SUSY Partition Functions and Higher Dimensional A-twist

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Based on
arXiv:1605.06531 with Cyril Closset
arXiv:1701.03171 with Cyril Closset and Brian Willett
arXiv:1707.nnnnn with Cyril Closset and Brian Willett

A-twist in Two Dimensions

- Consider an $\mathcal{N}=(2,2)$ theory on on $\mathbf{R}^2$ with R-symmetry $U(1)_V \times U(1)_A$

- It can be put on a Σ_g by [Witten 91]

$$U(1)_{E_A} = U(1)_E + U(1)_V \text{ A-twist}$$

$$U(1)_{E_B} = U(1)_E + U(1)_A \text{ B-twist}$$

- We can insert the half-BPS operators $\phi_{A,B}$ defined by

$$\{\widetilde{\mathcal{Q}}_+, \phi_A\} = \{\mathcal{Q}_-, \phi_A\} = 0$$

$$\{\widetilde{\mathcal{Q}}_+, \phi_B\} = \{\widetilde{\mathcal{Q}}_-, \phi_B\} = 0$$

  which form a (twisted) chiral ring: $\phi_i \phi_j = C_{ij}{}^k \phi_k$.

- The (A-) B-twisted theory only depends on the (twisted) F-term.

A-twist in Higer Dimensions

- Uplifting A-twisted theory to *3d* and *4d*

$$2d\ \mathcal{N}=(2,2) \xrightarrow[S^1]{} 3d\ \mathcal{N}=2 \xrightarrow[S^1]{} 4d\ \mathcal{N}=1$$

- *3d N=2* and *4d N=1* theories can be viewed as A-twisted theories with infinitely many KK modes.

- This point of view allows us to study higher dimensional theories defined on a large class of manifolds with different geometries.

- The twisted chiral rings in 2d uplift to the co-dimension two defects in higher dimensions. It provides a natural framework to study the algebra of these extended operators.

Three-dimensional SUSY Background

When can we define *3d N=2* theories on a curved space $\mathcal{M}_3$?

- When we have $U(1)_R$, we preserve two supersymmetries if $\mathcal{M}_3$ has a *U*(1) isometry [Closset-Dumitrescu-Festuccia-Komargodski 13]

$$K^\mu = \zeta\sigma^\mu\widetilde{\zeta}$$

- These manifolds are S^1 bundle over an orbifold. In this talk, we focus on a class of such manifolds with smooth base, labeled by two integers:

$$\boxed{S^1 \longrightarrow \mathcal{M}_{g,p} \longrightarrow \Sigma_g}$$

with metric $ds^2 = (d\phi + C(z,\bar{z}))^2 + 2g_{z\bar{z}}dzd\bar{z}$, $\qquad \frac{1}{2\pi}\int_{\Sigma_g} dC = p \in \mathbb{Z}$

This theory can be understood as a pull-back of the A-twist along the base Σ_g.

Partition Functions and Indices

- In this talk, we will write down the supersymmetric partition function

$Z[\mathcal{M}_{g,p}]$ of *3d N=2* theories

- This can be easily uplifted to *4d N=1* theories on $\mathcal{M}_{g,p_1,p_2}$

$$T^2 \longrightarrow \mathcal{M}_{g,p_1,p_2} \to \Sigma_g$$

With the *SL(2,Z)* action on the torus, it defines a *generalized index*

$$\boxed{\begin{aligned} &Z[\mathcal{M}_{g,p} \times S^1] = e^{-\beta E_{\mathcal{M}_{g,p}}} I[\mathcal{M}_{g,p}] \\ &\text{where } I[\mathcal{M}_{g,p}] = \mathrm{Tr}_{\mathcal{M}_{g,p}} \left[(-1)^F q^{2J+R} \prod_\alpha y_\alpha^{Q^\alpha} \right] \end{aligned}}$$

This quantity generalize the superconformal index

$$I_{S^3}(p,q,y) = \mathrm{Tr}_{S^3} \left[(-1)^F p^{J_3+J_3'+\frac{1}{2}R} q^{J_3-J_3'+\frac{1}{2}R} \prod_\alpha y_\alpha^{Q^\alpha} \right]$$

at $p=q$ limit. [Romelsberger 05]

Special Examples of $Z[\mathcal{M}_{g,p}\ (\times S^1)]$

- Chern-Simons theory on $\Sigma_g \times S^1$: Verlinde formula [Witten 86][Verlinde 88] [Blau-Thompson 93]

- $Z[S^3]$ [Kapustin-Willett-Yaakov 09]

- Topologically twisted indices $Z[\Sigma_g \times S^1],\ Z[\Sigma_g \times T^2]$ [Closset-HK 16] [Benini-Zaffaroni 15,16]

- Superconformal index $Z[S^3 \times S^1]$ [Romelsberger 05]

This framework will allow us to write down these results in a uniform way and show how they are related to each other.

$\mathcal{M}_{g,p}$ background is also considered by [Ohta-Yoshida 13][Nishioka-Yaakov 14]. The results do not agree with ours. For CS theory, the result reduces to the formula computed in [Blau-Thompson 06][Kallen 11]

Two-dimensional GLSM

After A-twisting,

- Vector multiplet $\mathcal{V} = (\sigma, \lambda, \tilde{\lambda}, \Lambda_z, \tilde{\Lambda}_{\bar{z}}, D + iF_{z\bar{z}})$ with gauge group $\mathbf{G}$
- Chiral multiplet $\Phi = (\phi, \psi_\pm, F)$ in a representation $\mathcal{R}$ of $\mathfrak{g}$
- Superpotential $\mathcal{W}(\Phi)$
- Twisted Superpotential $\widetilde{\mathcal{W}}(\Sigma)$,

  including FI term $\widetilde{\mathcal{W}}_{\mathrm{FI}}(\Sigma) = \sum_a t^a \Sigma_a$

Let's consider the Coulomb branch where we have $\mathbf{G} \longrightarrow U(1)^{\mathrm{rk}(\mathbf{G})}$

Coulomb Branch of GLSM

- Coulomb branch is parametrized by complex scalars $\sigma_{a=1,\cdots,\mathrm{rk}(\mathbf{G})}$.

  In addition, we have a quantized flux $\frac{1}{2\pi i}\int_{\Sigma_g}\sqrt{g}f^a_{z\bar{z}} = \mathfrak{m}^a \in \mathbf{Z}$.

- In this background, the low energy effective action can be written as

$$\boxed{\mathcal{S}_{\mathrm{eff}} = \int_{\Sigma_g}\left(f_a\frac{\partial\mathcal{W}}{\partial\sigma_a} + \widetilde{\Lambda}^a\Lambda_b\frac{\partial\mathcal{W}}{\partial\sigma_a\partial\sigma_a}\right) + \int_{\Sigma} d^2x\sqrt{g}R\Omega(\sigma) + \mathcal{Q}(\cdots)}$$

Note that it only depends on the *twisted effective superpotential* on $\mathbf{R}^2$:

$$\mathcal{W} = \mathcal{W}_{\mathrm{FI}} - \frac{1}{2\pi i}\sum_\rho(\rho(\sigma)+m)\left(\log[\rho(\sigma)+m]-1\right) - \frac{1}{2}\sum_{\alpha\in\mathfrak{g}_+}\alpha(\sigma))$$

and the *dilaton effective action*:

$$\Omega = -\frac{1}{2\pi i}\sum_\rho(r-1)\log\left(\rho(\sigma)+m\right) - \frac{1}{2\pi i}\sum_{\alpha\in\mathfrak{g}_+}\log\alpha(\sigma)$$

which tells us how the theory couples to the non-trivial curvature. [Witten 93] [Nekrasov-Shatashvilli 14]

Coulomb Branch and Vacua

- The quantum vacua of this theory is given by

$$\boxed{\exp\left(2\pi i \frac{\partial \mathcal{W}}{\partial \sigma_a}\right) = 1, \quad \sigma_a \neq \sigma_b \ (a \neq b)}$$

which we call the *Bethe equation*. For $\mathcal{N} = (2,2)$ theory, it gives the Bethe equation for the corresponding integrable system (XXX spin-chain). [Nekrasov-Shatashvilli 09]

- For $g > 0$, the Coulomb branch has a singularity where the non-abelian gauge symmetry enhances. We will discard these solutions fixed by the Weyl symmetry.

- Example: $\mathbf{G} = U(N)$ with L fundamental multiplets:

$$\prod_{i=1}^{L} \frac{\sigma_a - m_i + z/2}{\sigma_a - m_i - z/2} = e^{2\pi i t} \prod_{j \neq i} \frac{\sigma_i - \sigma_j + z}{\sigma_i - \sigma_j - z}$$

Computation of the partition function

Let's go back to the effective action

$$\mathcal{S}_{\text{eff}} = \int_{\Sigma_g} \left(f_a \frac{\partial \mathcal{W}}{\partial \sigma_a} + \widetilde{\Lambda}^a \Lambda_b \frac{\partial \mathcal{W}}{\partial \sigma_a \partial \sigma_a} \right) + \int_{\Sigma} d^2x \sqrt{g} R \Omega(\sigma) + \mathcal{Q}(\cdots)$$

Integrating out the zero modes for $\Lambda^a_z, \tilde{\Lambda}^a_{\bar{z}}$ and evaluating at the solution of the Bethe equation, we get

$$\boxed{Z_g = \sum_{\hat{\sigma} \in \mathcal{S}_{\text{BE}}} \mathcal{H}(\hat{\sigma})^{g-1} \prod_{a \in \text{flavour}} \Pi_a(\hat{\sigma})^{\mathfrak{n}_F}}$$

$$\mathcal{H}(\sigma) = e^{2\pi i \Omega(\sigma)} H(\sigma)$$

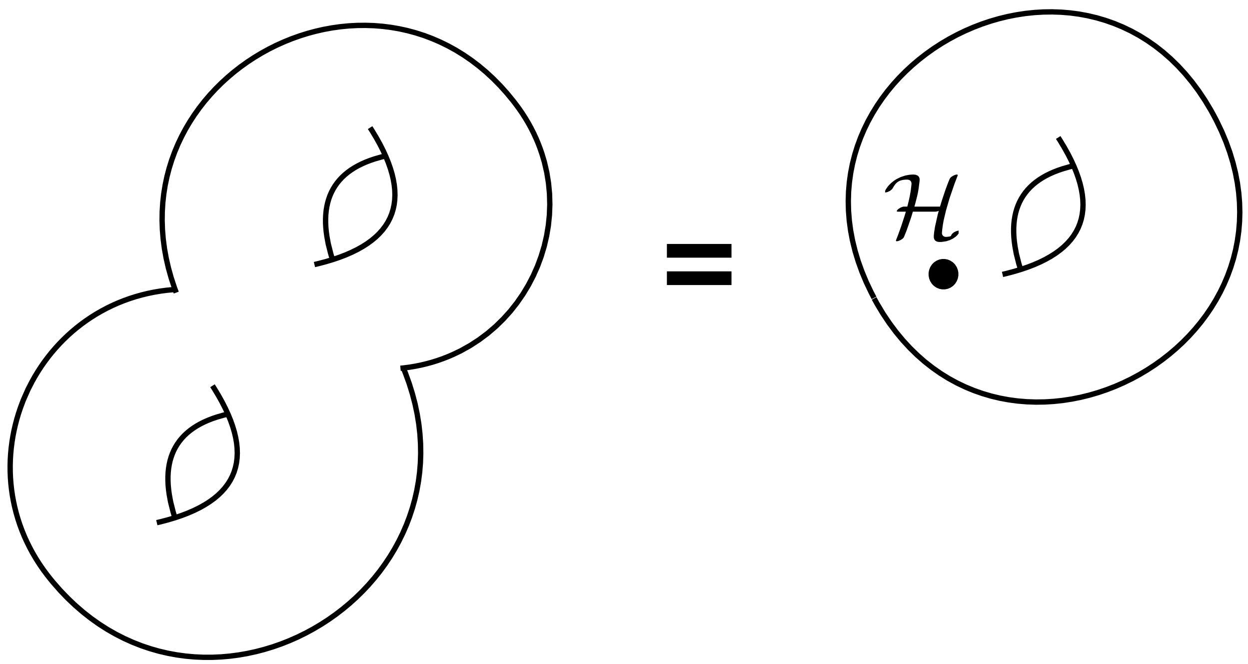
$$H(\sigma) = \det_{ab} \left(2\pi i \frac{\partial \mathcal{W}}{\partial \sigma_a \partial \sigma_b} \right) = \det_{ab} \left(\sum_{\rho} \frac{\rho^a \rho^b}{\rho(\sigma) + m} \right)$$

$$\Pi^a = \exp \left(2\pi i \frac{\partial \mathcal{W}}{\partial \sigma^F_a} \right)$$

This result was first obtained by [Vafa 91] for Landau-Ginzburg model, and later by [Melnikov-Plesser 05] for 2d GLSM with massive vacua. When $g = 0$, the full path integral derivation is given by [Closset-Cremonesi-Park 15] with Omega deformation.

Handle-gluing Operator

$$Z_g = \sum_{\hat{\sigma} \in \mathcal{S}_{\mathrm{BE}}} \mathcal{H}(\hat{\sigma})^{g-1} \prod_{a \in \mathrm{flavour}} \Pi_a(\hat{\sigma})^{\mathfrak{n}_F}$$



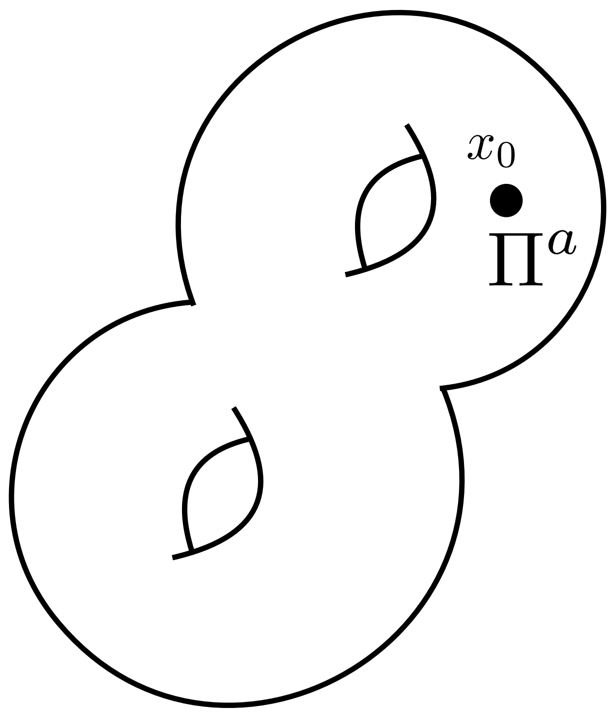
- Add one handle $g \to g+1$

$$\mathcal{H}(\sigma) = e^{2\pi i \Omega(\sigma)} H(\sigma)$$

$$H(\sigma) = \det_{ab}\left(2\pi i \frac{\partial \mathcal{W}}{\partial \sigma_a \partial \sigma_b}\right) = \det_{ab}\left(\sum_{\rho} \frac{\rho^a \rho^b}{\rho(\sigma)+m}\right)$$

Flux Operator

$$Z_g = \sum_{\hat{\sigma} \in \mathcal{S}_{\text{BE}}} \mathcal{H}(\hat{\sigma})^{g-1} \prod_{a \in \text{flavour}} \Pi_a(\hat{\sigma})^{\mathfrak{n}_F}$$



$$f^a_{z\bar{z}} = 2\pi i \; \mathfrak{n}^a_F \delta^2(x - x_0)$$

- Add one unit of flux $\mathfrak{n}^a_F \to \mathfrak{n}^a_F + 1$

$$\Pi^a = \exp\left(2\pi i \frac{\partial \mathcal{W}}{\partial \sigma^F_a}\right)$$

Correlation Functions

- One can insert a local BPS operator

$$\langle \mathcal{O}(\sigma) \rangle = \sum_{P(\hat{\sigma})=0} \mathcal{O}(\hat{\sigma}) \mathcal{H}(\hat{\sigma})^{g-1} \prod_{a \in \text{flavour}} \Pi_a(\hat{\sigma})^{\mathfrak{n}_F}$$

It follows that the correlation function satisfy

$$\boxed{\langle \mathcal{O}(\sigma) P(\sigma) \rangle = 0}$$

which gives the quantum chiral ring relation.

- After the 3d uplift, this idea can be used to find the Wilson loop algebra and the duality actions on the loop operators.

3d Theories on $\mathbf{R}^2 \times S^1$

- Consider the *3d N=2* theories on a circle. The classical coulomb branch is parameterized by

$$u_a = i\beta(\sigma_a + i a_a^0) \ \in \mathbf{C}^*, \quad a = 1, \cdots \mathrm{rk}(\mathbf{G})$$

- *3d* twisted effective superpotential can be obtained by summing over all Kaluza-Klein modes. We have

$$\mathcal{W}_{\mathrm{3d}} = \frac{1}{2} k^{ab} u_a u_b + \frac{1}{24} k_g + \frac{1}{(2\pi i)^2} \sum_\rho \mathrm{Li}_2(e^{2\pi i \rho(u)})$$

- Note that $\mathcal{W}_{3d}$ suffers from the branch cut ambiguity

$$\mathcal{W}_{3d}(\sigma) \to \mathcal{W}_{3d}(\sigma) + n^a \sigma_a + m^a, \ n^a, m^a \in \mathbf{Z}$$

- The dilaton effective action is

$$\Omega_{3d} = \sum_a k_{aR} u_a + \frac{1}{2} k_{RR} - \frac{1}{2\pi i}(r-1) \sum_\rho \log(1 - e^{2\pi i u}) - \frac{1}{2\pi i} \sum_{\alpha \in \mathfrak{g}} \log(1 - x^\alpha)$$

3d Theories on $\Sigma_g \times S^1$

- With this information, we can write down the full correlation function for the Wilson loops in $\Sigma_g \times S^1$:

$$\boxed{\langle W(x) \rangle_{\Sigma_g \times S^1} = \sum_{\hat{x} \in \mathcal{S}_{\text{BE}}} W(x) \mathcal{H}(\hat{x})^{g-1} \prod_{F \in \text{flavour}} \Pi_a(\hat{x})^{\mathfrak{n}_F}}$$

with

$$\mathcal{H}(x) = e^{2\pi i \Omega(u)} \det_{ab} \left(\frac{\partial^2 \mathcal{W}_{3d}}{\partial u_a \partial u_b} \right)$$

$$\Pi_a(x) = \exp\left(2\pi i \frac{\partial \mathcal{W}_{3d}}{\partial u_F^a} \right)$$

3d Theories on $\mathcal{M}_{g,p}$

- Let us consider the manifold with a non-trivial fiberation

$$S^1 \longrightarrow \mathcal{M}_{g,p} \longrightarrow \Sigma_g$$

with metric $ds^2 = (d\phi + C(z,\bar{z}))^2 + 2g_{z\bar{z}}dzd\bar{z}\,,\qquad \frac{1}{2\pi}\int_{\Sigma_g} dC = p \in \mathbb{Z}$

- In the two-dimensional point of view, this theory has an additional flavour symmetry, which is the $U(1)_{KK}$. We can turn on the background twisted vector multiplet for this symmetry, whose lowest component is C_μ. The twisted mass is given by

$$m_{KK} = \frac{1}{\beta}$$

- We will call the flux operator for $U(1)_{KK}$ the *Fibering operator*:

$$\boxed{\mathcal{F} = \exp\left(2\pi i \frac{\partial}{\partial m_{KK}}\left(m_{KK}\mathcal{W}_{3d}\right)\right)}$$

Fibering Operator

- More explicitly,

$$\mathcal{F} = \exp\left(2\pi i\left(\mathcal{W} - u_a\frac{\partial\mathcal{W}}{\partial u_a} - m_i\frac{\partial\mathcal{W}}{\partial m_i}\right)\right)$$

- A chiral multiplet contributes

$$\mathcal{F}_\Phi(u) = \exp\left(\frac{1}{2\pi i}\mathrm{Li}_2\left(e^{2\pi i u}\right) + u\log(1 - e^{2\pi i u})\right)$$

to $\mathcal{F}$. The fibering operator satisfies the difference equation

$$\mathcal{F}(u-1) = \Pi(u)\mathcal{F}(u)$$

- Now we can write down the full $\mathcal{M}_{g,p}$ partition function with Wilson loops:

$$\boxed{\langle W(u)\rangle_{\mathcal{M}_{g,p}} = \sum_{\hat{u}\in\mathcal{S}_{BE}} W(\hat{u})\mathcal{F}(\hat{u})^p\mathcal{H}(\hat{u})^{g-1}\prod_{a\in\text{flavour}}\Pi_a(\hat{u})^{\mathfrak{n}_a^F}}$$

[Closset-HK-Willett 17]

$\mathcal{M}_{g,p}$ Partition Function

$$\boxed{\langle W(u)\rangle_{\mathcal{M}_{g,p}} = \sum_{\hat{u}\in\mathcal{S}_{BE}} W(\hat{u})\mathcal{F}(\hat{u})^p\mathcal{H}(\hat{u})^{g-1}\prod_{a\in\text{flavour}}\Pi_a(\hat{u})^{\mathfrak{n}_a^F}}$$

- $\mathfrak{n}_a^F \in \mathbb{Z}_p$ is the flux for the torsion subgroup $\subset H^2(\mathcal{M}_{g,p},\mathbb{Z})$
- S^3 partition function [Kapustin-Willett-Yaakov 09] can be rewritten as

$$Z_{S^3} = \sum_{\hat{u}\in\mathcal{S}_{BE}} \mathcal{F}(\hat{u})\mathcal{H}(\hat{u})^{-1} = \langle\mathcal{F}(u)\rangle_{S^2_A\times S^1}$$

an operator insertion of the twisted index of [Benini-Zaffaroni 15]

Path Integral Derivation of $Z[\mathcal{M}_{g,p}]$

- We can compute the same quantity by honest localization computation of the UV lagrangian. As a result, we get an integral expression of the $Z[\mathcal{M}_{g,p}]$:

$$Z[\mathcal{M}_{g,p}] = \frac{1}{|W_{\mathbf{G}}|} \sum_{\mathfrak{m} \in \mathbb{Z}_p^r} \int_{C^\eta} d^r u \; \mathcal{F}(u)^p \Pi_a(u)^{\mathfrak{m}_a} \Pi_i(u)^{\mathfrak{n}_i} e^{2\pi i (g-1)\Omega(u)} H(u)^g$$

- Note that for p > 0, $u^a = i\beta(\sigma^a + i a_0^a)$ is valued in a complex plane $\mathbb{C}^r$. This is due to the fact that the integrand is invariant under the following large gauge transformation:

$$(u^a, \mathfrak{m}^a) \sim (u^a + 1, \mathfrak{m}^a + p)$$

  This can be gauge-fixed by declaring $\mathfrak{m} \in \mathbb{Z}_p^r$ and integrating over whole complex plane of $u \in \mathbb{C}^r$.

- C^η is the real r-dimensional contour which is given by the “JK-residue integral” similarly to [Benini-Hori-Eager-Tachikawa 13][Hori-HK-Yi 14].

- For $p > 0$, by a judicious choice of η, the contour $\mathcal{C}_\eta$ is continuously deformed to a real line integral under a favourable condition. Especially for $p = 1$, this agrees with the usual expression for the three-sphere partition function computed by [Kapustin-Willett-Yaakov 09]

- When $g - 1 = 0 \bmod p$, R-symmetry bundle trivializes. For such cases, we can relax the integrality condition for the R-charge and continuously vary it. [Jafferis 10][Hama-Hosomich-Lee 10]

Algebra of BPS Wilson loops

- Let us consider the 1/2-BPS Wilson loops with insertion

$$W(x) = \mathrm{Tr}_{\rho \in \mathcal{R}} x^{\rho}$$

- Classically, the algebra of the Wilson loops are given by

$$\mathbf{C}[x_1, x_1^{-1}, \cdots, x_r, x_r^{-1}]^{W_{\mathbf{G}}}$$

- The quantum algebra is conjectured and tested for a few examples in [Kapustin-Willett 13]. The proof directly follows from our formula:

$$\langle W(x) \rangle_{\mathcal{M}_{g,p}} = \sum_{P(\hat{x})=0} W(\hat{x}) \mathcal{F}(\hat{x})^p \mathcal{H}(\hat{x})^{g-1} \Pi(\hat{x})^{\mathfrak{n}}$$
$$\rightarrow \quad \langle W(x) P(x) \rangle_{\mathcal{M}_{g,p}} = 0$$

which tells us that the quantum algebra is

$$\boxed{\mathbf{C}[x_1, x_1^{-1}, \cdots, x_r, x_r^{-1}]^{W_{\mathbf{G}}} / I_P}$$

where I_P is the ideal generated by $P(x)$ with $x_a \neq x_b \; (a \neq b)$. [Closset-HK 16]

3d Dualities

- Seiberg-like dualities $\mathcal{T} = \mathcal{T}_D$ are encoded in the Bethe equations. For example, for the Aharony duality, we have

$$P(x) = C(y) \prod_{i=1}^{N_f} (x - \hat{x}_i) = C(y) \prod_{i=1}^{N_c} (x - \hat{x}_i) \prod_{i=1}^{N_f - N_c} (x - \hat{x}_{D,i}) = P_D(x)$$

- It gives a one-to-one map to the dual vacua $\mathcal{D} : \{\hat{x}\} \to \{\hat{x}_D\}$

- The statement of the dualities for all g, p:

$$\boxed{\begin{aligned} \Pi(\{\hat{x}\}, \nu) &= \Pi_D(\{\hat{x}_D\}, \nu) \\ \mathcal{H}(\{\hat{x}\}, \nu) &= \mathcal{H}_D(\{\hat{x}_D\}, \nu) \\ \mathcal{F}(\{\hat{x}\}, \nu) &= \mathcal{F}_D(\{\hat{x}_D\}, \nu) \end{aligned}}$$

  for all the solution sets $\{\hat{x}; \hat{x}_D\}$.

- We have proved the relations for $\Pi, \ \mathcal{H}$.

- $\mathcal{F} = \mathcal{F}_D$ follows from the identity of the dilogarithms proved in [Ray 91]

Duality Action on Wilson Line

- For the BPS Wilson loops, the statement of the duality is

$$\langle W(\hat{x}) \rangle = \langle W(\{\hat{x}_D\}) \rangle_D$$

- This relation provides a systematic way of obtaining the representation of the dual BPS Wilson loops. [Closset-HK 16]

- For example, $\mathbf{G} = U(3),\ N_f = 5,\ y_i = 1$

$$\begin{aligned}
\square^D + \square &= 5\ , \\
\ydiagram{1,1}^{\,D} + \square^D \otimes \square + \ydiagram{1,1} &= 10\ , \\
\ydiagram{1,1}^{\,D} \otimes \square + \square^D \otimes \ydiagram{1,1} + \ydiagram{1,1,1} &= 10\ , \\
\ydiagram{1,1}^{\,D} \otimes \ydiagram{1,1} + \square^D \otimes \ydiagram{1,1,1} &= 5\ , \\
\ydiagram{1,1}^{\,D} \otimes \ydiagram{1,1,1} &= 1\ .
\end{aligned}$$

One can use this relation iteratively to write down the dual Wilson loop:

$$\square^D = 5 - \square\ , \qquad \ydiagram{1,1}^{\,D} = 10 - 5\,\square + \ydiagram{2}\ , \qquad \cdots$$

4d Theories on $\mathcal{M}_{g,p_1,p_2}$

- All of the discussion so far can be generalized to *4d N=1* theories on four-manifolds labeled by three integers g, p_1, p_2. [Closset-HK-Willett]

$$T^2 \underset{p_1,p_2}{\longrightarrow} \mathcal{M}_{g,p_1,p_2} \longrightarrow \Sigma_g$$

which is realized by turning on the graviphoton background

$$\frac{1}{2\pi}\int_{\Sigma_g} dC_1 = p_1, \quad \frac{1}{2\pi}\int_{\Sigma_g} dC_2 = p_2, \quad p_1, p_2 \in \mathbb{Z}$$

for $U(1)_{KK_1} \times U(1)_{KK_2}$ symmetry.

- The Coulomb branch variable is $u = \tau a_1 - a_2$ with

$$a_1 = \frac{1}{2\pi}\int_{S^1_{\beta_1}} A_\mu dx^\mu, \quad a_{x^2} = \frac{1}{2\pi}\int_{S^1_{\beta_2}} A_\mu dx^\mu$$

- The twisted effective superpotential can be computed by summing over two KK towers. The chiral multiplet contributes

$$\mathcal{W}^{4d}_\Phi = \frac{1}{(2\pi i)^2}\sum_{m\in\mathbb{Z}} \mathrm{Li}_2(e^{2\pi i u} q^m), \quad q = e^{2\pi i \tau}$$

Partition Function on $\mathcal{M}_{g,p_1,p_2}$

- $\mathcal{W}$ can be rewritten as

$$\mathcal{W}_{\Phi}^{4d} = -\frac{u^3}{6\tau} + \frac{u^2}{4} - \frac{u\tau}{12} + \frac{1}{24} + \frac{1}{(2\pi i)^2}\sum_{m=0}^{\infty} \mathrm{Li}_2(e^{2\pi i u} q^m) - \mathrm{Li}_2(e^{-2\pi i u} q^{m+1})$$

- Following the same logic, the partition function can be written as

$$\boxed{Z[\mathcal{M}_{g,p_1,p_2}] = \sum_{\hat{x}\in\mathcal{S}_{BE}} \mathcal{F}_1(\hat{x})^{p_1}\mathcal{F}_2(\hat{x})^{p_2}\mathcal{H}(\hat{x})^{g-1}\prod_a \Pi_a(\hat{x})^{\mathfrak{n}_a}}$$

$$\mathcal{F}_1(u) = \exp\left(2\pi i \frac{\partial \mathcal{W}}{\partial \tau}\right) \qquad \mathcal{H}(u) = e^{2\pi I \Omega(u)} \det_{ab}\left(\frac{\partial^2 \mathcal{W}}{\partial u_a \partial u_b}\right)$$

$$\mathcal{F}_2(u) = \exp\left(2\pi i\left(\mathcal{W} - u_a \frac{\partial \mathcal{W}}{\partial u_a}\right)\right) \qquad \Pi_F(u) = \exp\left(2\pi i \frac{\partial \mathcal{W}}{\partial u_a^F}\right)$$

- The Bethe equation in 4d is an elliptic equation. For example, we have

$$\prod_{i=1}^{N_f} \frac{\theta(x^{-1}y_i;q)}{\theta(x\widetilde{y}_i;q)} = 1$$

for the SQCD with Sp gauge group. The well-definedness of this equation comes from the gauge anomaly cancellation condition.

Modular Transformation

- Note that the new formula is written in terms of u, ν valued in tori, it has a well defined modular transformation under $SL(2, \mathbb{Z})$:

$$S : (u, \tau) \to \left(\frac{u}{\tau}, -\frac{1}{\tau} \right) \ , \quad T : (u, \tau) \to (u, \tau + 1)$$

- This is distinguished feature compared to the usual superconformal index, which only had a real holonomy variable.

Modular Transformation

- Each contribution of the partition function transforms nicely under the modular transformation ($S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\widetilde{T} \equiv CSTS = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$):

$$
\begin{aligned}
S[\mathcal{F}_1] &= e^{\frac{\pi i}{3\tau}\mathcal{A}^{\mathbf{abc}}\mathbf{u_a u_b u_c}}\,\mathcal{F}_2{}^{-1}\,, & \widetilde{T}[\mathcal{F}_1] &= \mathcal{F}_1\mathcal{F}_2\,,\\
S[\mathcal{F}_2] &= e^{-\frac{\pi i}{3\tau^2}\mathcal{A}^{\mathbf{abc}}\mathbf{u_a u_b u_c}}\,\mathcal{F}_1\,, & \widetilde{T}[\mathcal{F}_2] &= \mathcal{F}_2\,,\\
S[\Pi_{\mathbf{a}}] &= e^{\frac{\pi i}{2}\mathcal{A}^{\mathbf{a}}}e^{\frac{\pi i}{\tau}\mathcal{A}^{\mathbf{abc}}\mathbf{u_b u_c}}\,\Pi_{\mathbf{a}}\,, & \widetilde{T}[\Pi_{\mathbf{a}}] &= e^{-\frac{\pi i}{6}\mathcal{A}^{\mathbf{a}}}\,\Pi_{\mathbf{a}}\,,\\
S[\mathcal{H}] &= e^{\frac{\pi i}{2}\mathcal{A}^{R}}e^{\frac{\pi i}{\tau}\mathcal{A}^{R\mathbf{bc}}\mathbf{u_b u_c}}\,\mathcal{H}\,, & \widetilde{T}[\mathcal{H}] &= e^{-\frac{\pi i}{6}\mathcal{A}^{R}}\,\mathcal{H}\,,
\end{aligned}
$$

where $\mathcal{A}^{abc} = \sum_I Q_I^a Q_I^b Q_I^c, \quad \mathcal{A}^a = \sum_I Q_I^a$

are the *anomaly coefficients* of the theory. Since they vanish for the gauge fugacity, the $SL(2,\mathbb{Z})$ transformation of the full partition function is well defined, and only depends on the t'Hooft anomaly of the theory.

- With this transformation, one can always map $(p_1, p_2) \to (p, 0)$ which defines an index on $\mathcal{M}_{g,p} \times S^1$.

Integral formula for $Z[\mathcal{M}_{g,p} \times S^1]$

- We can derive the same quantity by a direct localization computation. The final formula reads

$$Z[\mathcal{M}_{g,p} \times S^1] = \frac{1}{|W_{\mathbf{G}}|} \sum_{\mathfrak{m} \in \mathbb{Z}_p^r} \int_{C^\eta} d^r u \; \mathcal{F}_1(u)^p \Pi_a(u)^{\mathfrak{m}_a} \Pi_i(u)^{\mathfrak{n}_i} e^{2\pi i (g-1)\Omega(u)} H(u)^g$$

- For $g = 0, p = 1$, we can show that it reduces to the SCI in the limit $p = q$ with a unit circle contour integral.

Casimir Energy

- For $\mathcal{M}_{g,p} \times S^1$, the partition function can be written as

$$Z[\mathcal{M}_{g,p} \times S^1](\nu,\tau) = e^{2\pi i \tau \mathcal{E}_{\mathcal{M}_{g,p}}(\nu,\tau)} \mathcal{I}_{\mathcal{M}_{g,p}}(\nu,\tau)$$

where $\mathcal{I}_{\mathcal{M}_{g,p}}$ is generalized index,

$$\mathcal{I}_{\mathcal{M}_{g,p}}(\nu,\tau) = \mathrm{Tr}_{\mathcal{M}_{g,p}}\left[(-1)^F q^{2J+R} \prod_a y_a^{Q_a}\right] = \mathcal{I}^{(0)}(\nu) + \mathcal{O}(q)$$

and $\mathcal{E}_{\mathcal{M}_{g,p}}$ is the supersymmetric Casimir energy [Lorenzen-Martelli 15][Assel-Cassini-Di Pietro-Komargodski-Lorenzen-Martelli 15]. We find

$$\boxed{\begin{aligned} \mathcal{E}_{\mathcal{M}_{g,p}}(\nu;\tau) = \; & p\left(\frac{\mathcal{A}^{\alpha\beta\gamma}}{6\tau^3}\nu_\alpha\nu_\beta\nu_\gamma - \frac{\mathcal{A}^\alpha}{12\tau}\nu_\alpha\right) \\ & -(g-1)\left(\frac{\mathcal{A}^{\alpha\beta R}}{2\tau^2}\nu_\alpha\nu_\beta + \frac{\mathcal{A}^R}{12}\right) - \mathfrak{n}_\alpha\left(\frac{\mathcal{A}^{\alpha\beta\gamma}}{2\tau^2}\nu_\beta\nu_\gamma + \frac{\mathcal{A}^\alpha}{12}\right) \end{aligned}} \; .$$

Note that it is determined by anomalies of the theory. For the three sphere case, it agrees with the observation in [Bobev-Bullimore-Kim 15]

Cardy Formula

- We can study the expression for the $q \to 1$ limit, which corresponds to the three-dimensional limit. We first take the modular transformation then take the $q \to 0$ limit. We find a universal expression

$$\boxed{\log Z[\mathcal{M}_{g,p} \times S^1] = -\frac{2\pi i}{\tau}\left((1-g)\frac{\mathcal{A}^R}{12} + \frac{\mathcal{A}^\alpha}{12}\left(p\frac{\nu_\alpha}{\tau} - \mathfrak{n}_\alpha\right)\right) + \mathcal{O}(\beta_1^0)}$$

- This agrees with the formula

$$\log Z[\mathcal{M}_{g,p} \times S^1] = -\frac{\pi \mathrm{Tr}(R)}{24\beta_2} L_{M_{g,p}} + \mathcal{O}(\beta_2^0)$$
$$L_{M_{g,p}} = 4\beta_1(1-g)$$

given by [Di Pietro-Komargodski 14].

Witten Index of SQCD

- $\mathcal{M}_{1,0} \times S^1 = T^4$ partition function computes the Witten index of the theory. The expression reduces to

$$Z[T^4] = \sum_{P(x)=0} 1$$

- The number of solutions of the Bethe equation gives the Witten index.

- For SQCD with $\mathbf{G} = USp(2N_c),\ 2N_f$ flavours, we find

$$Z[T^4] = \binom{N_f - 2}{N_c}$$

- For SQCD with $\mathbf{G} = SU(N_c),\ N_f$ flavours, we *conjecture*

$$Z[T^4] = \binom{N_f - 2}{N_c - 1}$$

at *generic* fugacities. This formula agrees with the Seiberg dualities.

Summary

- We derived generalized supersymmetric partition functions of *3d N=2* theories and *4d N=1* theories, which uncover the relation between correlation functions on manifolds with different topologies.
- In the A-model point of view, the expression can be written as a sum over the Bethe vacua.
- The localization computation gives an integral expression, which turns out to be equivalent to the first expression.
- In 3d, it provides a useful tool to study the dualities and the algebra of the half-BPS Wilson loops.
- In 4d, the index has a well-defined modular transformation, Casimir energy and Cardy formula determined by the anomalies of the theory.
- We can compute the Witten index for 3d CS-YM-Matter theory and 4d SQCD, and checked that the Seiberg dualities holds for all g,p.

Future Directions

- What is the algebra of the surface operators in 4d? [work in progress]
- Relation between Casimir energy and anomaly, phases of the 4d partition functions [work in progress]
- Can we generalize this story to the most general Seifert manifold?
- Application to the 3d-3d correspondence?
- How can we introduce the squashing (for genus 0) in this story?
- What is the Hilbert space interpretation of the generalized index?