Modular Constraints on Conformal Field Theories with Currents

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Geometry of String and Gauge Theories
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- Two-dimensional CFTs with $c \geq 1$

• The two-dimensional CFTs with $c < 1$ are classified by minimal model. The CFT data(spectrum and OPE coefficients) for these models were analyzed.

• The situation of $c \geq 1$ CFTs different, because no degenerate state appears in the unitary irreducible representation.

• Examples of two-dimensional CFTs : Minimal models, Liouville theory, WZW models, etc.

• We consider two-dimensional CFTs on the torus, and investigate how the modular property constrains the structure of partition function.

• GOAL :

Modular Differential Equation $\bigoplus$ Modular Bootstrap	$\Longrightarrow$	Spectrum analysis, Find the modular invariant partition function

- The Character Decomposition
  - The Virasoro characters are defined by

$$\chi_0(\tau) = \frac{1}{\eta(\tau)} q^{-\frac{c-1}{24}}(1-q), \quad \chi_h(\tau) = \frac{1}{\eta(\tau)} q^{h-\frac{c-1}{24}}$$

For convenience, we mainly use the reduced character.

$$\hat{\chi}_0(\tau) = \tau^{\frac{1}{4}}\eta(\tau)\chi_0(\tau), \quad \hat{\chi}_h(\tau) = \tau^{\frac{1}{4}}\eta(\tau)\chi_h(\tau)$$

The torus partition function of unitary CFT admit the character decomposition,

$$\mathcal{Z}(\tau,\bar{\tau}) = \chi_0(\tau)\bar{\chi}_0(\bar{\tau}) + \sum_{h,\bar{h}} d(h,\bar{h})\chi_h(\tau)\bar{\chi}_{\bar{h}}(\bar{\tau}) + \sum_{j=1}^{\infty}\left[d(j)\chi_j(\tau)\bar{\chi}_0(\bar{\tau}) + \tilde{d}(j)\chi_0(\tau)\bar{\chi}_j(\bar{\tau})\right].$$

- Constraint from the modular invariance
  - $\mathcal{S}$- transformation : $\mathcal{Z}(\tau,\bar{\tau}) = \mathcal{Z}(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}})$

$$\boxed{\hat{\mathcal{G}}_0(\tau,\bar{\tau}) + \sum_{j=1}^{\infty}\left[d(j)\hat{\mathcal{G}}_j(\tau,\bar{\tau}) + \tilde{d}(j)\hat{\mathcal{G}}_{\bar{j}}(\tau,\bar{\tau})\right] + \sum_{h,\bar{h}} d(h,\bar{h})\hat{\mathcal{G}}_{h,\bar{h}}(\tau,\bar{\tau}) = 0}$$

where the function $\hat{\mathcal{G}}_\lambda(\tau,\bar{\tau})$ is defined as $\hat{\chi}_\lambda(\tau)\hat{\bar{\chi}}_\lambda(\bar{\tau}) - \hat{\chi}_\lambda(-\frac{1}{\tau})\hat{\bar{\chi}}_\lambda(-\frac{1}{\bar{\tau}})$.

- The Modular Bootstrap Equation(MDE)
  - Idea : n characters of rational conformal field theory(RCFT) are solutions of n-th order modular differential equation, [Mathur, Muhki, Sen 88]

$$\mathcal{D}_\tau^n \chi(\tau) + \sum_{k=0}^{n-1} \phi_k(\tau) \mathcal{D}_\tau^k \chi(\tau) = 0.$$

with $D_\tau \chi(\tau) \equiv \partial_\tau \chi(\tau) - \frac{\pi i r}{6} \chi(\tau)$.

- Second Order Modular Differntial Equation
  - Solve the second order differential equation,

$$D_\tau^2 \chi(\tau) + \hat{\mu} E_4(\tau) \chi(\tau) = 0,$$

with an ansatz $\chi_{\hat{\lambda}}(q) = q^\alpha (a_0 + a_1 q + a_2 q^2 + a_3 q^3 + a_4 q^4 + \cdots)$.

  - The coefficients are positive integer only for [Mathur, Muhki, Sen 88], [Tuite 08]

$$\boxed{c \in \{\frac{2}{5}, 1, 2, \frac{14}{5}, 4, \frac{26}{5}, 6, 7, \frac{38}{5}, 8\}}.$$

- Third Order Modular Differntial Equation
  - Solve the third order differential equation

$$D_\tau^3\chi(\tau) + \mu_1 E_4(\tau)D_\tau\chi(\tau) + \mu_2 E_6(\tau)\chi(\tau) = 0,$$

with an ansatz $\chi_{\hat\lambda}(q) = q^\alpha(a_0 + a_2q^2 + a_3q^3 + a_4q^4 + \cdots)$.

  - Each coefficients are positive integer only for [Mathur, Muhki, Sen 88], [Tuite 08]

$$\boxed{c \in \{-\frac{44}{5}, 8, 16, \frac{47}{2}, 24, 32, \frac{164}{5}, \frac{236}{7}, 40\}}.$$

  - The primary characters have the form of

$$\chi_{h_\pm}(\tau) = q^{h_\pm - \frac{c}{24}}\left[b_0 + b_1q + b_2q^2 + \cdots\right]$$

with $h_\pm(c) = \frac{c+4}{16} \pm \frac{\sqrt{368+24c-c^2}}{16\sqrt{31}}$.

  - The structure of primary characters is *undetermined* from the modular differential equation.

- Modular Bootstrap - Basic Strategy [Rattazzi, Rychkov, Tonni, Vichi 08], [Poland, Simmons-Duffin 10]
  - Apply the linear functional $\alpha\left[\hat{\mathcal{G}}(z,\bar{z})\right] \equiv \sum_{m,n}^{m+n=N} \alpha_{m,n}\partial_z^m\partial_{\bar{z}}^n\hat{\mathcal{G}}(z,\bar{z})$ to the modular bootstrap equation. ($\tau \equiv i\ e^z$, the crossing point : $z=0$)

$$\alpha\left[\hat{\mathcal{G}}_0(z,\bar{z})\right] + \sum_{j=1}^{j_{max}}\left(d(j)\ \alpha\left[\hat{\mathcal{G}}^j(z,\bar{z})\right] + \bar{d}(j)\ \alpha\left[\hat{\mathcal{G}}^{\bar{j}}(z,\bar{z})\right]\right) + \sum_{h,\bar{h}\in\mathcal{P}} d(h,\bar{h})\ \alpha\left[\hat{\mathcal{G}}^{h,\bar{h}}(z,\bar{z})\right] = 0.$$

  - Find $\alpha_{m,n}$ such that,

$$\begin{aligned} &\alpha\left[\hat{\mathcal{G}}_0(z,\bar{z})\right] > 0, \\ \text{and} \quad &\alpha\left[\hat{\mathcal{G}}^j(z,\bar{z})\right] \geq 0,\ \alpha\left[\hat{\mathcal{G}}^{\bar{j}}(z,\bar{z})\right] \geq 0 \quad \text{for} \quad j \in \mathbb{Z}, \\ \text{and} \quad &\alpha\left[\hat{\mathcal{G}}^{h,\bar{h}}(z,\bar{z})\right] \geq 0 \quad \text{for} \quad (h,\bar{h}) \in \mathcal{P} \end{aligned}$$

If we find such $\alpha_{m,n}$, then we conclude no modular invariant partition function exist.

- Inputs [Collier, Lin, Yin 16]

• Scalar Gap Problem

In this problem, we impose a gap Δ_s only to the scalar operator. Namely,

$$\Delta \geq \Delta_s \text{ for } j = 0, \quad \Delta \geq j \text{ for } j \neq 0.$$

• Maximal Gap Problem

In this problem, we impose a gap Δ_m.

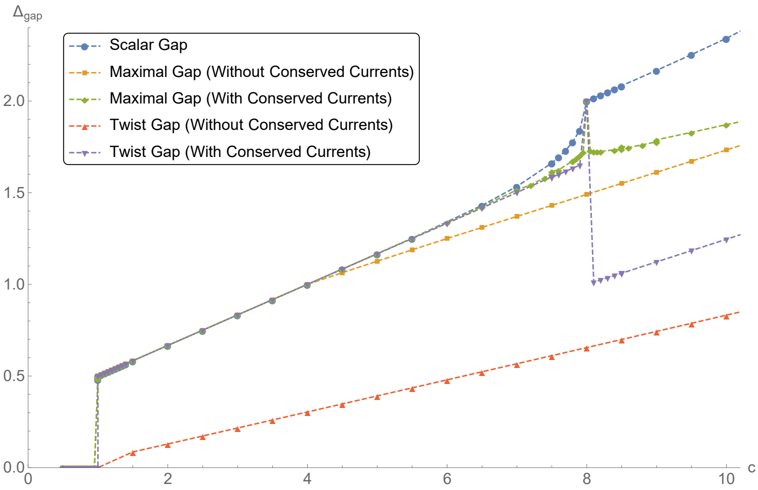
$$\Delta \geq \text{Max}(j, \Delta_m).$$

• Twist Gap Problem

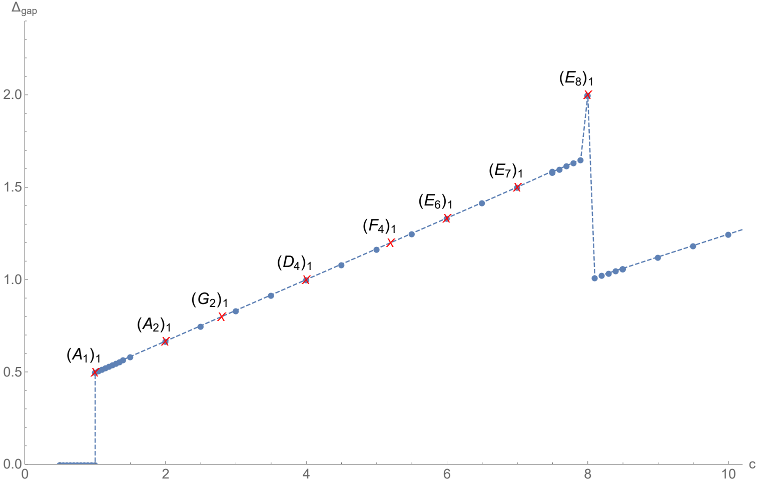
In this problem, we impose a gap Δ_t to the twist, $t \equiv \Delta - j$.

$$\Delta \geq j + \Delta_t.$$

- The Numerical Result ($c \leq 8$)



- The Numerical Result ($c \leq 8$), Twist Gap



- Expected CFTs on the bound (Twist Gap)
  - For Wess-Zumino-Witten model,

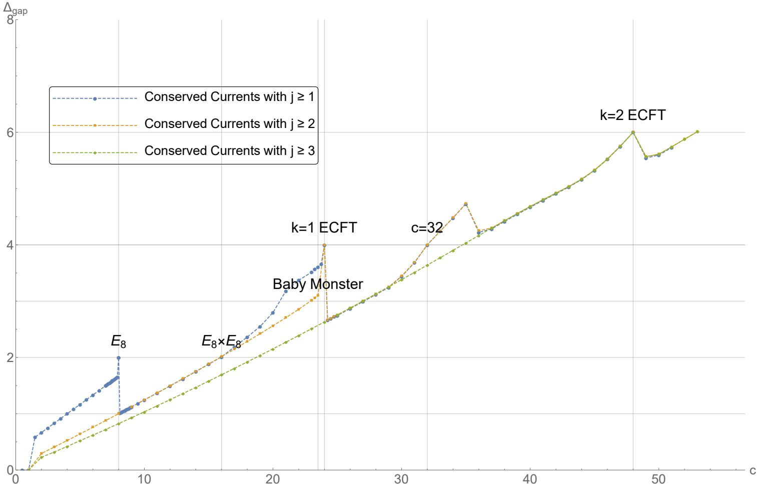
$$c = \frac{k \, \mathrm{dim}\hat{\mathfrak{g}}}{k+g}, \qquad h_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2(k+g)}$$

  - The twist gap problem realize level-1 WZW models on the boundary!

Central Charge	Lowest Primary	Expected CFT
$c = 1$	$\Delta_t = 1/2$	$SU(2)_1$ WZW model
$c = 2$	$\Delta_t = 2/3$	$SU(3)_1$ WZW model
$c = 14/5$	$\Delta_t = 4/5$	$(G_2)_1$ WZW model
$c = 4$	$\Delta_t = 1$	$SO(8)_1$ WZW model
$c = 26/5$	$\Delta_t = 6/5$	$(F_4)_1$ WZW model
$c = 6$	$\Delta_t = 4/3$	$(E_6)_1$ WZW model
$c = 7$	$\Delta_t = 3/2$	$(E_7)_1$ WZW model
$c = 8$	$\Delta_t = 2$	$(E_8)_1$ WZW model

  - They are two-channel RCFTs, solution of the second order MDE.

- The Numerical Result (Twist Gap)



- The Numerical Result (Twist Gap)
  - When the holomorphic currents are included from $j = 1$,

Central Charge	Lowest Primary	Expected CFT
$c = 16$	$\Delta_t = 2$	$(E_8 \times E_8)_1$ WZW model
$c = 24$	$\Delta_t = 4$	Monster CFT
$c = 32$	$\Delta_t = 4$	$k = 4/3$ ECFT
$c = 48$	$\Delta_t = 6$	$k = 2$ ECFT

  - The unique modular invariant partition function at $c = 24$ is,

$$\begin{aligned}\mathcal{Z}_{k=1}(q, \bar{q}) &= (j(q) - 744)(\bar{J}(\bar{q}) - 744) \\ &= (1 + 196884q^2 + \cdots)(1 + 196884\bar{q}^2 + \cdots)\end{aligned}$$

  - When the holomorphic currents are included from $j = 2$,

Central Charge	Lowest Primary	Automorphism
$c = 8$	$\Delta_{gap} = 1$	$2 \cdot O^+(10, 2)$
$c = 16$	$\Delta_{gap} = 2$	$2^{16} \cdot O^+(10, 2)$
$c = 47/2$	$\Delta_{gap} = 3$	Baby Monster

- Finding the degeneracy bound [Rattazzi, Rychkov, Vichi 10]
    - Rewrite the modular bootstrap equation as

$$\alpha\left[\hat{\mathcal{G}}_0(z,\bar{z})\right] + d(h^*,\bar{h}^*)\ \alpha\left[\hat{\mathcal{G}}^{h^*,\bar{h}^*}(z,\bar{z})\right] + \alpha\left[\hat{\mathcal{G}}^{rest}(z,\bar{z})\right] = 0,$$

$$\alpha\left[\hat{\mathcal{G}}^{rest}(z,\bar{z})\right] \equiv \sum_{j=1}^{j_{max}} \left(d(j)\ \alpha\left[\hat{\mathcal{G}}^{j}(z,\bar{z})\right] + \bar{d}(j)\ \alpha\left[\hat{\mathcal{G}}^{\bar{j}}(z,\bar{z})\right]\right) + \sum_{h,\bar{h}\in\mathcal{P}} d(h,\bar{h})\ \alpha\left[\hat{\mathcal{G}}^{h,\bar{h}}(z,\bar{z})\right],$$

and solve the following problem.

$$\begin{aligned} \text{Maximize}\quad & \alpha\left[\hat{\mathcal{G}}_0(z,\bar{z})\right], \quad \text{such that} \quad \alpha\left[\hat{\mathcal{G}}^{h^*,\bar{h}^*}(z,\bar{z})\right] = 1 \\ \text{and}\quad & \alpha\left[\hat{\mathcal{G}}^{j}(z,\bar{z})\right] \geq 0,\ \alpha\left[\hat{\mathcal{G}}^{\bar{j}}(z,\bar{z})\right] \geq 0 \quad \text{for}\ \ j\in\mathbb{Z}, \\ \text{and}\quad & \alpha\left[\hat{\mathcal{G}}^{h,\bar{h}}(z,\bar{z})\right] \geq 0 \quad \text{for}\ \ (h,\bar{h})\in\mathcal{P} \end{aligned}$$

- This gives the maximum bound of the degeneracy of state with $(h^*,\bar{h}^*)$.

$$\boxed{d(h^*,\bar{h}^*) \leq -\alpha\left[\hat{\mathcal{G}}_0(z,\bar{z})\right]}$$

- Extremal Functional Method [Paulos, El-Showk 14]
  - Suppose the degeneracy saturate the maximum bound. Then,

$$\sum_{j=1}^{j_{max}} \left(d(j)\ \alpha^* \left[\hat{\mathcal{G}}^j(z,\bar{z}) \right] + \bar{d}(j)\ \alpha^* \left[\hat{\mathcal{G}}^{\bar{j}}(z,\bar{z}) \right] \right) + \sum_{h,\bar{h}\in\mathcal{P}} d(h,\bar{h})\ \alpha^* \left[\hat{\mathcal{G}}^{h,\bar{h}}(z,\bar{z}) \right] = 0$$

  - Therefore,

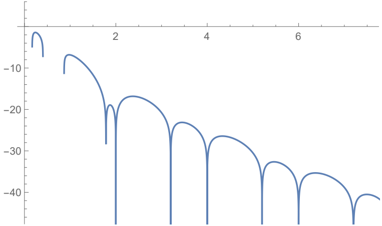
$$d(h,\bar{h})\ \alpha^* \left[\hat{\mathcal{G}}^{h,h}(z,\bar{z}) \right] = 0, \quad \text{for} \quad {}^\forall (h,\bar{h}) \in \mathcal{P}.$$

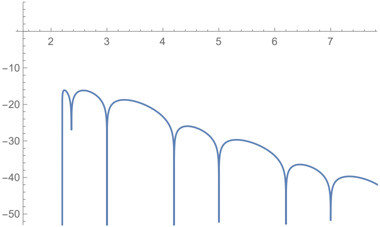
Idea : Find the states such that $\alpha^* \left[\hat{\mathcal{G}}^{h,h}(z,\bar{z}) \right] = 0$!

- Spectrum Analysis

1. Apply EFM and find the states such that makes $\alpha^* \left[\hat{\mathcal{G}}^{h,h}(z,\bar{z}) \right] = 0$.
2. For those states, repeat the degeneracy analysis.
3. Find the consistent character decomposition.

- F_4 example
  - The EFM analysis applied to $(F_4)_1$ WZW gives,





  - From the EFM analysis, the low-lying spectrum of spin-0 and spin-1 sector are,

$$\Delta_{j=0} \in \{\frac{6}{5} + 2n, 2 + 2n \Big| n \in \mathbb{Z}^+\}, \quad \Delta_{j=1} \in \{\frac{11}{5} + 2n, 3 + 2n \Big| n \in \mathbb{Z}^+\}.$$

  - The primary characters of F_4 :

$$\chi^{F_4}_{[0]}(q) \equiv \chi_{1;0,0,0,0}(q) = 1 + 52q + 377q^2 + 1976q^3 + 7852q^4 + \mathcal{O}(q^5),$$
$$\chi^{F_4}_{[4]}(q) \equiv \chi_{0;0,0,0,1}(q) = q^{\frac{3}{5}}\left(26 + 299q + 1702q^2 + 7475q^3 + 27300q^4 + \mathcal{O}(q^5)\right).$$

- F_4 example (continued)
  - For the each low-lying spectrum, the maximum degeneracies are,

$(h,\bar{h})$	Max. Deg	$(h,\bar{h})$	Max. Deg	$(h,\bar{h})$	Max. Deg
$(\frac{3}{5},\frac{3}{5})$	676.0000	$(1,1)$	2704.0000	$(1,0)$	52.00028
$(\frac{3}{5},\frac{8}{5})$	7098.0001	$(2,1)$	16848.001	$(2,0)$	324.0007
$(\frac{3}{5},\frac{13}{5})$	35802.002	$(3,1)$	80444.061	$(3,0)$	1547.0091
$(\frac{8}{5},\frac{8}{5})$	74529.0001	$(2,2)$	104976.005	$(4,0)$	5499.0126

  - The relation between *partition function* and *reduced partition function* is given by,

$$\hat{\mathcal{Z}}^{\mathcal{W}_2}_{F_4}(q,\bar{q}) = |\tau|^{\frac{1}{2}}\eta(\tau)^2\bar{\eta}(\bar{\tau})^2\mathcal{Z}_{F_4}(q,\bar{q}) - \underbrace{(1-q)(1-\bar{q})}_{\text{Vaccum contribution}}.$$

  - Our numerical result agrees to the following diagonal form partition function.

$$\boxed{\mathcal{Z}_{F_4}(q,\bar{q}) = |\chi^{F_4}_{[0]}(q)|^2 + |\chi^{F_4}_{[4]}(q)|^2}$$

- The Result Summary ($c \leq 8$)
  - The numerical result confirms the structure of modular invariant partition function, in terms of the character. In case of $(G_2)_1, (F_4)_1$ and $(E_7)_1$, it have the diagonal form. [Gannon 92]

$$\mathcal{Z}_{G_2}(q,\bar{q}) = |\chi^{G_2}_{[0]}(q)|^2 + |\chi^{G_2}_{[1]}(q)|^2$$
$$\mathcal{Z}_{F_4}(q,\bar{q}) = |\chi^{F_4}_{[0]}(q)|^2 + |\chi^{F_4}_{[4]}(q)|^2$$
$$\mathcal{Z}_{E_7}(q,\bar{q}) = |\chi^{E_7}_{[0]}(q)|^2 + |\chi^{E_7}_{[6]}(q)|^2$$

  - In case of E_6,

$$\mathcal{Z}_{E_6}(q,\bar{q}) = \chi^{E_6}_{[0]}(q)\bar{\chi}^{E_6}_{[0]}(\bar{q}) + \chi^{E_6}_{[1]}(q)\bar{\chi}^{E_6}_{[5]}(\bar{q}) + \chi^{E_6}_{[5]}(q)\bar{\chi}^{E_6}_{[1]}(\bar{q})$$

  $\chi^{E_6}_{[1]}(q)$ and $\chi^{E_6}_{[5]}(q)$ are complex conjugate to each other, their characters are identical.

  - The $(A_1)_1, (A_2)_1, (G_2)_1, (D_4)_1$ and $(E_8)_1$ WZW models are realized via the scalar gap problem. [Collier, Lin, Yin 16]

  - The twist gap problem realize WZW models with Deligne's exceptional series on the numerical boundary!

- $(E_{7,1/2})_1$ WZW model?
  - $E_{7,1/2}$ is non-simple Lie algebra, its subalgebra is E_7. It splits into $E_7 \oplus 56 \oplus \mathbb{R}$.
  - The degeneracy analysis at $c = \frac{38}{5}$ gives, [Cohen, Man de 96], [Landsberg, Manivel 06]

$(h,\bar{h})$	Max. Deg	$(h,\bar{h})$	Max. Deg	$(h,\bar{h})$	Max. Deg
$(\frac{4}{5},\frac{4}{5})$	3249.0004	$(1,1)$	36100.000	$(1,0)$	190.00412
$(\frac{4}{5},\frac{9}{5})$	59565.012	$(2,1)$	501600.00	$(2,0)$	2640.0481
$(\frac{9}{5},\frac{9}{5})$	1092025.06	$(2,2)$	6969600.01	$(3,0)$	19285.021

- From the second order MDE, structure of the vacuum character and primary character of $h = \frac{4}{5}$ are given by,

$$\chi_0^{E_{7,1/2}}(q) = 1 + 190q + 2831q^2 + 22306q^3 + 129276q^4 + 611724q^5 + \mathcal{O}(q^5),$$

$$\chi_{\frac{4}{5}}^{E_{7,1/2}}(q) = q^{\frac{4}{5}}\left(57 + 1102q + 9367q^2 + 57362q^3 + 280459q^4 + 1181838q^5 + \mathcal{O}(q^6)\right).$$

- If there is $(E_{7,1/2})_1$ WZW model, the modular invariant partition function may have the following diagonal form.

$$\boxed{\mathcal{Z}_{E_7}(q,\bar{q}) = \chi_0^{E_{7,1/2}}(q)\bar{\chi}_0^{E_{7,1/2}}(\bar{q}) + \chi_{\frac{4}{5}}^{E_{7,1/2}}(q)\bar{\chi}_{\frac{4}{5}}^{E_{7,1/2}}(\bar{q})}$$

- Cousins of Extremal Conformal Field Theories

• The extremal conformal field theory is originally suggested by Witten, as a candidate for dual CFT of pure gravity in AdS_3. It admits holomorphic factorization, and the central charge is quantized by $c = 24k$. [Witten 07]

• The buliding block of the partition function is Klein-j function, defined by Eisenstein series of weight 4 and weight 6.

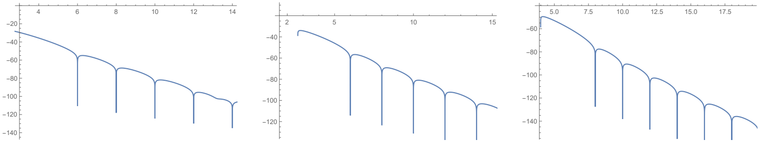
$$J(q) \equiv j(q) - 744 = 1728\frac{E_4^3}{E_4^3 - E_6^2} - 744$$
$$= q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

• For $c = 24(k = 1)$, the unique modular invariant partition function is given by $\mathcal{Z}_{k=1}(q, \bar{q}) = J(q)\bar{J}(\bar{q})$. For $c = 48(k = 2)$, the modular invariant partition function is given by $\mathcal{Z}_{k=2}(q, \bar{q}) = (J(q)^2 - 393767)(\bar{J}(\bar{q})^2 - 393767)$.

• Extended to $c = 8m$, the modular invariant partition functions are constructed by fractional power of j-function. It has the structure of [Avramis, Kehagias, Mattheopoulou 07]

$$\mathcal{Z}_{8m}(\tau) = j^{m/3}(\tau)\sum_{r=0}^{[m/3]} a_r j^{-r}(\tau).$$

- Examine ECFTs via the modular bootstrap
  - The twist gap problem realize the ECFTs with $c = 24, 32, 48$ on its boundary.
  - The EFM analysis suggests that all of them have the states with integer Δ.



  - Reading the degeneracy, we confirm the following structure of the modular invariant partition function.

$$c = 24 : \mathcal{Z}_{k=1}(q,\bar{q}) = J(q)\bar{J}(\bar{q})$$

$$c = 32 : \mathcal{Z}_{k=\frac{4}{3}}(q,\bar{q}) = (J(q)^{\frac{4}{3}} - 992J(q)^{\frac{1}{3}})(J(\bar{q})^{\frac{4}{3}} - 992J(\bar{q})^{\frac{1}{3}})$$

$$c = 48 : \mathcal{Z}_{k=2}(q,\bar{q}) = (J(q)^2 - 393767)(\bar{J}(\bar{q}) - 393767)$$

- Gapped CFT
  - Recall the vacuum character from third order differential equation. with an ansatz $\chi_{\hat{\lambda}}(q) = q^{\alpha}(a_0 + a_2 q^2 + a_3 q^3 + a_4 q^4 + \cdots)$. We refer those CFTs as gapped CFTs.

$$\chi_0^{c=8}(q) = 1 + 156q^2 + 1024q^3 + 6790q^4 + 32768q^5 + \mathcal{O}(q^6)$$
$$\chi_0^{c=16}(q) = 1 + 2296q^2 + 65536q^3 + 1085468q^4 + 12320768q^5 + \mathcal{O}(q^6)$$
$$\chi_0^{c=47/2}(q) = 1 + 96256q^2 + 9646891q^3 + 366845011q^4 + 8223700027q^5 + \mathcal{O}(q^6)$$

  - In the mathematics, the corresponding vertex operator algebra was constructed.

Exceptional Vertex Operator Algebras and the Virasoro Algebra

Michael P. Tuite

$C = 8, d_2 = 155$: This can be realized as the fixed point free lattice VOA V_L^+ (fixed under the automorphism lifted from the reflection isometry of the lattice L) for the rank 8 even lattice $L = \sqrt{2}E_8$. The automorphism group is $O_{10}^+(2).2$ [**G**].

$C = 16, d_2 = 2295$: The VOA V_L^+ for the rank 16 Barnes-Wall even lattice $L = \Lambda_{16}$ whose automorphism group is $2^{16}.O_{10}^+(2)$ [**S**].

$C = 23\frac{1}{2}, d_2 = 96255$: This can be realized as the integrally graded subVOA of Höhn's Baby Monster Super VOA $VB^{\natural}$ whose automorphism group is the Baby Monster group $\mathbb{B}$ [**Ho2**].

- The partition function of $c = 8$ gapped CFT
  - The degeneracy analysis with imposing conserved currents from $j = 2$ gives,

$(h, \bar{h})$	Max. Deg	$(h, \bar{h})$	Max. Deg	$(h, \bar{h})$	Max. Deg
$(\frac{1}{2}, \frac{1}{2})$	496.0000000	$(1, 1)$	33728.00000	$(2, 0)$	155.000000
$(\frac{1}{2}, \frac{3}{2})$	17360.00000	$(2, 1)$	505920.0000	$(3, 0)$	868.000000
$(\frac{3}{2}, \frac{3}{2})$	607600.0009	$(2, 2)$	7612825.000	$(4, 0)$	5610.00000

  - The MDE determines primary character up to the overall constants a_0 and a_1.

$$\chi_1^{c=8}(\tau) = a_0 q^{1/2}\Big(1 + 36q + 394q^2 + 2776q^3 + 15155q^4 + 69508q^5 + \mathcal{O}(q^6)\Big),$$
$$\chi_2^{c=8}(\tau) = a_1 q^1\Big(1 + 16q + 136q^2 + 832q^3 + 4132q^4 + 17696q^5 + \mathcal{O}(q^6)\Big)$$

  - Finally, we suggest that modular invariant partition function reads,

$$\mathcal{Z}_{c=8} = \chi_0^{c=8}(\tau)\bar{\chi}_0^{c=8}(\bar{\tau}) + 496\chi_1^{c=8}(\tau)\bar{\chi}_1^{c=8}(\bar{\tau})|_{a_0=1} + 33728\chi_2^{c=8}(\tau)\bar{\chi}_2^{c=8}(\bar{\tau})|_{a_1=1}.$$
$$= 1 + \underbrace{496}_{1+155+340} q^{\frac{1}{2}}\bar{q}^{\frac{1}{2}} + \underbrace{17856}_{2\times 155+2\times 868+15810} q^{\frac{3}{2}}\bar{q}^{\frac{1}{2}} + \underbrace{33728}_{2108+31620} q\bar{q} + \underbrace{539648}_{539648} q^2\bar{q} + \cdots$$

- The partition function of $c = 16$ gapped CFT
  - The degeneracy analysis with imposing conserved currents from $j = 2$ gives,

$(h, \bar{h})$	Max. Deg	$(h, \bar{h})$	Max. Deg	$(h, \bar{h})$	Max. Deg
$(\frac{3}{2}, \frac{3}{2})$	32505856.0032	$(1, 1)$	134912.0000	$(2, 0)$	2295.00000
$(\frac{3}{2}, \frac{5}{2})$	1657798656.0001	$(2, 1)$	18213120.00	$(3, 0)$	63240.0000
$(\frac{3}{2}, \frac{7}{2})$	34228666368.005	$(2, 2)$	2464038225.003	$(4, 0)$	1017636.00

  - The MDE determines primary character up to the overall constants b_0 and b_1.

$$\chi_1^{c=16} = b_0 q^1 \Big(1 + 136q + 4132q^2 + 67712q^3 + 770442q^4 + 6834240q^5 + \mathcal{O}(q^6)\Big),$$
$$\chi_2^{c=16} = b_1 q^{3/2} \Big(1 + 52q + 1106q^2 + 14808q^3 + 147239q^4 + 1183780q^5 + \mathcal{O}(q^6)\Big)$$

  - Finally, we suggest that modular invariant partition function reads,

$$\mathcal{Z}_{c=16} = \chi_0^{c=16}(\tau)\bar{\chi}_0^{c=16}(\bar{\tau}) + 134912\chi_1^{c=16}(\tau)\bar{\chi}_1^{c=16}(\bar{\tau})|_{b_0=1} + 32505856\chi_2^{c=16}(\tau)\bar{\chi}_2^{c=16}(\bar{\tau})|_{b_1=1}$$
$$= 1 + \underbrace{2296}_{2\times 1+186+2108} q^2 + \underbrace{65536}_{2\times 1+186+14756+50592} q^3 + \underbrace{134912}_{186+340+868+22858+110670} q\bar{q} + \cdots$$

Baby Monster CFT [Höhn 07]

- The degeneracies with $c = \frac{47}{2}$ gives,

$(h, \bar{h})$	Max. Deg $(h, \bar{h})$	$(h, \bar{h})$	Max. Deg
$(\frac{3}{2}, \frac{3}{2})$	19105641.026984403127	$(\frac{5}{2}, \frac{5}{2})$	1298173112605.3499336
$(2, 2)$	9265025041.322733803	$(\frac{31}{16}, \frac{31}{16})$	9265217540.6086142750
$(\frac{5}{2}, \frac{3}{2})$	4980203754.2560961756	$(\frac{47}{16}, \frac{31}{16})$	1011288637613.8107313

- The character from 3rd MDE is given by,

$$\chi_0^{c=47/2} = q^{48/48} a_0 \Big((1 + 96256 q^2 + 9646891 q^3 + 366845011 q^4 + \mathcal{O}(q^5) \Big)$$

$$\chi_1^{c=47/2} = q^{25/48} a_1 \Big(1 + \frac{785}{3} q + \frac{44393}{3} q^2 + 418441 q^3 + \frac{23301881}{3} q^4 + \mathcal{O}(q^5) \Big)$$

$$\chi_2^{c=47/2} = q^{23/24} a_2 \Big(1 + \frac{5177}{47} q + 4372 q^2 + 100627 q^3 + 1625207 q^4 + \mathcal{O}(q^5) \Big)$$

- Corresponding modular invariant partition function reads,

$$\mathcal{Z}_{c=47/2} = \chi_0^{c=47/2}(\tau) \bar{\chi}_0^{c=16}(\bar{\tau}) \Big|_{a_0=1} + \chi_1^{c=47/2}(\tau) \bar{\chi}_1^{c=16}(\bar{\tau}) \Big|_{a_1=4371} + \chi_2^{c=47/2}(\tau) \bar{\chi}_2^{c=16}(\bar{\tau}) \Big|_{a_2=96256}$$

$$= 1 + \underbrace{96256}_{1+96255} q^2 + \underbrace{9646891 q^3}_{2\times 1 - 4371 + 2\times 96255 + 9458750} + \underbrace{19105641 q^{3/2} \bar{q}^{3/2}}_{1+96255+9458750+9550635} + \cdots .$$

- Bootstrapping with $\mathcal{W}$-algebra
  - We consider the following reduced $\mathcal{W}$-algebra character.

$$\chi_{h,w}(q) = \mathrm{Tr}_{h,w}(q^{L_0 - \frac{c}{24}} \bar{q}^{L_0 - \frac{c}{24}})$$

  - In case of the $\mathcal{W}_{2,3}$-algebra,

$$\chi_0(\tau) = \frac{q^{-\frac{c-2}{24}}(1-q)^3(1+q)}{\eta(\tau)^2}, \qquad \chi(\tau) = \frac{q^{h-\frac{c-2}{24}}}{\eta(\tau)^2}$$

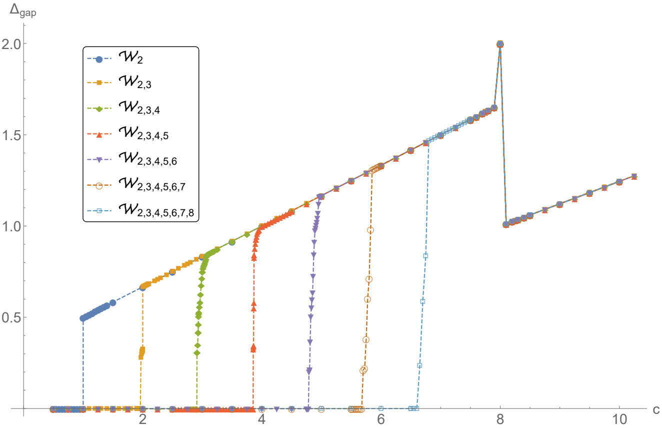
because of

$$\langle 0 | L_1 L_{-1} | 0 \rangle = 0, \quad \langle 0 | W_1 W_{-1} | 0 \rangle = 0, \quad \langle 0 | W_2 W_{-2} | 0 \rangle = 0.$$

  - For general rank-k case, the character for general rank-k $\mathcal{W}_{f_1, f_2, \ldots f_k}$-algebra is,

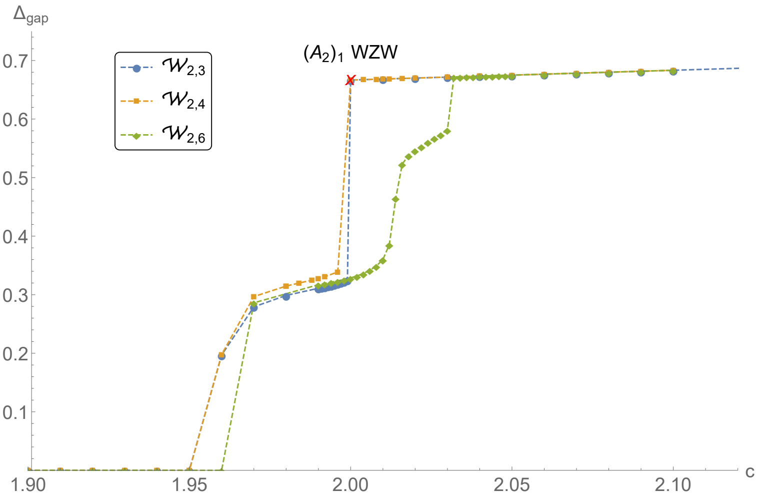
$$\chi(\tau) = \frac{q^{h-\frac{c-N+1}{24}}}{\eta(\tau)^{N-1}}, \quad \chi_0(\tau) = \frac{q^{-\frac{c-N+1}{24}}}{\eta(\tau)^{N-1}} \prod_{j=1}^{k} \prod_{i=1}^{f_j - 1} (1 - q^i).$$

- The Numerical Bounds(Twist Gap)



- The behavior of $c \geq k$ is identical to the Virasoro results. On the other hand, the numerical bound at $c \leq k$ is collapsed.

- Numerical bound with Rank-2 $\mathcal{W}$-algebra



- Rank-2 $\mathcal{W}$-algebra and $(A_2)_1$ WZW model

• In case of $\mathcal{W}_{2,3}$ and $\mathcal{W}_{2,4}$ they realize the $(A_2)_1$ WZW model at the end of the unitary world.

• For both cases, the reduced partition functions are,

$$\hat{\mathcal{Z}}^{\mathcal{W}_{2,3}}_{c=2} = 8q + 4q^3 + 7q^4 + 12q^7 + \cdots$$
$$\hat{\mathcal{Z}}^{\mathcal{W}_{2,4}}_{c=2} = 8q + 5q^3 + 5q^4 + 2q^6 + 11q^7 + \cdots$$

Namely, the character decomposition with positive integer coefficients is available.

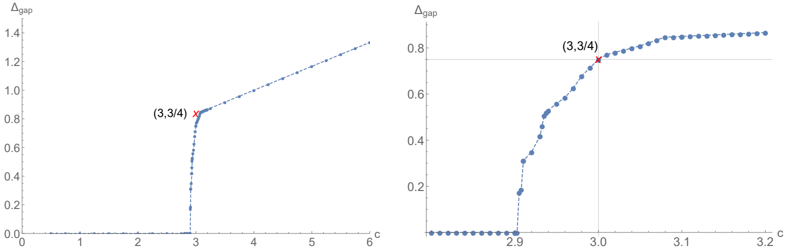
• On the other hand, $\mathcal{W}_{2,6}$ rule out the $(A_2)_1$ WZW model.

• In the eye of character decomposition,

$$\hat{\mathcal{Z}}^{\mathcal{W}_{2,6}}_{c=2} = 8q + 5q^3 + 6q^4 - q^5 + 12q^7 + 2q^8 + \cdots$$

The decomposition have negative integer coefficient at q^5 order. Hence, character decomposition with $\mathcal{W}_{2,6}$ algebra do not consistent with $(A_2)_1$ WZW model.

- Numerical bound with Rank-3 $\mathcal{W}$-algebra



- Among the level-1 WZW models with Delign's exceptional series, no CFT with $c = 3$, while rank-3 $\mathcal{W}_{2,3,4}$ algebra make cliff around $c \sim 3$. Useless $\mathcal{W}_{2,3,4}$?
- The numerical boundary pass through $(c = 3, \Delta = \frac{3}{4})$, the data for $(A_3)_1$ WZW model. The character of $(A_3)_1$ is solution of third order modular differential equation.
- CLAIM : The $\mathcal{W}_{2,3,4}$ algebra EXCLUSIVELY realize $(A_3)_1$ WZW model !

- Analysis on $(A_3)_1$ WZW model
  - The numerical analysis on the degeneracy of low-lying states gives

$(h,\bar{h})$	Max. Deg	$(h,\bar{h})$	Max. Deg	$(h,\bar{h})$	Max. Deg
$(\frac{3}{8},\frac{3}{8})$	32.00000	$(\frac{1}{2},\frac{1}{2})$	36.000000	$(1,1)$	225.00714
$(\frac{3}{8},\frac{11}{8})$	96.00000	$(\frac{1}{2},\frac{3}{2})$	48.00000	$(1,2)$	75.00020
$(\frac{11}{8},\frac{11}{8})$	288.01585	$(\frac{3}{2},\frac{3}{2})$	64.11818	$(2,2)$	25.00500

  - The characters of (A_3) affine Lie algebra reads,

$$\chi^{A_3}_{[0]}(q) = 1 + 15q + 51q^2 + 172q^3 + 453q^4 + 1128q^5 + \mathcal{O}(q^6),$$
$$\chi^{A_3}_{[1]}(q) = q^{\frac{3}{8}}\left(4 + 24q + 84q^2 + 248q^3 + 648q^4 + 1536q^5 + \mathcal{O}(q^3)\right),$$
$$\chi^{A_3}_{[2]}(q) = q^{\frac{1}{2}}\left(6 + 26q + 102q^2 + 276q^3 + 728q^4 + 1698q^5 + \mathcal{O}(q^3)\right),$$
$$\chi^{A_3}_{[3]}(q) = q^{\frac{3}{8}}\left(4 + 24q + 84q^2 + 248q^3 + 648q^4 + 1536q^5 + \mathcal{O}(q^3)\right).$$

  - Again, the numerical analysis is consistent with the following diagonal structure.

$$\boxed{\mathcal{Z}_{A_3}(q,\bar{q}) = |\chi^{A_3}_{[0]}(q)|^2 + |\chi^{A_3}_{[1]}(q)|^2 + |\chi^{A_3}_{[2]}(q)|^2 + |\chi^{A_3}_{[3]}(q)|^2}$$

- Conclusion and Outlook

• The two-channel RCFTs($k = 1$ WZW models with Deligne's exceptional series) and the three-channel RCFTs(cousins of extremal conformal field theories) are analyzed via modular bootstrap. It turns out that twist gap problem with holomorphic currents realize those theories on the numerical bound

• The modular invariant partition function for special class of three-channel RCFTs($c = 8, c = 16, c = \frac{47}{2}$) are suggested. The coefficients in partition function decomposed in terms of $O^+(10, 2)$ or baby monster group.

• Extension to the $\mathcal{W}$-algebra cases. We expect the refined unitary bound will be $c \geq k$.

• Application to the supersymmetric cases : Super WZW models, super extremal conformal field theory?