# Dualities and exact methods in SUSY gauge theories

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What happens at low energy in asymptotically free theories?

- Do we have confinement? what type of confinement? mass gap? chiral symmetry breaking?
- If we start in the UV with a given gauge and flavor group how are these realised in the vacuum?
- Can we derive the mass spectrum of the hadrons and their excitations?
- Are there different phases? Interacting fixed point CFT theories?
- ...

Answering these questions is hard. We expect/hope that strongly coupled phases could be described by a new set of emergent degrees of freedom in terms of which the theory is weakly coupled.

# Dualities

In presence of extra symmetries such as supersymmetry or conformal symmetry we have more control and sometimes it is possible to realise this paradigm in the context of dualities. Two main examples:

- IR dualities: we can describe the strongly coupled phase of a gauge theory in terms of a different weakly coupled theory.
- Holographic dualities the emergent fields live in a space with one extra dimension and the dual theory is a gravity theory.

# Large N holographic dualities

't hooft '74: the large N expansion of a gauge theory

$$\sum_{g\geq 0} N^{2-2g} f_g(\lambda),$$

with $\lambda = g_{YM}^2 N$, looks like a string expansion.

Maldacena '97: The IIB string on $AdS_5 \times S_5$ is equivalent to the 4d $\mathcal{N} = 4$ Super-Yang-Mills (SYM) theory.

Weaker form: at low energies the IIB string reduces to supergravity on $AdS_5 \times S_5$, in gauge theory this limit corresponds to large N and large $\lambda$.

Holographic dualities: classical super-gravity in $d + 1$ provides the weakly coupled description of a $d$-dimensional SUSY gauge theory at very strong coupling!

Key role of Integrability: the planar sector of SYM is solvable, for example it is possible to compute the scaling dimensions of local operators $\Delta[\mathcal{O}] = f_{\mathcal{O}}(\lambda)$.

AdS/CFT correspondences have been formulated also in less SUSY cases and in other dimensions. Although people are looking for integrability a key test remains the comparison of semiclassical supergravity computations in $AdS_d$ to large N strong coupling results on the $d-1$ gauge theory side.

The holy grail would be an exact gauge theory quantity (valid for generic $g_{YM}^2$) which can efficiently be studied at large N and large $\lambda$.

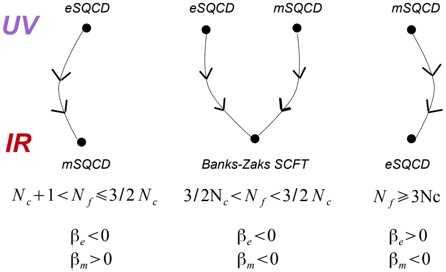
In 2007 Pestun applied the method of localisation to the path integral of 4d SUSY theories on the sphere $S^4$ and found the holy grail!

# IR dualities

First example for $\mathcal{N} = 1$ SQCD Seiberg'94.

eSQCD: $SU(N_c)$ with $N_f$ becomes IR free for $N_f > 3N_c$. In the conformal window $3/2N_c < N_f < 3N_c$ it flows to an IR SCFT. Below $N_f < 3/2N_c$ mesons go beyond the unitarity bound ($\Delta < 1$).

For $N_f > N_c + 1$ eSQCD has a dual description as mSQCD: $SU(N_f - N_c)$ with $N_f$ flavors and extra singlet $\Phi$ with $\mathcal{W} = \Phi_{i,j} q_i \tilde{q}_j$.




In the confining region $N_c + 1 < N_f \leq 3/2N_c$ the emergent weakly coupled dof are those of the magnetic dual mQCD which is IR free there.

The Seiberg duality has been generalized to other gauge groups and has cousins in 3d and 2d.

Tests have to be non-perturbative, typical strategies:

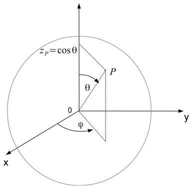
- Anomaly matching
- Map the gauge invariant operators of the electric theories to those of the magnetic duals
- Consistency checks: integrate out some of the fields and see what happens in the magnetic dual. Connect to the phases where an effective description is available ($\mathcal{W}$ constrained by holomorphy and symmetries).

Holy grail: an RG invariant (independent on $g_{YM}$) quantity that can be computed from the UV Lagrangian.

This object exist! It is a Witten Index counting protected operators [Romelsberger'05,'07,Doland-Osborn'08] and coincides with the path integral on $S^3 \times S^1$, it can also be computed via localisation.

# Localisation: a simple example

Suppose we want to calculate the average on a two-sphere of $e^{it\ f(\theta,\phi)}$ with $f(\theta,\phi) = \cos\theta$:




$$I = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta\ e^{it\cos\theta}\,.$$

This integral is simple to evaluate and yields:

$$I = 2\pi \int_{-1}^{1} d(\cos\theta)\ e^{it\cos\theta} = \tfrac{2\pi}{it}(e^{it} - e^{-it})\,.$$

# 1-loop exact

We could have tried to calculate this integral in the stationary phase approximation:

$$I(t) = \int d^n x \; g(x) \; e^{it\, f(x)} \approx \left(\tfrac{2\pi}{t}\right)^{\frac{n}{2}} \sum_i g(x_i) e^{it\, f(x_i)} \frac{e^{i\sigma_i \pi/4}}{\det(f''(x_i))^{1/2}}$$

$\sigma = \sigma^+ - \sigma^-$.

In general this is approximation and there are $1/t$ corrections.
Surprisingly in our case the stationary phase approximation is exact!

$$I = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \; e^{it\cos\theta} = \tfrac{2\pi}{it}(e^{it} - e^{-it})$$

The integral receives contributions only from the stationary points of the height function: the north and south poles of the sphere $\theta = 0, \pi$.

WHY?

The reason is the rotational symmetry about the $z$ axis, it is a circle or $U(1)$ action that sends $\phi \to \phi + \mathit{const}$.
The stationary points, the north and south poles, are also the fixed points of this $U(1)$ action.

In the '80s mathematicians Duistermaat-Heckman, Berline-Vergne and Atiyah-Bott discovered a class of integrals which localise: they receive contributions only from the fixed points of a $U(1)$ action.

In the '80s Witten applied localisation to the path integrals of SUSY QM and 2d and 4d QFTs and more recently in 2003 Nekrasov used it to calculate 4d instanton partition functions.

In 2007 Pestun applied localisation to the path integral of 4d theories with 8 supercharges on the sphere $S^4$. The compactness of the space serves as an IR regulator.

# SUSY Localisation

Suppose a theory has a BRST like super-symmetry $Q$ s.t. $Q^2 = 0$ and consider the vev of an operator invariant under $Q$:

$$\langle \mathcal{O} \rangle = \int DX \; \mathcal{O} \; e^{-S[X]} .$$

The vev is invariant if $S[X] \to S[X] + tQV$:

$$\frac{d}{dt}\langle \mathcal{O} \rangle_t \equiv \frac{d}{dt} \int DX \; \mathcal{O} \; e^{-S[X]-tQV} = -\int DX \; Q \left( V\mathcal{O} \; e^{-S[X]-tQV} \right) = 0$$

since $QS[X] = Q\mathcal{O} = 0$, the last equality holds provided the measure is not-anomalous and there are no-boundaries.

Since $\langle \mathcal{O} \rangle_0 = \langle \mathcal{O} \rangle_\infty$ the path integral will localise to the sum of the contributions of the saddle points points $X_0$ where $QV = 0$.

After expanding about the saddle points $X = X_0 + \frac{\chi}{\sqrt{t}}$ one is left to calculate quadratic integrals in the fluctuations $\chi$ yielding ratios of bosonic and fermionic determinants which we denote $Z_{1loop}$:

$$\langle \mathcal{O} \rangle = \lim_{t \to \infty} \int DX \ \mathcal{O} \ e^{-S[X] - tQV} = \int DX_0 \ \mathcal{O}_{|X=X_0} \ e^{-S[X_0]} Z_{1loop}[X_0]$$

The saddle point values $X_0$ are matrices in the Cartan of the gauge group.

The path integral is reduced to a matrix integral!

Subsequently localisation has been applied to a multitude of SUSY theory on compact spaces of various dimensions. Spheres $S^d$, $d = 2, .., 7$, indices $S^d \times S^1$, $d = 1, .., 5$ and more general manifolds.

# 2007-2017: 10 years of susy localization

The abundance of results obtained in the past 10 years and their many applications have been partially collected in a review:

**Localization techniques in quantum field theories**

**Vasily Pestun[1], Maxim Zabzine[2], Francesco Benini[3,4], Tudor Dimofte[5], Thomas T. Dumitrescu[6], Kazuo Hosomichi[7], Seok Kim[8], Kimyeong Lee[9], Bruno Le Floch[10], Marcos Mariño[11], Joseph A. Minahan[2], David R. Morrison[12], Sara Pasquetti[13], Jian Qiu[2,14,15], Leonardo Rastelli[16], Shlomo S. Razamat[17], Silvu S. Pufu[18], Yuji Tachikawa[19], Brian Willett[20], Konstantin Zarembo[21,2]**

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# Localization industry

Localising a SUSY theory on a curved background is a two step process:

- I) Formulate the theory on the curved background preserving some SUSY.

  First works did so by adding extra $1/r$ terms to the action.

  Systematic approach: start from off-shell supergravity, which is later decoupled leaving behind a rigid susy theory with extra background fields [Festuccia-Seiberg'11].

- II) Calculate determinants $\rightarrow$ special functions!

# The special functions feast

A surprising harmony emerges from the calculation of determinants.

For example for spheres with *squashing* parameters $\epsilon_1 \cdots \epsilon_r$ the localised path integrals is written in terms of two types of special functions:

$$\Upsilon_r(x|\epsilon) = \gamma_r\,(x|\epsilon)\,\gamma_r(\sum_i \epsilon_i - x|\epsilon)^{(-1)^r}\,, \quad S_r(x|\epsilon) = \gamma_r(x|\epsilon)\gamma_r(\sum_i \epsilon_i - x|\epsilon)^{(-1)^{r-1}}\ .$$

with

$$\gamma_r(x|\epsilon) = \prod_{n_1 \cdots n_r = 0}^{\infty} (x + n_1\epsilon_1 + \cdots + n_r\epsilon_r)\,.$$

For even and odd spheres an $SU(N)$ vector multiplet contributes as:

$$Z_{S^{2r}} = \int \prod_{i=1}^{N} da_i \prod_{i<j} \Upsilon_r(a_i - a_j|\epsilon)\ e^{P_r(a)} + \cdots\ ,$$

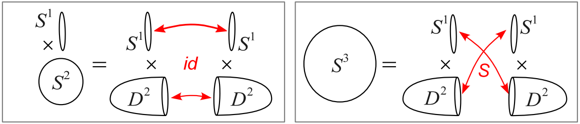
$$Z_{S^{2r-1}} = \int \prod_{i=1}^{N} da_i \prod_{i<j} S_r(a_i - a_j|\epsilon)\ e^{P_r(a)} + \cdots\ .$$

The degree $r$ polynomials $P_r(a)$ come from the classical action (Yang-Mills, Chern-Simons, FI couplings).

The dots indicate possible non-perturbative contributions (instantons, sectors with non-zero magnetic flux, $\cdots$).

These special functions have been studied by mathematicians, they have amazing properties. For example they have funny factorisation properties leading to an intriguing physical interpretation.

Consider the 3-manifolds $S^3$ and $S^2 \times S^1$, they can be obtained by gluing solid tori $D^2 \times S^1$ with appropriate identification of the boundaries:




It turns out that localised path integrals on these spaces enjoy a similar property [SP'12][Beem-Dimofte-SP'12]:

$$Z_{S^2\times S^1} = \sum_{\alpha} \left|\left|\mathcal{B}_{\alpha}^{3d}\right|\right|_{id}^{2}, \qquad Z_{S^3} = \sum_{\alpha} \left|\left|\mathcal{B}_{\alpha}^{3d}\right|\right|_{S}^{2}.$$

The 3d holomorphic blocks $\mathcal{B}_{\alpha}^{3d}$ are $D^2 \times S^1$ partition functions, $\alpha$ labels SUSY vacua.

This property holds for more general manifolds in various dimensions.

As more and more localisation results were emerging it appeared that partition functions are indeed a sort of topological objects, they do not depend on the metric.

However they depend on other geometric data, on various couplings and on parameters that could be turned on for the flavor symmetries.

Moreover some of the partition functions are RG invariants ($g_{YM}^2$ independent) such as $S^3 \times S^1$ while others such as $S^4$ depend on $g_{YM}^2$.

# Making order: when, what, how?

In parallel to derivation of the multitude of exact results it has also been developed a systematic approach to answer the following questions:

- When can we formulate a SUSY theory on curved space $M^d$ preserving some SUSY?
  For example for 4d theories one needs $M^4$ to be a complex manifold, [Klare-Tomasiello-Zaffaroni'12,Dumitrescu-Festuccia-Seiberg'12].
- What additional data on top of those already present in flat space enter the Lagrangian $\mathcal{L}_{M_d}$?
  Typically one needs to turn on extra background fields for example following the Festuccia-Seiberg appraoch.
- How does supersymmetry constrain the dependence of the localised partition function $Z_{M^d}$ on the data in $\mathcal{L}_{M_d}$?
  For example $Z_{M^3\times S^1}$ with $M^3 \sim \Sigma \times S^1$ can be shown to be metric and gauge coupling independent but depends on the complex structure and on the fugacities for the flavor symmetries. [Closset-Dumitrescu-Festuccia-Komargodski'14].

# Applications

- Many highly non-trivial tests of holographic dualities for 3d, 4d and 5d theories. (Required advanced large N matrix model techniques).
- Tests and new proposal for IR dualities in various dimensions using RG flow invariant partition functions (heavy use of mighty integral identities for special functions).
- Correspondences: inspired by the 6d perspective, connect gauge theories localised on curved spaces to low dimensional models.

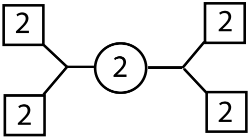
# 6=4+2 perspective

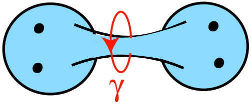
Think of 4d SUSY theories as compactifications of a mother 6d theory (describing coincident M5 branes) on a 2d surface $C_{g,n}$. [Gaiotto'09].

These 4d theories *remember their origin* and are called $\mathcal{T}_{g,n}$.

For example an $\mathcal{N}=2$ SQCD theory can be obtained from 2 M5 on a sphere with 4 punctures:

$\mathrm{SU}(2), \mathrm{N_f}=4$



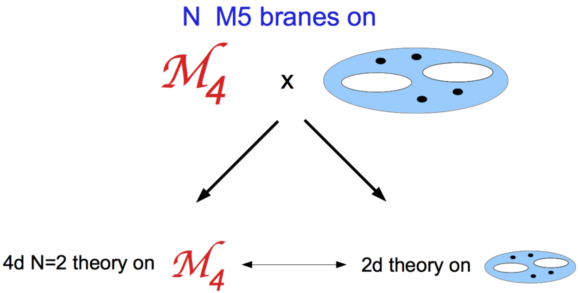

Tubes are like gauge groups, punctures carry flavor symmetries.

Different "pairs of pants" decompositions correspond to S-duality frames of the gauge theory.

→ This construction indicates that we can encounter exotic strongly coupled non-Lagrangian theories as we move in the parameter space of familiar theories!

# Correspondences

Physics intuition suggests that the 4d theories $\mathcal{T}_{g,n}$ should have a 2d avatar:




- For $M^4 = S^4$ the 2d avatar is the Liouville/Toda CFT theory [Alday-Gaiotto-Tachikawa'09].
- For $M^4 = S^3 \times S^1$ the 2d avatar is a 2d TQFT [Gadde-Pomoni-Rastelli-Razamat'09].

$\rightarrow$ Import new technologies from the 2d world in the study of 4d theories!

# Other applications

- 2d localisation on $S^2$, $T^2$ has important math applications!
- localisation and BH entropy [Dabholkar-Gomes-Murthy'11].
- F-extremization: the 3-sphere partition of $\mathcal{N}=2$ theories $F = -\log Z_{S^3}$ is extremized by the exact superconformal R-Symmetry [Jafferis'10].
- 3d c-theorem or F-theorem: $F = -\log Z_{S^3}$ is a monotonic function decreasing along the RG flow $F_{UV} > F_{IR}$ [Jafferis-Klebanov-Pufu-Safdi'11].

$\rightarrow$ Integrated Strategy to expand and better understand our understanding of dualities.

# 3D non-SUSY dualities

In the past two-years building on the results for SUSY IR dualities, holographic large N dualities for 3d Chern-Simons+ matter theories and the dualities describing phases of condensed matter systems, a web of dualities for non-susy 3d theories has been formulated [Aharony'15] [Seiberg-Senthil-Wang-Witten'16], [Karch-Tong'16]+many others.

In the abelian case the fundamental duality is a bosonization:

$$U(1) + 1\ \mathrm{fermion} + CS \leftrightarrow\ \mathrm{Wilson-Fisher\ scalar} + CS$$

By combining this duality with its time reversed version one can generate other dualities. For example one can re-derive the particle-vortex duality relating a gauged scalar to a Wilson-Fisher scalar.

There are also non-abelian generalisations.

Are we going to see also 4d non-susy dualities?

# 3d dualities with monopole potentials

3d gauge theories admit Monopole operators, local disorder operators, creating a magnetic flux on a 2-sphere surrounding the insertion point.

What happen if we add monopole operators to the Lagrangian?

It's tricky, monopoles are not polynomial in the elementary fields.

Why do we care?

New non-trivial fixed point theories $\mathcal{T}_{\mathfrak{M}}$, new dualities.

How can we reach $\mathcal{T}_{\mathfrak{M}}$?

Conservative approach: go to the fixed point $\mathcal{T}_0$ of the theory without monopole deformations. If $\Delta[\mathfrak{M}] < 3$ (relevant def.) then we can turn it on and initiate a flow to $\mathcal{T}_{\mathfrak{M}}$. Other possibilities?

Some of these questions can be addressed in the SUSY case combining: SUSY dualities and localisation/special functions techniques + F-theorem analysis

# Conservative approach

Consider the $\mathcal{N}=1$ SQCD $U(N_c)$with $N_f$ flavors $Q, \tilde{Q}$ and $\mathcal{W}=0$. We want to consider the deformation

$$\mathcal{W}_{\rm mon} = \mathfrak{M}^+ + \mathfrak{M}^-$$

Monopole charges under any Abelian symmetry are computed by

$$\delta Q(\mathfrak{M}) = -\frac{1}{2} \sum_{\text{fermions } \psi} Q(\psi) \left|\rho_\psi(\mathfrak{m})\right| ,$$

where the fermions $\psi$ transform with $\rho_\psi$ under the gauge group.

In particular the dimension $R[\mathfrak{M}^\pm]$ is given in terms of $R_Q$, in turned determined by F-maximization [Jafferis].

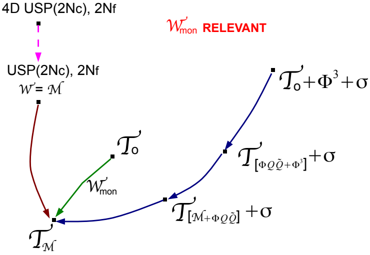
$\mathcal{W}_{\rm mon}$ is a good deformation in $\mathcal{T}_0$ if $\mathfrak{M}^\pm$ are relevant and above the unitarity bound:

$$1/2 < R[\mathfrak{M}^\pm]_{\mathcal{T}_0} < 2$$

this condition holds in a very narrow window of values of $N_f$ at fixed $N_c$!

There alternative ways to reach $\mathcal{T}_{\mathfrak{M}}$. The first one is to from 4d, as a bonus we also get a dual description $\mathcal{T}'_{\mathfrak{M}}$ [Benini-Benvenuti-SP].

# Possible RG flows leading to $\mathcal{T}_{\mathfrak{M}}$:



# $\mathcal{T}_{\mathfrak{M}}$ and its dual $\mathcal{T}'_{\mathfrak{M}}$ from 4d

The starting point is a version of the 4d Seiberg $\mathcal{N} = 1$ duality known as Intriligator-Pouliot duality:

$$USP(2N_c),\ 2N_f \text{ flavors } Q_i \leftrightarrow USP(2N_f - 2N_c - 4),\ 2N_f \text{ flavors } q_i,\ \ \mathcal{W} = \sum_{a<b} M^{ab} q_a q_b$$

When this dual pair is compactified on $R^3 \times S^1$ we obtain a 3d duality between:

$\mathcal{T}_1$: $USP(2N_c)$ with $2N_f$ flavors and $\mathcal{W}_1 = Y$

and

$\mathcal{T}_2$: $USP(2N_f - 2N_c - 4)$ with $2N_f$ flavors and $\mathcal{W}_2 = \sum_{a<b} M^{ab} q_a q_b + \tilde{Y}$

Key observation: in the reduction non-perturbaive monopoles $Y, \tilde{Y}$ are generated and contribute to the superpotential breaking the $U(1)_A$ symmetry, [Aharony-Hanany-Intriligator-Seiberg-Strassler], [Aharony-Razamat-Seiberg-Willett].

# $\mathcal{T}_{\mathfrak{M}}$ and its dual $\mathcal{T}'_{\mathfrak{M}}$ from 4d

The $\mathcal{T}_1 = \mathcal{T}_2$ duality reduces to the Aharony duality in a suitable mass deformation [Aharony-Razamat-Seiberg-Willett].

Considering a different mass deformation we propose the following duality between:

$\mathcal{T}_{\mathfrak{M}}$ : $U(N_c)$ with $N_f$ flavors $Q, \tilde{Q}$ and $\mathcal{W} = \mathfrak{M}^+ + \mathfrak{M}^-$

and

$\mathcal{T}'_{\mathfrak{M}}$ : $U(N_f - N_c - 2)$ with $N_f$ flavors $q, \tilde{q}$ and $\mathcal{W} = \hat{\mathfrak{M}}^+ + \hat{\mathfrak{M}}^- + Mq\tilde{q}$

The localised partition function of these dual pair agree and we can perform many consistency checks.

# Electric theory: $U(N_c), N_f$ and $\mathcal{W} = \mathfrak{M}^+ + \mathfrak{M}^-$

Imposing that monopole operators have $R$-charge 2 we find

$$R[\mathfrak{M}^\pm] = N_f(1 - R_Q) - (N_c - 1) = 2 \to R_Q = 1 - \frac{N_c + 1}{N_f}.$$

The only gauge invariant SUSY operators are $N_f^2$ mesons $Q_i \tilde{Q}_j$ of dimension $R[Q_i \tilde{Q}_j] = 2\left(1 - \frac{N_c+1}{N_f}\right)$ with $Rank[Q_i \tilde{Q}_j] \leq N_c$.

Mesons should be above their unitarity bound:

$$R[Q_i \tilde{Q}_j] = 2(1 - \frac{N_c + 1}{N_f}) > 1/2 \to N_f \geq \frac{4(N_c+1)}{3}.$$

Mesons can be mapped into the singlets of the dual theory.

# Phase space

As we vary $N_f$ and $N_c$ we encounter various phases:

- $N_f \geq N_c + 3$, $\mathcal{T}_{\mathfrak{M}}$ is interacting with $\mathcal{T}'_{\mathfrak{M}}$ dual:
  - $N_f > \frac{4}{3}(N_c + 1)$, $\mathcal{T}_{\mathfrak{M}}$ is completely interacting.
  - $N_c + 3 \leq N_f \leq \frac{4}{3}(N_c + 1)$, $\mathcal{T}_{\mathfrak{M}}$ has an interacting SCFT and a decoupled free sector: mesons are below the unitarity bound and the cubic superpotential $M_{ab} q_a \tilde{q}_b$ is irrelevant so the singlets decouple.
- $N_f = N_c + 2$, $\mathcal{T}_{\mathfrak{M}}$ is a Wess-Zumino model with $\mathcal{W} = \det(M)$, irrelevant for $N_c \geq 3$.
- $N_f = N_c + 1$ deformed Higgs branch $\det(M) = 1$.
- $N_f \leq N_c$ there are no supersymmetric vacua.

# 3D completion for $N_f \leq 3N_c + 2$

Start from $\mathcal{T}_0$, $N_f = 3N_c + 2$ plus $N_f^2$ chirals $\Phi_{ij}$ with $\mathcal{W} = \Phi_{ij}^3$.

- Couple $\Phi_{ij}$ to $\mathcal{T}_0$ by $\mathcal{W} = \sum_{ij=1}^{N_f} \left( \Phi_{ij} Q_i \tilde{Q}_j + \Phi_{ij}^3 \right)$.
- Flow to $\mathcal{T}_{\Phi_{ij} Q_i \tilde{Q}_j + \Phi_{ij}^3}$ with $R[Q] = R[\Phi_{ij}] = 2/3$ and $R[\mathfrak{M}^{\pm}] = N_f \big(1 - R[Q]\big) - N_c + 1 = \frac{5}{3} < 2$.
- We add $\mathfrak{M}^+ + \mathfrak{M}^-$, this sets $R[Q]_{\mathcal{T}} = \frac{2N_c+1}{3N_c+2}$ and $R[\Phi_{ij}] = 2 - 2R[Q] > 2/3$, the $N_f^2$ terms $\Phi_{ij}^3$ must be dropped.
- add $N_f^2$ extra chirals $\sigma_{ij}$ and couple them linearly to the $\Phi_{ij}$'s. Both $\Phi_{ij}$ and $\sigma_{ij}$ become massive and integrating them out we get $\mathcal{T}_{\mathfrak{M}}$.

For $N_f = 3N_c + 2$ we can reach $\mathcal{T}_{\mathfrak{M}}$ by a chain of 3D RG flows:

$$\mathcal{T}_0 \oplus \mathcal{T}_{\Phi^3} \oplus \sigma \;\rightarrow\; \mathcal{T}_{\Phi q \tilde{q} + \Phi^3} \oplus \sigma \;\rightarrow\; \mathcal{T}_{\mathfrak{M} + \Phi q \tilde{q}} \oplus \sigma \;\rightarrow\; \mathcal{T}_{\mathfrak{M}}$$

$\rightarrow$Lower $N_f$ via mass deformations.