta H}\right]$

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$\mathcal{N}$=4 Pure Yang-Mills

- Upon localization computation, the twisted partition function can be reorganized as

$$\Omega^{G}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{|W|}\sum_{w}{}' \frac{1}{\det(\mathbf{y}^{-1} - \mathbf{y}w)}$$

where the sum is only over elliptic Weyl elements.

- An elliptic Weyl element $w \in W$ is defined by absence of a unit eigenvalue, i.e., $\det(1-w) \neq 0$.

G	W	Elliptic Weyl Elements
$SU(N)$	S_N	$(123\cdots N)$
$SO(4)$	$Z_2 \times S_2$	$(\dot{1})(\dot{2})$
$SO(5)/Sp(2)$	$(Z_2)^2 \times S_2$	$(1\dot{2}), \quad (\dot{1})(\dot{2})$
$SO(6)$	$(Z_2)^2 \times S_3$	$(1\dot{2})(\dot{3})$
$SO(7)/Sp(3)$	$(Z_2)^3 \times S_3$	$(\dot{1}\dot{2}\dot{3}), \quad (12\dot{3}), \quad (1\dot{2})(\dot{3}), \quad (\dot{1})(\dot{2})(\dot{3})$
$SO(8)$	$(Z_2)^3 \times S_4$	$(\dot{1}\dot{2}\dot{3})(\dot{4}), \quad (12\dot{3})(\dot{4}), \quad (1\dot{2})(3\dot{4}), \quad (\dot{1})(\dot{2})(\dot{3})(\dot{4})$

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[Yi `97] [Green, Gutperle `97]

$$\Omega^G = \mathcal{I}^G_{\text{bulk}} = -\delta\mathcal{I}^G = -\delta\mathcal{I}^{U(1)^r/W} = \mathcal{I}^{U(1)^r/W}_{\text{bulk}}$$

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$\mathcal{N}$=4 Pure Yang-Mills

G=SU(p)

- For G=SU(p), one can see that $\Omega^{SU(p)}_{\mathcal{N}=4}(\mathbf{y}) = \dfrac{\mathbf{y}-\mathbf{y}^{-1}}{p(\mathbf{y}^p - \mathbf{y}^{-p})} \quad \to \dfrac{1}{p^2}$

- Such an expression appears naturally in the wall-crossing algebra and leads to rational invariant.
  - We will see how this arises by computing the twisted partition function of non-primitive quiver theories.

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- Such an expression appears naturally in the wall-crossing algebra and leads to rational invariant.
  - We will see how this arises by computing the twisted partition function of non-primitive quiver theories.
  - For a general gauge group G, one may attempt to use $\Omega^{G}_{\mathcal{N}=4}(\mathbf{y})$ to form an analogous object.
  - In particular, for orientifold theories one could uniquely define the quantity $\Xi^{(p)}_{\mathcal{N}=4}(\mathbf{y}) \equiv \Omega^{G^{(p)}_{\mathrm{orientifold}}}_{\mathcal{N}=4}(\mathbf{y})$ for $G^{(p)}_{\mathrm{orientifold}} = O(2p), O(2p+1), Sp(p)$

$\mathcal{N}$=8 Pure Yang-Mills

- The twisted partition function computation leads to

$$\Omega^G_{\mathcal{N}=8}(\mathbf{y},x) = \frac{1}{|W|}\sum_w{}' \frac{1}{\det(\mathbf{y}^{-1}-\mathbf{y}w)}\cdot\frac{\det(\mathbf{y}^{-1}x^{1/2}-\mathbf{y}x^{-1/2}w)}{\det(x^{1/2}-x^{-1/2}w)}$$

- The above gives the $\mathcal{N}$=8 equivariant version of the asymptotic contribution:

$$\Delta^G_{\mathcal{N}=4}(\mathbf{y}) \equiv \frac{1}{|W|}\sum_w{}' \frac{1}{\det(\mathbf{y}^{-1}-\mathbf{y}w)}$$

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$\mathcal{N}$=16 Pure Yang-Mills

- The $\mathcal{N}$=16 equivariant version of the asymptotic contribution is as straightforward:

$$\Delta^G_{\mathcal{N}=4}(\mathbf{y}) \equiv \frac{1}{|W|}\sum_w{}' \frac{1}{\det(\mathbf{y}^{-1}-\mathbf{y}w)} = \Omega^G_{\mathcal{N}=4}(\mathbf{y})$$

$$\Delta^G_{\mathcal{N}=8}(\mathbf{y},x) \equiv \frac{1}{|W|}\sum_w{}' \frac{1}{\det(\mathbf{y}^{-1}-\mathbf{y}w)} \cdot \frac{\det(\mathbf{y}^{-1}x^{1/2}-\mathbf{y}x^{-1/2}w)}{\det(x^{1/2}-x^{-1/2}w)} = \Omega^G_{\mathcal{N}=8}(\mathbf{y},x)$$

$$\Delta^G_{\mathcal{N}=16}(\mathbf{y},x) \equiv \frac{1}{|W|}\sum_w{}' \frac{1}{\det(\mathbf{y}^{-1}-\mathbf{y}w)} \cdot \prod_{a=1}^{3} \frac{\det(\mathbf{y}^{\frac{R_a-2}{2}}x^{\frac{F_a}{2}}-\mathbf{y}^{-\frac{R_a-2}{2}}x^{-\frac{F_a}{2}}w)}{\det(x^{\frac{F_a}{2}}-x^{-\frac{F_a}{2}}w)}$$

- With $\mathcal{N}$=16, we do expect a bound state, however. [Witten \`95]

$\mathcal{N}$=16 Pure Yang-Mills

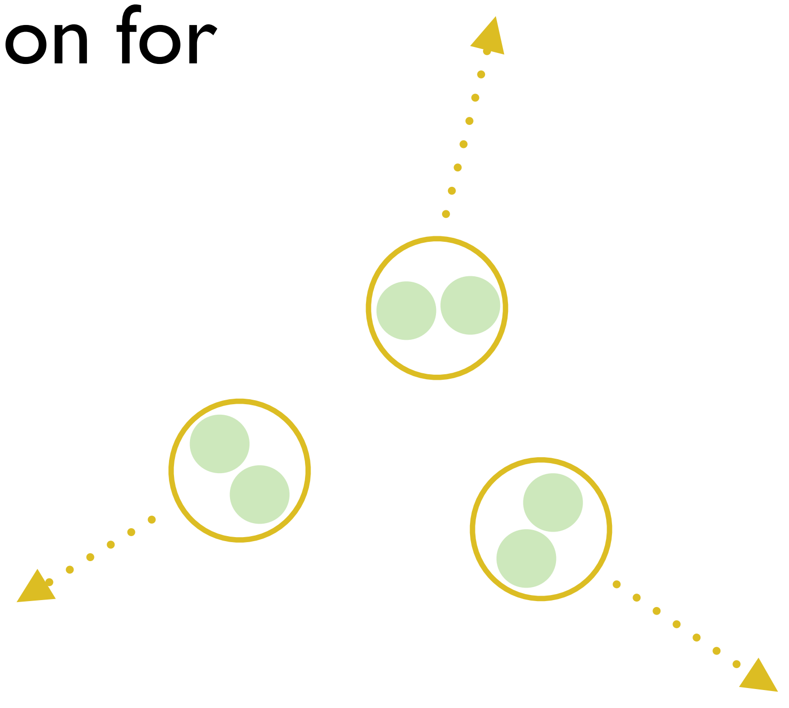
G=SU(p): D0 bound state problem

- For G=SU(p), we can compute the twisted partition function and express it as

$$\Omega^{SU(p)}_{\mathcal{N}=16}(\mathbf{y},x) = 1 \; + \sum_{p'|p,\ p'>1} \Delta^{SU(p')}_{\mathcal{N}=16}(\mathbf{y},x)\cdot 1$$

- This can be thought of as the equivariant version for

$$\Omega^{SU(p)}_{\mathcal{N}=16}\Big|_{\mathbf{y}\to 1, x\to 1} = \sum_{p'|p} \frac{1}{p'^2}$$



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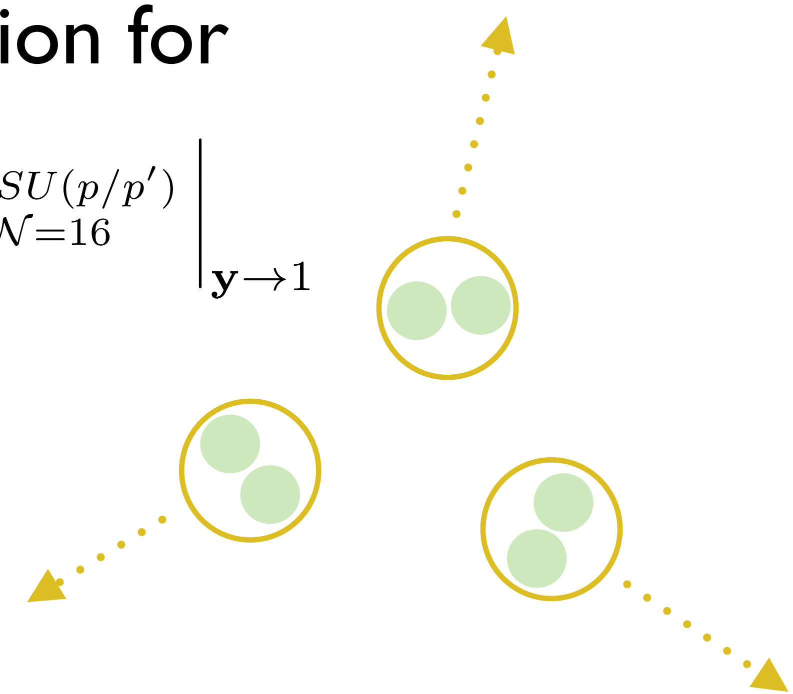
$$\Omega^{SU(p)}_{\mathcal{N}=16}\Big|_{\mathbf{y}\to 1,x\to 1} = \sum_{p'|p}\frac{1}{p'^2} = \mathcal{I}^{SU(p)}_{\mathcal{N}=16}\Big|_{\mathbf{y}\to 1} + \sum_{p'|p,\ p'>1}\frac{1}{p'^2}\cdot \mathcal{I}^{SU(p/p')}_{\mathcal{N}=16}\Big|_{\mathbf{y}\to 1}$$



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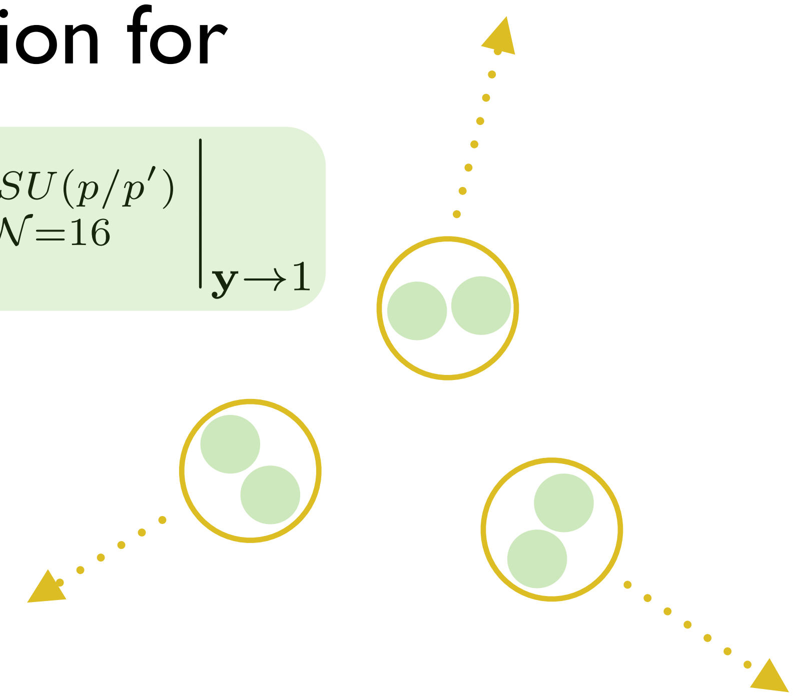
showing that $\mathcal{I}^{SU(p)}_{\mathcal{N}=16} = 1$ for all ranks.



$\mathcal{N}$=16 Pure Yang-Mills

General G

- One is naturally led to try giving a similar interpretation for other gauge groups.

- Presuming an analogous partial-bound-state structure, we can read off the true Witten index by decomposing the twisted partition function.

$\mathcal{N}$=16 Pure Yang-Mills

General G

$$\Omega^{SO(5)/Sp(2)}_{\mathcal{N}=16} = 1 + 2\Delta^{SO(3)/Sp(1)}_{\mathcal{N}=16} + \Delta^{SO(5)/Sp(2)}_{\mathcal{N}=16}$$

$$\Omega^{G_2}_{\mathcal{N}=16} = 2 + 2\Delta^{SU(2)}_{\mathcal{N}=16} + \Delta^{G_2}_{\mathcal{N}=16}$$

$$\Omega^{SO(7)}_{\mathcal{N}=16} = 1 + 3\Delta^{SO(3)}_{\mathcal{N}=16} + \left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + \Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(7)}_{\mathcal{N}=16}$$

$$\Omega^{Sp(3)}_{\mathcal{N}=16} = 2 + 3\Delta^{Sp(1)}_{\mathcal{N}=16} + \left(\Delta^{Sp(1)}_{\mathcal{N}=16}\right)^2 + \Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(3)}_{\mathcal{N}=16}$$

$$\Omega^{SO(8)}_{\mathcal{N}=16} = 2 + 4\Delta^{SO(3)}_{\mathcal{N}=16} + 2\left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + \left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^3 + 3\Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(8)}_{\mathcal{N}=16}$$

$$\Omega^{SO(9)}_{\mathcal{N}=16} = 2 + 4\Delta^{SO(3)}_{\mathcal{N}=16} + 2\left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + 2\Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(3)}_{\mathcal{N}=16}\cdot\Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(7)}_{\mathcal{N}=16} + \Delta^{SO(9)}_{\mathcal{N}=16}$$

$$\Omega^{Sp(4)}_{\mathcal{N}=16} = 2 + 5\Delta^{Sp(1)}_{\mathcal{N}=16} + 2\left(\Delta^{Sp(1)}_{\mathcal{N}=16}\right)^2 + 2\Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(1)}_{\mathcal{N}=16}\cdot\Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(3)}_{\mathcal{N}=16} + \Delta^{Sp(4)}_{\mathcal{N}=16}$$

$\mathcal{N}$=16 Pure Yang-Mills

General G

Witten Index obtained via twisted partition function!

$$\Omega^{SO(5)/Sp(2)}_{\mathcal{N}=16} = 1 + 2\Delta^{SO(3)/Sp(1)}_{\mathcal{N}=16} + \Delta^{SO(5)/Sp(2)}_{\mathcal{N}=16}$$

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$$\Omega^{SO(7)}_{\mathcal{N}=16} = 1 + 3\Delta^{SO(3)}_{\mathcal{N}=16} + \left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + \Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(7)}_{\mathcal{N}=16}$$

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$\mathcal{N}$=16 Pure Yang-Mills

General G

It is crucial that we have fully equivariant indices.

$$\frac{53}{32}\,\Omega^{SO(5)/Sp(2)}_{\mathcal{N}=16} = 1 + 2\Delta^{SO(3)/Sp(1)}_{\mathcal{N}=16} + \Delta^{SO(5)/Sp(2)}_{\mathcal{N}=16}$$

$$\frac{395}{144}\,\Omega^{G_2}_{\mathcal{N}=16} = 2 + 2\Delta^{SU(2)}_{\mathcal{N}=16} + \Delta^{G_2}_{\mathcal{N}=16}$$

$$\frac{267}{128}\,\Omega^{SO(7)}_{\mathcal{N}=16} = 1 + 3\Delta^{SO(3)}_{\mathcal{N}=16} + \left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + \Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(7)}_{\mathcal{N}=16}$$

$$\frac{395}{128}\,\Omega^{Sp(3)}_{\mathcal{N}=16} = 2 + 3\Delta^{Sp(1)}_{\mathcal{N}=16} + \left(\Delta^{Sp(1)}_{\mathcal{N}=16}\right)^2 + \Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(3)}_{\mathcal{N}=16}$$

$$\frac{3755}{1024}\,\Omega^{SO(8)}_{\mathcal{N}=16} = 2 + 4\Delta^{SO(3)}_{\mathcal{N}=16} + 2\left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + \left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^3 + 3\Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(8)}_{\mathcal{N}=16}$$

$$\frac{7555}{2048}\,\Omega^{SO(9)}_{\mathcal{N}=16} = 2 + 4\Delta^{SO(3)}_{\mathcal{N}=16} + 2\left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + 2\Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(3)}_{\mathcal{N}=16}\cdot\Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(7)}_{\mathcal{N}=16} + \Delta^{SO(9)}_{\mathcal{N}=16}$$

$$\frac{8067}{2048}\,\Omega^{Sp(4)}_{\mathcal{N}=16} = 2 + 5\Delta^{Sp(1)}_{\mathcal{N}=16} + 2\left(\Delta^{Sp(1)}_{\mathcal{N}=16}\right)^2 + 2\Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(1)}_{\mathcal{N}=16}\cdot\Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(3)}_{\mathcal{N}=16} + \Delta^{Sp(4)}_{\mathcal{N}=16}$$

$$\Omega^G_{\mathcal{N}}(\mathbf{y},x)\Big|_{\mathbf{y}\to 1, x\to 1} \sim \mathcal{I}^G_{\mathcal{N},\mathrm{bulk}}\Big|_{\mathbf{y}\to 1}$$

	$\mathcal{N}=4,8$	$\mathcal{N}=16$
$SU(N)$	$\frac{1}{N^2}$	$\sum_{p\mid N}\frac{1}{p^2}$
$SO(4)$	$\frac{1}{16}$	$\frac{25}{16}$
$SO(6)=SU(4)$	$\frac{1}{16}$	$\frac{21}{16}$
$SO(8)$	$\frac{59}{1024}$	$\frac{3755}{1024}$
$SO(5)$	$\frac{5}{32}$	$\frac{53}{32}$
$SO(7)$	$\frac{15}{128}$	$\frac{267}{128}$
$SO(9)$	$\frac{195}{2048}$	$\frac{7555}{2048}$
$Sp(2)$	$\frac{5}{32}$	$\frac{53}{32}$
$Sp(3)$	$\frac{15}{128}$	$\frac{395}{128}$
$Sp(4)$	$\frac{195}{2048}$	$\frac{8067}{2048}$
G_2	$\frac{35}{144}$	$\frac{395}{144}$

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cf.
[Moore, Nekrasov, Shatashvili `98]
[Kac, Smilga `99]
[Staudacher `00]
[Pestun `02]

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[Pestun `02]

[SJL, Yi `16]

$$\Omega^G_{\mathcal{N}} \dashrightarrow \mathcal{I}^G_{\mathcal{N},\text{bulk}}$$

[Hwang, Yi `17]

$$\mathcal{Z}^G_{\mathcal{N},\text{matrix model}} \not\dashrightarrow \mathcal{I}^G_{\mathcal{N},\text{bulk}}$$

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$\mathcal{N}$=16 Pure Yang-Mills

Generating Functions for SU, SO, Sp

$$\sum_N \mathcal{I}^{SU(N)}_{\mathcal{N}=16} t^N = \frac{1}{1-t}$$

$$\sum_N \mathcal{I}^{SO(N)}_{\mathcal{N}=16} t^N = \prod_{n=1}^{\infty} (1+t^{2n-1})$$

$$\sum_N \mathcal{I}^{Sp(N)}_{\mathcal{N}=16} t^{2N} = \prod_{n=1}^{\infty} (1+t^{2n})$$

$\mathcal{N}=16$ Pure Yang-Mills

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$\mathcal{N}$=16 Pure Yang-Mills

Bound states of D-particles via M/IIA duality

$$\sum_N \mathcal{I}^{SU(N)}_{\mathcal{N}=16} t^N = \frac{1}{1-t}$$

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M on	$\mathbf{S}^1 \times \mathbb{R}^{9,1}$	$\mathbf{S}^1 \times \mathbb{R}^{0,1} \times \mathbb{R}^9/\mathbb{Z}_2$
IIA on	$\mathbb{R}^{9,1}$	$\mathbb{R}^{0,1} \times \mathbb{R}^9/\mathbb{Z}_2$

M-theory origin of IIA forming an infinite tower of multi D-particle bound states [Witten `95]

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- IIA must remember the M theory origin by forming an infinite tower of D-particle bound states moving freely along $\mathbb{R}^{9,1}$

[Yi `97] [Sethi, Stern `97] [Gutperle, Green `97]

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- IIA must remember the M theory origin by forming an infinite tower of D-particle bound states along fixed points of the orbifold

[Dasgupta, Mukhi `95]
[Kac, Smilga `99] [Kol, Hanany, Rajaraman `99] $\cdots$
[SJL, Yi `17]

Twisted Partition Function

Noncompact Dynamics and Localization

Non-compact Chirals

Free Chiral; U(1); ADHM

Pure Yang-Mills

D0 mechanics

Quivers

Rational invariants and nonprimitive dynamics

Summary and Outlook

BPS Quivers

- BPS states as D-branes wrapping various cycles.
- Low-energy D-brane dynamics by a quiver gauge theory.
- E.g. IIB on CY_3: one-particle BPS states seen as a D3-brane wrapping a SLag.

$\rightarrow$ *$\mathcal{D}$=0+1 quiver theory for particle-like BPS states in $\mathcal{D}$=3+1* [Denef `02]

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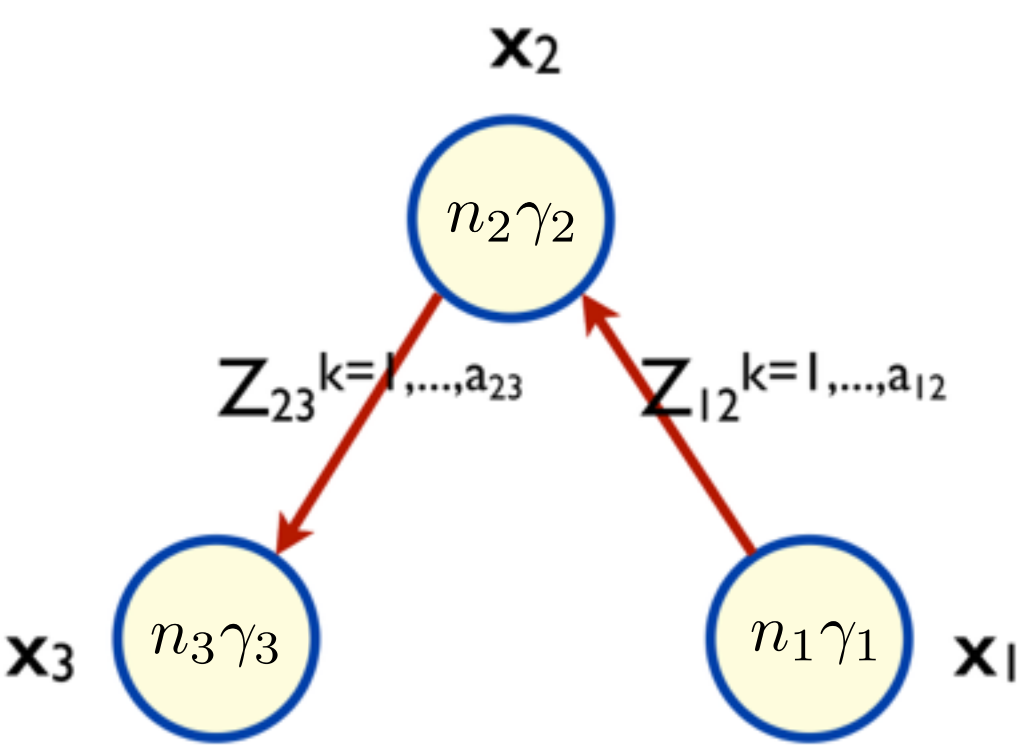
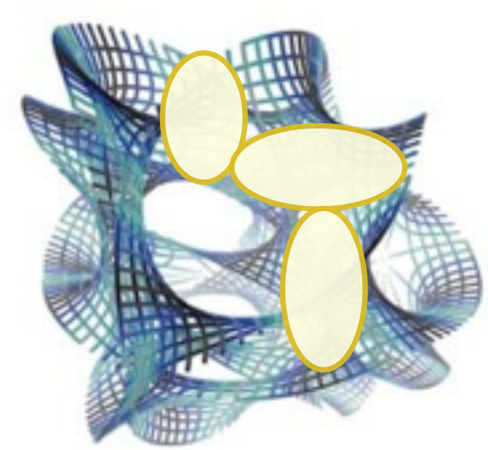


$$a_{vw} = \langle \gamma_v, \gamma_w \rangle$$

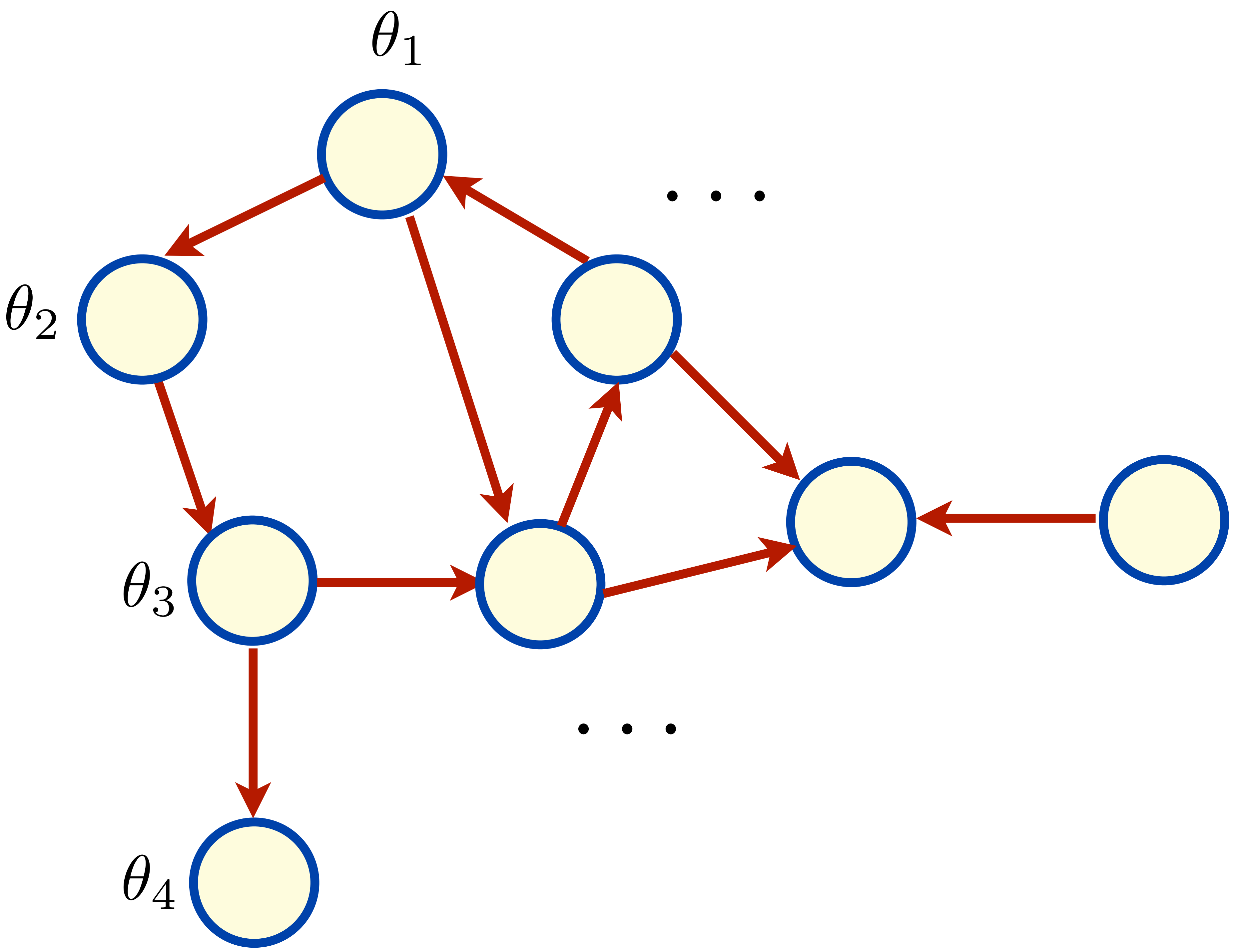


x1 x2 x3

$$U(n_1) \times U(n_2) \times U(n_3)$$

$$Z_{12}^{1,2,\ldots,a_{12}} \quad Z_{23}^{1,2,\ldots,a_{23}}$$

BPS Quivers



Wall Crossing

Phenomenon

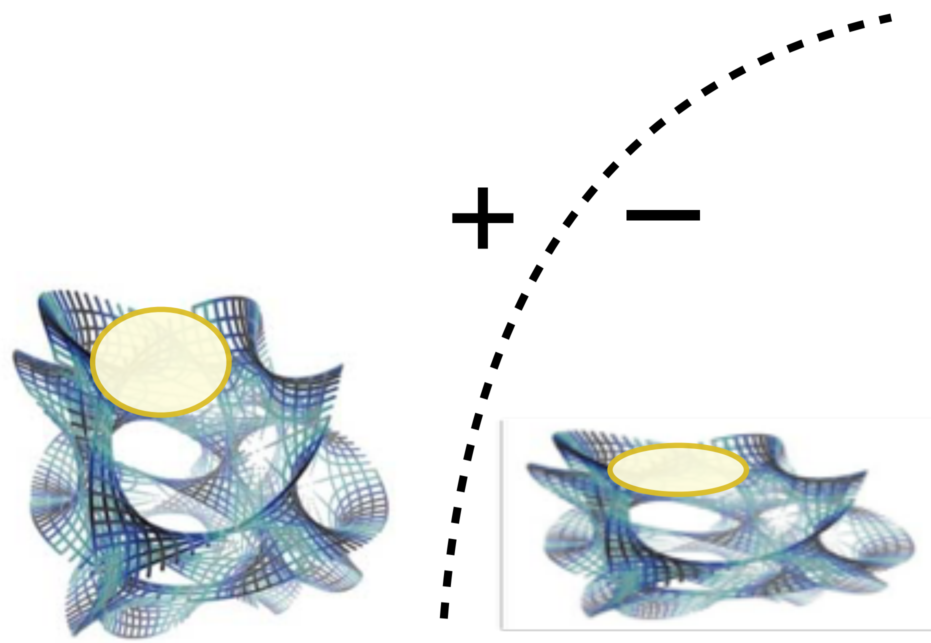
- BPS objects (dis)appear as the relevant CY geometry is deformed across a wall

γ_1

γ_2

γ_3

$\vdots$



γ_1	$\gamma_1 + \gamma_2$
γ_2	$\gamma_2 + \gamma_3$
γ_3	$\gamma_1 + \gamma_3$
$\vdots$	$\vdots$

$$\gamma = (e, m) \in \mathbb{Z}^r \times \mathbb{Z}^r$$

Wall Crossing

Algebra and Formula

[Kontsevich, Soibelman `08]
[Gaiotto, Moore, Neitzke `08]

- $[V_\gamma, V_{\gamma'}] = (-1)^{\langle \gamma,\gamma' \rangle} \langle \gamma, \gamma' \rangle V_{\gamma+\gamma'}$ **where** $\langle \gamma, \gamma' \rangle = e \cdot m' - m \cdot e'$

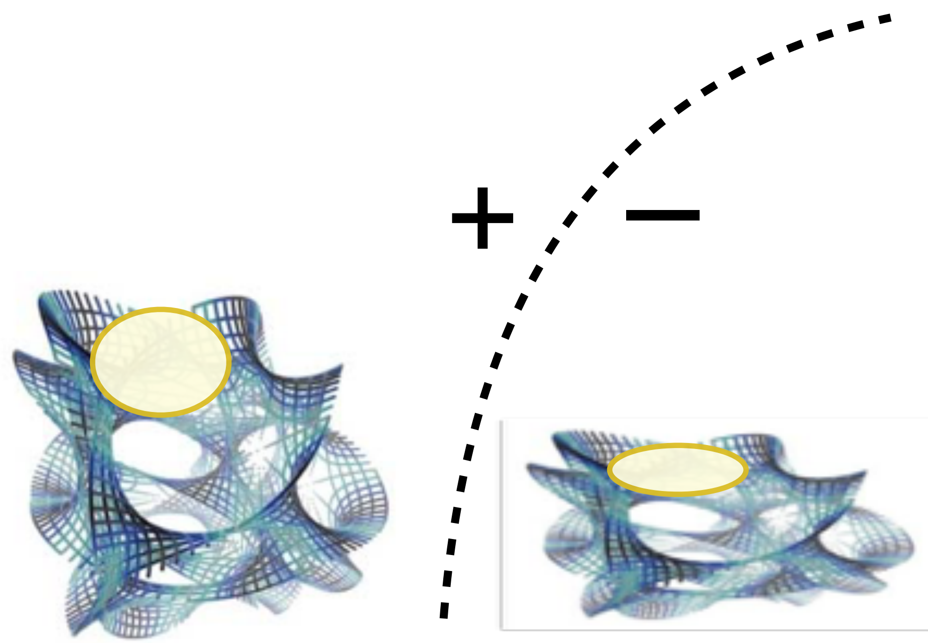
$$\prod_{\gamma}^{\circlearrowleft} K_\gamma^{\mathcal{I}^+(\gamma)} = \prod_{\gamma}^{\circlearrowright} K_\gamma^{\mathcal{I}^-(\gamma)} \quad \textbf{where} \quad K_\gamma \equiv \mathrm{Exp}\left(\sum_{n=1}^{\infty} \frac{V_{n\gamma}}{n^2} \right)$$

γ_1

γ_2

γ_3

$\vdots$



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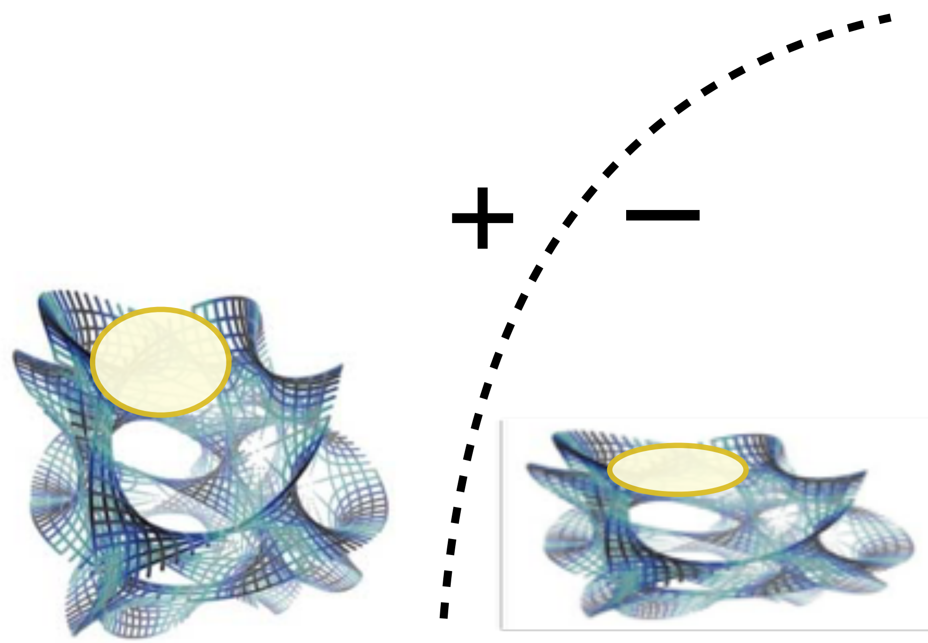
γ_1

γ_2



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Wall Crossing

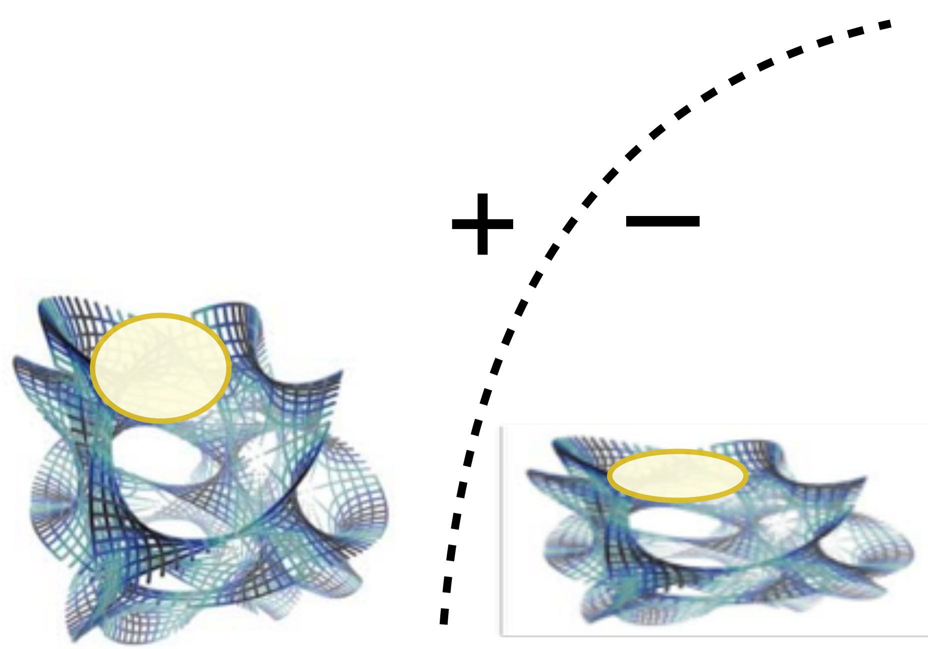
Example 1

- $[V_{\gamma_1}, V_{\gamma_2}] = -V_{\gamma_1+\gamma_2}$ $\qquad\qquad \langle \gamma_1, \gamma_2 \rangle = 1$

$$K_{\gamma_1} K_{\gamma_2} = K_{\gamma_2} K_{\gamma_1+\gamma_2} K_{\gamma_1}$$

γ_1

γ_2



γ_1

$\gamma_1 + \gamma_2$

γ_2

Wall Crossing

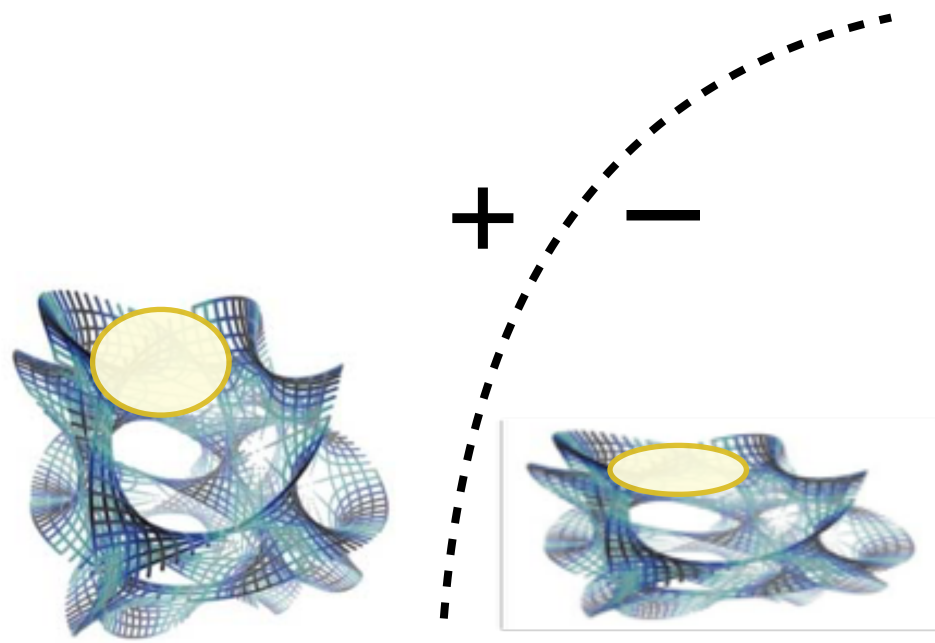
Example

- $[V_{\gamma_1}, V_{\gamma_2}] = 2V_{\gamma_1+\gamma_2}$ $\langle \gamma_1, \gamma_2 \rangle = 2$

$$K_{\gamma_1} K_{\gamma_2} = K_{\gamma_2} K_{\gamma_1+2\gamma_2} K_{2\gamma_1+3\gamma_2} \cdots K^{-2}_{\gamma_1+\gamma_2} \cdots K_{3\gamma_1+2\gamma_2} K_{2\gamma_1+\gamma_2} K_{\gamma_1}$$

γ_1

γ_2



γ_1

$(n+1)\gamma_1 + n\gamma_2 \quad , \quad n \geq 1$

$\gamma_1 + \gamma_2$

$n\gamma_1 + (n+1)\gamma_2 \quad , \quad n \geq 1$

γ_2

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rational invariants

Rational Invariants

Quiver dynamics

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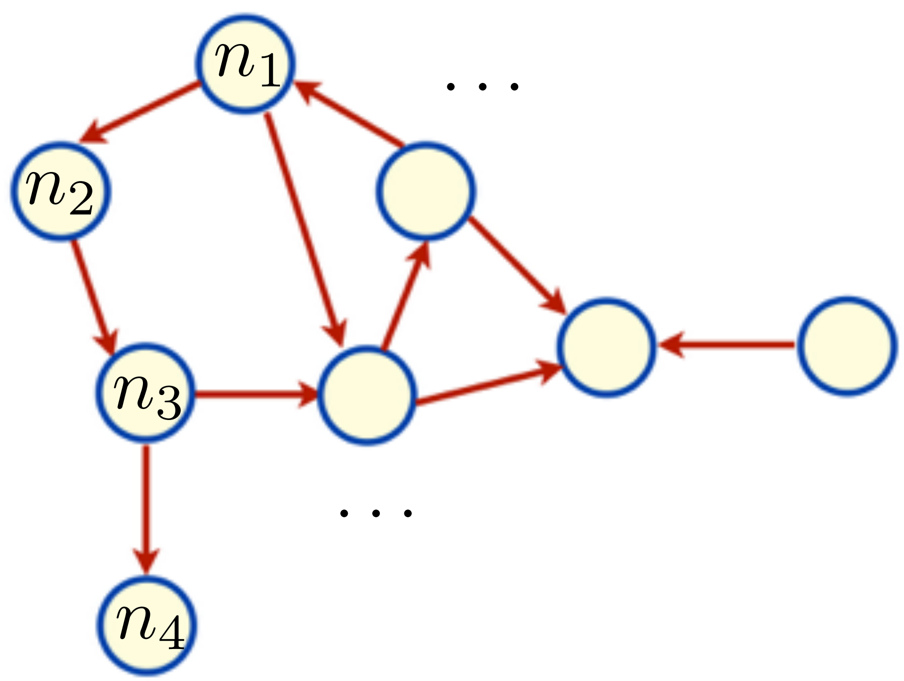

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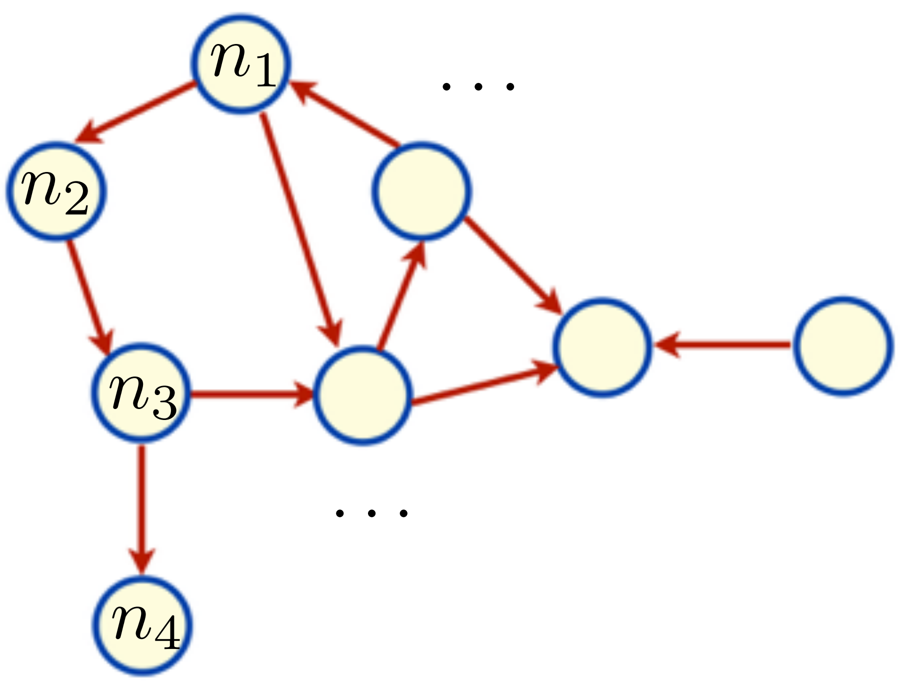

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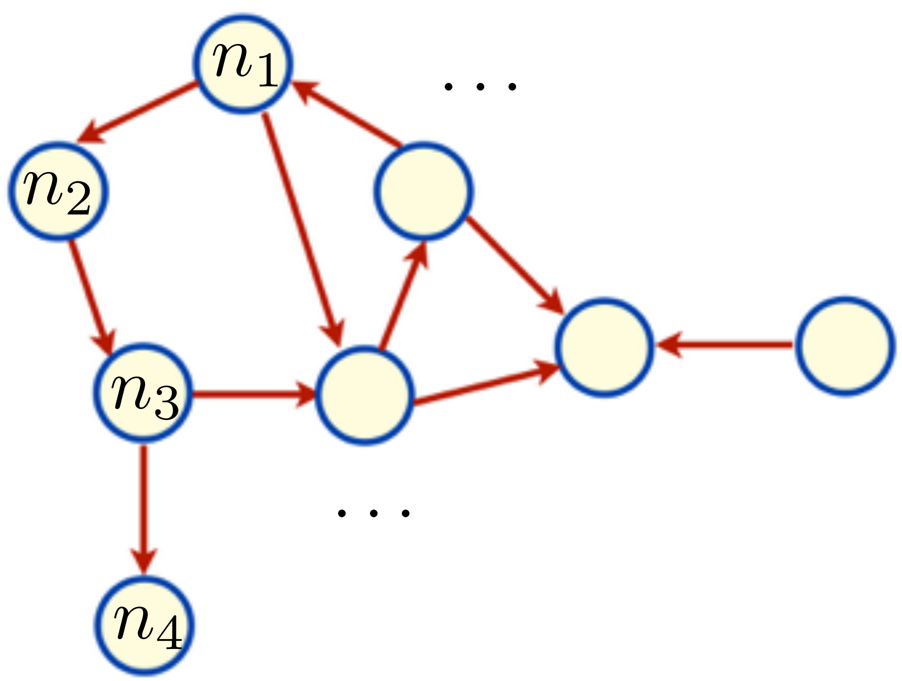

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- Reproduces the index ($\sim \Omega$) for primitive cases

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$$\Omega(\Gamma;\, \mathbf{y}) = \sum_{p|\Gamma} \mathcal{I}(\Gamma/p;\, \mathbf{y}^p) \cdot \frac{\mathbf{y} - \mathbf{y}^{-1}}{p\,(\mathbf{y}^p - \mathbf{y}^{-p})}$$

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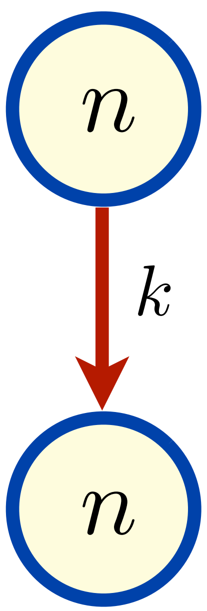
Moebius function

Nonprimitive Dynamics

Example: n-Kronecker quivers



$$\mathcal{I}(\mathcal{Q}_n^k;\,\mathbf{y}) = \sum_{p|n} \mu(p)\cdot\Omega(\mathcal{Q}_{n/p}^k;\,\mathbf{y}^p)\cdot\frac{\mathbf{y}-\mathbf{y}^{-1}}{p\left(\mathbf{y}^p-\mathbf{y}^{-p}\right)}$$

$$\mathcal{I}(\mathcal{Q}_2^1;\,\mathbf{y}) = 0$$

$$\mathcal{I}(\mathcal{Q}_2^2;\,\mathbf{y}) = 0$$

$$\mathcal{I}(\mathcal{Q}_2^3;\,\mathbf{y}) = -\chi_{5/2}(\mathbf{y}^2)$$

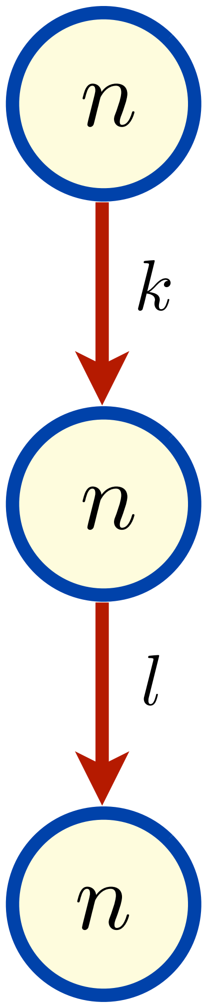
$$\mathcal{I}(\mathcal{Q}_2^4;\,\mathbf{y}) = -\chi_{9/2}(\mathbf{y}^2)-\chi_{5/2}(\mathbf{y}^2)$$

$$\mathcal{I}(\mathcal{Q}_2^5;\,\mathbf{y}) = -\chi_{13/2}(\mathbf{y}^2)-2\chi_{9/2}(\mathbf{y}^2)-\chi_{5/2}(\mathbf{y}^2)$$

where $\chi_{(b-1)/2}(\mathbf{y}^2) = \dfrac{\mathbf{y}^b-\mathbf{y}^{-b}}{\mathbf{y}-\mathbf{y}^{-1}}$

Nonprimitive Dynamics

Example: 3-node chain quivers



$$\mathcal{I}(\mathcal{Q}_n^{k,l};\, \mathbf{y}) = \sum_{p|n} \mu(p) \cdot \Omega(\mathcal{Q}_{n/p}^{k,l};\, \mathbf{y}^p) \cdot \frac{\mathbf{y} - \mathbf{y}^{-1}}{p\left(\mathbf{y}^p - \mathbf{y}^{-p}\right)}$$

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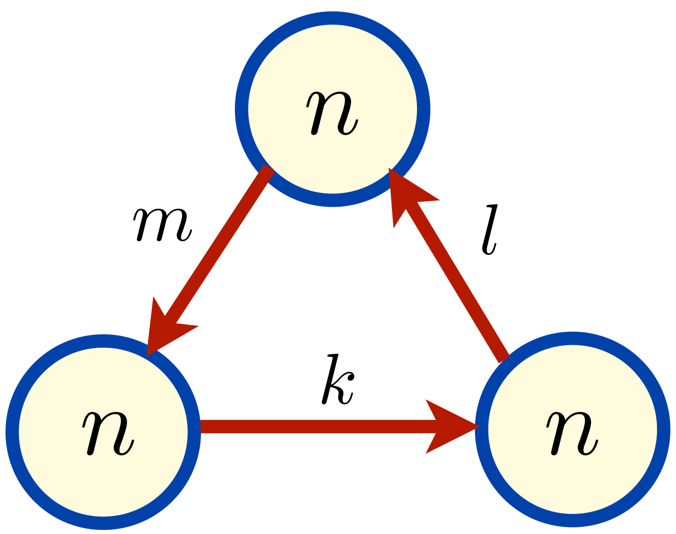
Nonprimitive Dynamics

Example: triangle quivers

$$\mathcal{I}(\mathcal{Q}_n^{k,l,m};\,\mathbf{y}) = \sum_{p|n} \mu(p)\cdot\Omega(\mathcal{Q}_{n/p}^{k,l,m};\,\mathbf{y}^p)\cdot\frac{\mathbf{y}-\mathbf{y}^{-1}}{p\left(\mathbf{y}^p-\mathbf{y}^{-p}\right)}$$



$$\mathcal{I}(\mathcal{Q}_2^{1,1,-1};\,\mathbf{y}) = 0$$

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Summary and Outlook

- *Witten index*($\mathcal{I}$) of susy QMs can address bound state problems.
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Thank You!