Dimensional reduction and dualities in $2d$

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based on:
arXiv:1611.02763 - O. Aharony, S. Razamat, N. Seiberg, BW
and work in progress
Over the last several decades we have accumulated a large web of supersymmetric dualities of QFTs known in various numbers of dimensions and amounts of supersymmetry.

Examples of exact (conformal) dualities:
- **Mirror symmetry** of $2d \mathcal{N} = (2, 2)$ Calabi-Yau sigma models
- **$S$-duality** of $4d \mathcal{N} = 2$ theories.

Examples of IR dualities:
- **Hori-Vafa duality** - $2d \mathcal{N} = (2, 2)$ linear sigma models $\leftrightarrow$ Landau-Ginzburg models.
- **3d mirror symmetry** - duality exchanging Higgs and Coulomb branches of IR SCFT of $3d \mathcal{N} = 4$ and $\mathcal{N} = 2$ gauge theories
- **Seiberg duality** of $4d \mathcal{N} = 1$ QCD, eg:

$$SU(N_c) + N_f \text{ fundamental flavors} \leftrightarrow$$

$$SU(N_f - N_c) + N_f \text{ fundamental flavors} + N_f^2 \text{ mesons and } W = q^a M^b_a \tilde{q}_b$$

also $3d \mathcal{N} = 2$ and $2d \mathcal{N} = (2, 2)$ versions.

Many others
Origin of dualities

- Where do they come from?
  - **String theory** - Many of these dualities were found by taking low energy limits of string theory constructions and applying string dualities.
  - **Compactification** - 4d S-duality from 6d $\mathcal{N} = (2,0)$ theory on $\Sigma_{g,s}$ [Gaiotto]
  - **RG flow** - We can flow from new dualities to old ones by turning on relevant parameters and flowing.
  - **Dimensional reduction** - we can compactify dual theories to lower dimensions and flow to lower dimensional effective descriptions.

In this talk we will ask the following questions:

- How can we describe the compactification of a $d$-dimensional SQFT to $d - 1$ dimensions?
- When does a $d$-dimensional duality imply a duality in $d - 1$ dimensions (and when does it not)?

In this way we can hope to better organize the web of dualities, and possibly find new ones.
Reduction and duality

- If two $d$-dimensional theories are exactly equivalent, then they are also equivalent on $\mathbb{R}^{d-1} \times S^1$. At low energies compared to $1/R$, we find equivalent $d-1$ dimensional theories.

- However, for IR dualities this is no longer true. If $\mu_a$ are relevant parameters which initiate a flow from the UV theory to the IR, we find on each side a family of theories parameterized by:

  $$\gamma_a \equiv \mu_a R^{\text{dim}(\mu_a)}$$

- For $\gamma_a \to 0$, we are reducing the UV descriptions, and have a $d-1$-dimensional Lagrangian, but no duality.

- For $\gamma_a \to \infty$, we are reducing the IR descriptions. Then we have a duality, but may not have a useful Lagrangian description.
Reduction and duality

- If two $d$-dimensional theories are exactly equivalent, then they are also equivalent on $\mathbb{R}^{d-1} \times S^1_R$. At low energies compared to $1/R$, we find equivalent $d-1$ dimensional theories.

- However, for IR dualities this is no longer true. If $\mu_a$ are relevant parameters which initiate a flow from the UV theory to the IR, we find on each side a family of theories parameterized by:

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- For $\gamma_a \to 0$, we are reducing the UV descriptions, and have a $d-1$-dimensional Lagrangian, but no duality.

- For $\gamma_a \to \infty$, we are reducing the IR descriptions. Then we have a duality, but may not have a useful Lagrangian description.

- However, we can attempt to construct a UV theory in $d-1$ dimensions which flows to the latter theory.
In [Aharony,Razamat,Seiberg,BW] this problem was studied for $d = 4$ and $\mathcal{N} = 1$ supersymmetric gauge theories.

One finds that in a $4d$ gauge theory on $\mathbb{R}^3 \times S^1_R$, instanton effects generate a superpotential in the effective $3d$ description at low energies:

$$W = \eta Y, \quad \eta = e^{-4\pi/(Rg_3^2)}$$

Here $Y$ is a chiral operator parameterizing the $3d$ Coulomb branch.

We can engineer a UV description in $3d$ by adding a superpotential $W = \eta Y_{mon}$, where $Y_{mon}$ is a monopole operator which flows to $Y$ at low energies.

Claim: the $3d$ theories one gets by dimensionally reducing a $4d$ IR dual pair and adding the appropriate $\eta Y$ superpotential are IR dual in $3d$. 
As an example, consider Seiberg duality between $4d \mathcal{N} = 1$ $SU(N_c)$ gauge theory with $N_f$ fundamental flavors and $SU(N_f - N_c)$ with $N_f$ flavors and $N_f^2$ mesons $M$, with $W = M q \bar{q}$.

**Claim 1**: The $3d \mathcal{N} = 2$ dimensional reductions of these theories, with $\eta Y$ superpotential, are IR dual.

**Claim 2**: By starting with this $3d$ duality with $N_f + 1$ flavors and giving one flavor a large real mass, one derives the following duality:

Theory $A$ : $SU(N_c)$ with $N_f$ flavors (and no superpotential)

Theory $B$ : $U(N_f - N_c)$ with $N_f$ flavors, mesons $M$, det fields $b, \tilde{b}$, and singlet $Y$, with $W = M q \bar{q} + Y b \tilde{b} + \tilde{V}_+ + \tilde{V}_-$

All known $3d$ Seiberg-like dualities can be generated in this way (see also [Amariti,Klare],[Niarchos],[Nii]).
New features in two dimensional QFTs

- For the rest of the talk, we will focus on reductions to 2d SQFTs, which have many features qualitatively different from higher dimensions.

- In higher dimensions, there is typically a moduli space of inequivalent vacua. One can focus on a point in moduli space, and at low energies the theory here is generically free (metric is irrelevant).

- In 2d, there is no moduli space of vacua. Instead, it is a target space for low energy fluctuations, and the quantum theory explores all of it.

- The metric on moduli space is often marginal, and different UV metrics lead to different CFTs.

- Another new feature is that a UV theory with multiple branches may flow to decoupled CFTs, eg, $\mathcal{N} = (4, 4) \ U(1)$ with $N_f$ hypermultiplets [Aharony, Berkooz].
New features in two dimensional dualities

There are several known dualities between $\mathcal{N} = (2, 2)$ QFTs, including exact (conformal) dualities:

- T-duality
- Hori-Kapustin duality
- Mirror symmetry

There are also IR, Seiberg-like dualities due to [Hori,Tong],[Hori], [Benini,Park,Zhao].

For IR dualities, the asymptotic behavior of the Kahler metric may not match, and one needs a refined notion of duality [Aharony,Razamat,Seiberg,BW]
We will start with a $3d \mathcal{N} = 2$ gauge theory. We consider two types of relevant parameters in the UV:

\begin{align*}
gauge couplings & \quad g_{3j}^2, \quad \text{real masses } m_a \\
\end{align*}

Correspondingly, we can define two types of parameters for the theory on $\mathbb{R}^2 \times S^1_R$:

\begin{align*}
\gamma_j &= g_{3j}^2 R, \quad t_a = m_a R
\end{align*}

As we’ll see, the former control the asymptotic behavior of the target space metric ($D$-terms), while the latter give rise to protected Kahler moduli (twisted $F$-terms).
Example: free $U(1)$ gauge theory

First consider the free 3d $\mathcal{N} = 2$ $U(1)$ gauge theory, with action:

$$S = \int d^3x \frac{1}{g_3^2} (F_{\mu\nu}^2 + (\partial_\mu \sigma)^2 - i\lambda^\dagger \gamma^\mu \partial_\mu \lambda)$$

We can dualize the gauge field to a scalar $\phi$ by writing $d\phi = \ast F$. Then quantization of flux identifies $\phi \sim \phi + g_3$, and the moduli space is a cylinder:

Now we place this theory on $\mathbb{R}^2 \times S^1_R$. Then we find a sigma model with cylinder target space, which, in 2d normalization, has radius $\sqrt{g_3^2 R} = \sqrt{\gamma}$. 
Alternatively, we describe the $3d$ theory on a circle in terms of a twisted chiral superfield:

$$\Sigma = \sigma + iA_3 + \ldots$$

Large gauge transformations identify $\Sigma \sim \Sigma + \frac{i}{R}$. In the $2d$ normalization, the radius of the cylinder becomes $\frac{1}{\sqrt{\gamma}}$. Thus we find a T-dual description of the first cylinder.

Two lessons:

1. $3d$ EM duality reduces to T-duality. [Aganagic, Hori, Karch, Tong].
2. The $2d$ theory depends importantly on $\gamma = g_3^2 R$. 
Example: pure $SU(2)$ gauge theory

- Here the Coulomb branch is parameterized by a chiral monopole operator:
  \[ Y = e^X \sim e^{\frac{1}{g_3^2} \sigma + i\gamma} \]

  Asymptotically we find a free $U(1)$ theory, and the metric is cylindrical,
  \[ X \sim X + ig_3. \]

- A non-perturbative superpotential is generated which causes runaway behavior in 3d [Affleck, Harvey, Witten]:
  \[ W = \frac{1}{Y} = e^{-X} \]

- Compactifying on a circle of radius $R$, we find a Liouville theory with radius
  \[ \sqrt{\gamma} = \sqrt{g_3^2 R}. \]
  This is dual to a cigar sigma model with asymptotic radius $1/\sqrt{\gamma}$.

- In the $\gamma \to 0$ we obtain the 2d $SU(2)$ theory, and the dual description becomes a free (twisted) chiral multiplet, $\Phi$.

- Thus the 2d $SU(2)$ theory is equivalent to a free twisted chiral multiplet, which we can identify with:
  \[ \Phi \equiv \text{Tr}\Sigma^2 \]
Example: $U(1) \ N_f = 1$

- Next consider an interacting gauge theory: SQED with chirals $Q$ and $\tilde{Q}$ of charge 1 and $-1$. The relevant parameters are:
  - gauge coupling $g_3^2$,
  - Fayet-Iliopolous (FI) parameter $\zeta$,
  - mass $m$

- The potential for the scalar fields is:
  \[ V = \frac{g_3^2}{8}(|Q|^2 - |\tilde{Q}|^2 - \frac{\zeta}{2\pi})^2 + \sigma^2(|Q|^2 + |\tilde{Q}|^2) \]

- Then the moduli space of vacua consists of two branches:
  - A Higgs branch, where $\sigma = 0$, but $Q$, $\tilde{Q}$ are non-zero,
  - When $\zeta = 0$, a Coulomb branch, where $Q = \tilde{Q} = 0$, but $\sigma$ is non-zero.
While the Coulomb branch is asymptotically a cylinder of radius $g_3$, the Higgs branch is independent of $g_3$. 

In the IR limit, $g_3 \rightarrow \infty$, these three branches appear symmetrically.

We find an IR dual description as an WZ model with superpotential $W = XYZ$. 
Reducing $U(1) \ N_f = 1$

- If we place this theory (with $\zeta = 0$) on $\mathbb{R}^2 \times S^1_R$, we find at energies below $1/R$ an effective 2d description.

- Asymptotically on the Coulomb branch this is a sigma model with radius $\sqrt{g_3^2 R} = \sqrt{\gamma}$.

- For finite $\gamma$, this theory is not equivalent to the 2d XYZ theory.

- We can try to engineer a 2d UV description. A natural guess is as 2d SQED with one flavor. However note:

$$g_2^2 = \frac{1}{R} g_3^2$$

So to obtain a finite $g_2$, we must take $\gamma \to 0$.

- In particular, 2d SQED is not equivalent to XYZ! (e.g., note they have different numbers of branches in the IR.)
Since the problem was with the Coulomb branch, we might look for deformations which lift it.

The FI parameter $\zeta$ does precisely this. Let us then instead take:

$$t = \zeta R$$

non-zero and finite. Here $t$ is the complexified 2d FI parameter, $t = r + i\theta$.

Then we claim one finds the 2d $U(1)$ theory with non-zero FI parameter $t$.

On the XYZ side, this gives a large mass to two of the chirals, $X$ and $Y$, and they can be integrated out. We find a single free chiral $Z$.

This leads to the 2d duality:

$U(1)$ with chirals of charge 1 and $-1$, $t \neq 0 \leftrightarrow$ a free chiral

which is a special case of a duality of [Benini,Park,Zhao].
More precisely, one can check that the asymptotic behavior on the moduli space of these theories are not the same:

\[ ds_{UV}^2 = dv^2 + \Omega(v) d\theta^2 \]

\[ \Omega(v) = \begin{cases} 
\frac{v^2}{\sqrt{1 + v^2 / r^2}} & \text{free chiral} \\
\frac{v^2}{r} & \text{U(1) gauge theory} 
\end{cases} \]

where \( r \) is the real FI parameter. Since the asymptotic behavior is not renormalized, the duality looks wrong.

However, for any \( r \neq 0 \), if one focuses on a fixed region on the moduli space and waits enough RG time, the metrics agree:
A useful tool in studying the reduction is the effective twisted superpotential of a 3d theory on $\mathbb{R}^2 \times S^1_R$ [Nekrasov,Shatashvili].

This is a function of the 2d twisted chiral multiplet, $\Sigma$, for dynamical $(u_a)$ and background (flavor) $(m_i)$ gauge multiplets:

$$\mathcal{W}(u_a, m_i) = \sum_{l \in \text{chirals}} \mathcal{W}^\Phi(Q^a_l u_a + q^i_l m_i) + \frac{R}{2} \sum_{a,b} k^{ab} u_a u_b + R \zeta^a u_a$$

where $\mathcal{W}^\Phi(u) = \frac{1}{(2\pi i)^2 R} \text{Li}_2(e^{2\pi i R u})$.

This determines the supersymmetric vacua of the theory, determined by the "Bethe equations"

$$1 = \exp \left( 2\pi i \partial_{u_a} \mathcal{W} \right) = \prod_{l} (1 - x_b Q^b_l v_i q^i_l) Q^a_l \prod_{b} x_b k^{ab} z^a$$

where $x_a = e^{2\pi i R u_a}$, $v_i = e^{2\pi i R m_i}$, $z^a = e^{2\pi i R \zeta^a}$.

We must throw out vacua with enhanced $SU(2)$ gauge symmetry [Hori,Tong] because they have a SUSY breaking potential:

$$\mathcal{W} = c_0 + c_2 u^2 + \ldots = c_0 + c_2 \text{Tr} \Sigma^2 + \ldots = c_0 + c_2 \Phi + \ldots$$
Twisted superpotential and reduction

- In the limit $R \to 0$, we expect this to approach the twisted superpotential of the 2d low energy description.

- For example:

$$
\mathcal{W}^\Phi(u) \xrightarrow{R \to 0} \mathcal{W}^\Phi_{(2d)}(u) + \log(2\pi R)u
$$

where $\mathcal{W}^\Phi_{(2d)}(u) = u(\log u - 1)$ is the twisted superpotential of a 2d chiral multiplet, and the non-zero KK modes generate a divergent FI term.

- This was used by [Aganagic, Hori, Karch, Tong] to study the reduction of 3d mirror symmetry.

- The limit we take depends strongly on how we scale the background and dynamical parameters as we take $R \to 0$.

- In the previous example, one finds:

$$
\mathcal{W}^{U(1)}_{(3d)}(u, m, \zeta = \frac{t}{R}) \xrightarrow{R \to 0} \mathcal{W}^{U(1)}_{(2d)}(u, m, t) + \ldots
$$

$$
\mathcal{W}^{XYZ}_{(3d)}(m, \zeta = \frac{t}{R}) \xrightarrow{R \to 0} \mathcal{W}^{Z}_{(2d)}(m, t) + \ldots
$$

consistent with our prediction.
Another 2d limit we can take is holding all the masses finite. Then one finds as $R \to 0$, the vacuum behaves as:

$$<u>_{vac} \sim \frac{1}{R}$$

Thus it is natural to define a new field, $X = Ru$, and one finds:

$$\mathcal{W}^U_{(3d)}(u = \frac{X}{R}, m, \zeta) \quad \longrightarrow \quad \mathcal{W}_{(2d)} = \zeta X + m \log \sinh \frac{X}{2} + ...$$

giving a massive Landau-Ginzburg model for the twisted chiral field $X$. On the dual side, one finds the 2d XYZ model.

One can check that this matches the mass-deformed XYZ theory in 2d. E.g., one can compute their $S^2$ partition functions and check:

$$Z_{S^2}(\zeta, m)[\text{LG model}] = Z_{S^2}(\zeta, m)[\text{XYZ}]$$

However, the duality of the massive theories does not imply a duality at zero mass, as the Kahler metrics disagree too drastically.
Next let us consider a more complicated example, $U(N_c)$ gauge theory with $N_f$ pairs of (anti-)fundamental chirals.

This theory has the following IR-dual description [Aharony]:

$$U(N_f - N_c) + N_f \text{ flavors} + N_f^2 \text{ mesons and singlets } \tilde{V}_\pm$$

with superpotential $W = Mq\tilde{q} + V_+ \tilde{V}_- + V_- \tilde{V}_+$

As before, reducing the undeformed theory leads to problems with the Coulomb branch, so we set $t = \zeta r$ finite.

This gives a mass to the fields $\tilde{V}_\pm$, and we arrive at the following 2d duality, found previously by [Benini, Park, Zhao]:

$$U(N_c) + N_f \text{ flavors} \leftrightarrow U(N_f - N_c) + N_f \text{ flavors} + \text{ mesons}$$
Next we consider the same theory with the addition of a SUSY Chern-Simons term for the gauge field, defined for $k \in \mathbb{Z}$:

$$S_{CS} = \frac{k}{4\pi} \int d^3x \text{Tr}\left( \epsilon^{\mu\nu\rho} (A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho) + 2\sigma D + \lambda \dagger \lambda \right)$$

Then is theory has the following IR-dual description [Giveon, Kutasov]:

$$U(|k| + N_f - N_c) - k + N_f \text{ flavors } + N_f^2 \text{ mesons, with } W = Mq\tilde{q}$$

The effect of the Chern-Simons term in the 3$d$ theory on a circle naively vanishes as we take $R \to 0$. Proceeding as before, we seem to find a 2$d$ duality:

$$U(N_c) + N_f \text{ flavors } \leftrightarrow U(|k| + N_f - N_c) + N_f \text{ flavors } + \text{ mesons}$$

for any $k \in \mathbb{Z}$, which is clearly wrong.
To see what happened, we look at the SUSY vacua of this theory, which are at solutions to:

$$1 = \exp\left(2\pi i \partial_{u_a} \mathcal{W}\right) = z x_a^k \prod_{i=1}^{N_f} \frac{x_a - \tilde{\mu}_i}{1 - x_a m_i}$$

This has $k + N_f$ solutions for each $a$, so that $N_{vac} = {N_f + k \choose N_c}$.

As $R \to 0$, the $k + N_f$ solutions split as:

$$\hat{u} = \left\{ \hat{u}_l, \quad l = 1, \ldots, N_f, \right\} \cup \left\{ -\frac{\zeta + n}{R}, \quad n = 0, \ldots, k - 1 \right\}$$

Then we must choose the eigenvalues from these two sets. Taking $\ell$ from the first set, we find a $U(\ell)$ gauge theory with $N_c - \ell$ decoupled massive fields. Summing over all ways of assigning eigenvalues, we find:

$$U(N_c)_{k, N_f} \xrightarrow{R \to 0} U(N_c)_{N_f} \oplus U(N_c - 1)_{N_f} \oplus \ldots \oplus U(N_c - k)_{N_f}$$

On the dual side, we have a similar direct sum, and the duality correctly maps the terms:

$$U(k + N_f - N_c)_{k, N_f} \xrightarrow{R \to 0} U(k + N_f - N_c)_{N_f} \oplus U(k + N_f - N_c - 1)_{N_f} \oplus \ldots \oplus U(N_f - N_c)_{N_f}$$
These examples lead to a general prescription for describing the reduction using the twisted superpotential:

1. Choose a desired scaling of background (real mass) parameters, $m_i = m_i(\hat{m}_i, R)$.

2. Then, for a given vacuum, one may have to rescale dynamical (gauge) parameters, $u_i = u_i(\hat{u}_i, R)$, in order to follow it in the $R \to 0$ limit.

3. With these variables, one finds a regular limit:

$$\mathcal{W}_{(3d)}(u_i = u_i(\hat{u}_i, R), m_i = m_i(\hat{m}_i, R)) \xrightarrow{R \to 0} \mathcal{W}_{(2d)}(\hat{u}_i, \hat{m}_i) + \ldots$$

which describes the reduction in the given region of parameter space. We interpret $\mathcal{W}_{(2d)}$ as arising from a particular 2d gauge theory or LG model.

4. If this does not account for all vacua, repeat for any remaining vacua. One finds in general a direct sum of 2d theories.

5. Finally, while this gives a good description of the massive theory, one must perform further checks to see if it holds in the massless limit.

Applying this to both sides of a 3d duality gives a conjectural 2d duality.
A related set of observables for studying the reduction are supersymmetric partition functions on $\mathcal{M}_{d-1} \times S^1_R$.

Formally, one finds:

$$\lim_{R \to 0} \mathcal{Z}_{\mathcal{M}_{d-1} \times S^1_R}[\mathcal{T}^{UV}_d] = \lim_{R \to 0} \mathcal{Z}_{\mathcal{M}_{d-1} \times S^1_R}[\mathcal{T}^{IR}_d]$$

$$= f_{\text{div}}(R) \mathcal{Z}_{\mathcal{M}_{d-1}}[\mathcal{T}^{IR}_{d-1}] = f_{\text{div}}(R) \mathcal{Z}_{\mathcal{M}_{d-1}}[\hat{\mathcal{T}}^{UV}_{d-1}]$$

where $f_{\text{div}}(R)$ is a divergent prefactor rising from integrating out KK modes (eg, [Di Pietro, Komargodski]).

Identities of partition functions in $d$ dimensions therefore imply identities in $d - 1$.

For $d = 3$, we can consider at the $\Sigma_g \times S^1$ and (untwisted) $S^2 \times S^1$ partition functions, but note there is no $2d$ uplift of the elliptic genus.

Elliptic genus is also sensitive to asymptotic metric [Troost],[Murthy]
For general $g$, one finds:

\[ Z_{\Sigma g \times S^1_R} = \sum_{\hat{u}_a \in S_{\text{vac}}} \mathcal{H}(\hat{u}_a, m_i)^{g-1} \]

where the sum is over supersymmetric vacua, and $\mathcal{H}$ can be expressed in terms of the twisted superpotential.

Following the general reduction procedure, one can show:

\[ Z_{\Sigma g \times S^1_R} \xrightarrow{R \to 0} \sum_{I \in \text{theories}} f^{(I)}(m_i, R) Z_{\Sigma g}^{(I)} \]

Typically one term in the sum is dominant, and we only see its contribution.

In this way one recovers identity of $\Sigma_g$ partition functions of $2d$ dual theories, as discussed by [Closset,Mekareeya,Park]
For $g = 0$, one may also define the $S^2 \times S^1$ index, computed by an integral:

$$I_{S^2 \times S^1}^R(m_i) = \int \! du_a \ I_{S^2 \times S^1}^{UV}(u_a, m_i)$$

At the level of the integrand, one finds:

$$I_{S^2 \times S^1}^{UV}(u_a, m_i) \to_{R \to 0} I_{S^2}^{UV}(u_a, m_i)$$

implying that one gets the $S^2$ partition function in the limit.

However, one finds that the integral and $R \to 0$ limit do not commute unless one performs the appropriate change of variables, $u_a \to \hat{u}_a$. Then one finds

$$I_{S^2 \times S^1} \to_{R \to 0} \sum_{l \in \text{theories}} \tilde{f}^{(l)}(m_i, R) I_{S^2}^{(l)}$$

This implies the identity of $S^2$ partition functions of 2d dual theories [Benini,Cremonesi].
Some other examples

- **3d duality:** Abelian mirror symmetry
  \[ \Rightarrow 2d \text{ duality:} \text{ Hori-Vafa/Hori-Kapustin duality (as shown by [Aganagic et al])} \]

- **3d duality:** \( SU(N_c)_k (N_f, N_a) \) (anti-)fundamental chirals dual to \( SU(N_f - N_c)_{-k} \) with \((N_f, N_a)\) for \( N_f > N_a + 1, \ k < \frac{N_f - N_a}{2} \). [Aharony,Fleischer]
  \[ \Rightarrow 2d \text{ duality:} \text{ } SU(N_c) (N_f, N_a) \text{ dual to } SU(N_f - N_c) (N_f, N_a) \text{ for } N_f > N_a + 1 \rightarrow \text{ generalization of [Hori,Tong].} \]

- **3d duality:** \( Sp(2N_c)_k 2N_f \) flavors dual to \( Sp(2(N_f + k - N_c - 1))_k + 2N_f \) flavors
  \[ \Rightarrow 2d \text{ duality:} \]
  - For \( 2k \) odd, gives \( Sp(2N_c) 2N_f \) flavors dual to \( Sp(2(N_f - N_c - \frac{1}{2})) + 2N_f \) flavors [Hori].
  - For \( 2k \) even, we formally find \( Sp(2N_c) \leftrightarrow Sp(2(N_f - N_c - 1)) \), however, this does not give a massless duality.

- \( S^2 \) partition functions match in these examples, as required by reducing the 3d index identities.

- Elliptic genera can also be tested \( \rightarrow \text{ fails when we do not have a massless duality.} \)
Summary

- We have seen various physical subtleties arise when studying the compactification of IR dualities of 3d theories.
- Because of the dependence of the 2d theory on the gauge coupling, naive reduction of dualities does not work, but sometimes fixes are available.
- The effective twisted superpotential and supersymmetric partition functions give useful tools for studying the reduction.
- We have recovered known dualities in 2d, and found new ones.

Open questions

- Better understanding the reduction of theories with Coulomb branches.
- Reducing other dualities from 3d to 2d, eg, nonabelian mirror symmetry, dualities derived from class S in 4d.
- Study reductions between other dimensions - e.g., 4 → 2 [Gadde, Razamat, BW]