Gauged Linear Sigma Models, Calabi-Yaus and Hemisphere Partition Function

[M. Romo, E. Scheidegger, JK: arXiv:1602.01382 [hep-th], in progress]

[K. Hori, JK: arXiv:1612.06214 [hep-th]]

[D. Erkinger, JK: arXiv:1704.00901 [hep-th]]

[R. Eager, K. Hori, M. Romo, JK: in progress]

Johanna Knapp

TU Wien

CERN, July 20, 2017

Outline

CYs and GLSMs

- A Calabi-Yau space (CY) can be realized as the low energy configuration of an $\mathcal{N} = (2,2)$ supersymmetric gauge theory in two dimensions - the gauged linear sigma model (GLSM).

  [Witten 93]

- The choice of gauge group and matter content determines the CY.
- Hypersurfaces and complete intersections in toric ambient spaces are realized by GLSMs with gauge groups $G = U(1)^k$.
- Non-abelian gauge groups lead to exotic CYs (e.g. determinantal varieties).
- Phases:
  - The FI-parameters and the θ-angles can be identified with the the Kähler moduli of the CY.
  - By tuning these couplings we can probe the Kähler moduli space.

GLSM Data

- G . . . a compact Lie group (gauge group)
- V . . . space of chiral fields $\phi_i \in V$
- $\rho_V : G \to GL(V)$ . . . faithful complex representation
  - CY condition: $G \to SL(V)$
- $R : U(1) \to GL(V)$ . . . R-symmetry
  - R_i . . . R-charges
  - R and ρ_V commute
  - charge integrality: $R(e^{i\pi}) = \rho_V(J)$ for $J \in G$
- $T \subset G$ . . . maximal torus
  - Lie algebras: $\mathfrak{g} = Lie(G)$, $\mathfrak{t} = Lie(T)$
  - $Q_i^a \in \mathfrak{t}_{\mathbb{C}}^*$ . . . gauge charges of chiral fields

GLSM Data (ctd.)

- $t \in \mathfrak{g}_{\mathbb{C}}^*$ . . . FI-theta parameter
  - $t^a = \zeta^a - i\theta^a$
  - $t^a \leftrightarrow$ Kähler moduli of the CY
- $\sigma \in \mathfrak{t}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ . . . scalar component of the vector multiplet
- $W \in \mathrm{Sym}(V^*)$ . . . superpotential
  - G-invariant
  - R-charge 2
  - non-zero for compact CYs
- Classical Potential

$$U = \frac{1}{8e^2}|[\sigma,\bar{\sigma}]|^2 + \frac{1}{2}\left(|Q(\sigma)\phi|^2 + |Q(\bar{\sigma})\phi|^2\right) + \frac{e^2}{2}D^2 + |F|^2$$

Classical Equations of Motion

- Higgs branch: $\sigma = 0$
- D-terms

$$\mu(\phi) = \zeta$$

  - $\mu : V \to \mathfrak{g}^*$ ... moment map
  - G is broken to a subgroup
- F-terms

$$dW = 0$$

- Phases: parameter space gets divided into chambers
- Classical Vacua

$$X_\zeta = \{dW^{-1}(0)\} \cap \mu^{-1}(\zeta)/G$$

  - Ideal I_ζ: $\phi \in V$ where the quotient is ill-defined

Coulomb branch

- Coulomb branch: $\phi = 0$, $\zeta = 0$
- σ fields can take any value classically.
- One-loop corrections generate en effective potential.

$$\widetilde{W}_{eff} = -t(\sigma) - \sum_i Q_i(\sigma)(\log(Q_i(\sigma)) - 1) + \pi i \sum_{\alpha>0} \alpha(\sigma)$$

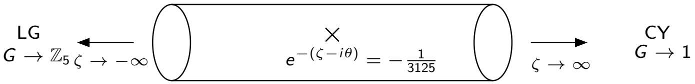
  - Q_i: weights of the matter representation ρ_V
  - $\alpha > 0$: positive roots
- Lifted except for points, lines, etc. $\rightarrow$ discriminant
  - One can smoothly interpolate between the phases.
  - The CY undergoes a topological transition.
- Mixed Branches if $\mathrm{rk}\, T > 1$.

Example 1: Quintic – $G = U(1)$

- Field content: $\phi = (p, x_1, \ldots, x_5) \in \mathbb{C}(-5) \oplus \mathbb{C}(1)^{\oplus 5}$
- Potential: $W = pG_5(x_1, \ldots, x_5)$
- D-term: $-5|p|^2 + \sum_{i=1}^{5} |x_i|^2 = \zeta$
- F-terms: $G_5(x_1, \ldots, x_5) = 0$, $p\frac{\partial G_5}{\partial x_i} = 0$
- $\zeta \gg 0$: $p = 0$, Quintic $G_5 = 0$ in $\mathbb{P}^4$
- $\zeta \ll 0$: Landau-Ginzburg orbifold with potential G_5
- Landau-Ginzburg/CY correspondence

[Witten 93][Herbst-Hori-Page 08][Kontsevich,Orlov]

- Moduli space



Example 2: Rødland model – $G = U(2)$

- Field content: $(p^1, \ldots, p^7, x_1, \ldots x_7) \in (\det^{-1} S)^{\oplus 7} \oplus S^{\oplus 7}$ with $S \simeq \mathbb{C}^2$ fundamental representation
- Potential:

$$W = \sum_{i,j,k=1}^{7} \sum_{a,b=1}^{2} A_k^{ij} p^k x_i^a \varepsilon_{ab} x_k^b = \sum_{i,j=1}^{7} A^{ij}(p)[x_i x_j]$$

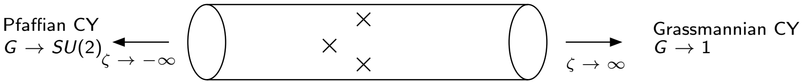
- D-terms:

$$\begin{aligned} -\sum_{i=1}^{7} |p^i|^2 + \sum_{j=1}^{7} |x_i|^2 &= \zeta \\ xx^\dagger - \frac{1}{2}||x||^2 \mathbf{1}_2 &= 0 \end{aligned}$$

Example 2: ctd.

- $\zeta \gg 0$: Complete intersection of codimension 7 in $G(2,7)$: $\sum_{i,j=1}^{7} A_k^{ij}[x_i x_j] = 0$
- $\zeta \ll 0$: Pfaffian CY in $\mathbb{P}^6$: $\mathrm{rk} A(p) = 4$; G broken to $SU(2) \rightarrow$ strongly coupled! (go to dual) [Hori '11]
- Grassmannian/Pfaffian correspondence

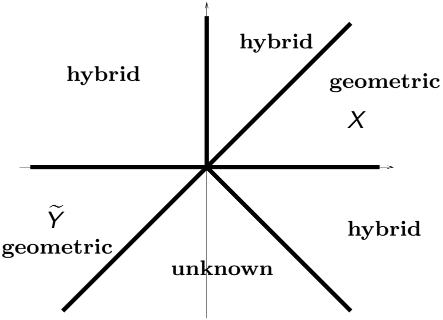
  [Hori-Tong '06][Rødland,Kuznetsov,Addington-Donovan-Segal,Halpern-Leistner,Ballard-Favero-Katzarkov]
- Moduli space



- The two CYs are not birational.

Example 3: Non-abelian two parameter model

- A two-parameter non-abelian model with two geometric phases. [Hori-JK arXiv:1612.06214 [hep-th]]
  - $G = (U(1)^2 \times O(2))/H$
  - free $\mathbb{Z}_2$-quotient of a codimension 5 complete intersection in a toric variety $\leftrightarrow$ determinantal variety
  - Phases



Example 3: (ctd.)

- The CYs in the geometric phases are not birational.
  - different fundamental groups
- There is a phase which has both strongly and weakly coupled components.
- The discriminant locus has a mixed Coulomb-confining branch that is mapped to a mixed Coulomb-Higgs branch under Hori's duality.
- There is an extra phase boundary due to non-abelian D-term equations.
- Mirror symmetry calculations work here because one phase is "almost" toric.

Further non-abelian examples

- One-parameter models with $(U(1) \times O(2))/H$
  - Reye congruence $\leftrightarrow$ determinantal quintic [Hosono-Takagi 11-14][Hori 11]
  - Hybrid model $\leftrightarrow$ determinantal variety in $\mathbb{P}^{11222}$ [Hori-JK 13]
- Models with $G = U(1) \times SU(2)$ and $U(2)$
  - Hybrid models $\leftrightarrow$ Pfaffian CYs in weighted $\mathbb{P}^4$

    [Kanazawa 10][Hori-JK '13]
- PAX/PAXY [Jockers et al. '12]
- SSSM [Gerhardus-Jockers '15]

B-branes in the GLSM

- B-branes in the GLSM are G-invariant matrix factorizations of the GLSM potential with R-charge 1

[Herbst-Hori-Page '08][Honda-Okuda,Hori-Romo '13]

- Data:
  - $\mathbb{Z}_2$-graded Chan-Paton space: space $M = M^0 \oplus M^1$
  - Matrix Factorization: $Q \in End^1(M)$ with

$$Q^2 = W \cdot \mathrm{id}_M$$

  - G-action: $\rho : G \to GL(M)$ with

$$\rho(g)^{-1} Q(g\phi)\rho(g) = Q(\phi)$$

  - R-action: $r_* : u(1)_R \to gl(M)$ with

$$\lambda^{\mathbf{r}_*} Q(\lambda^R \phi)\lambda^{-\mathbf{r}_*} = \lambda Q(\phi)$$

Examples

- No classification for matrix factorizations! However, there are some canonical examples.
- Example 1: Quintic

$$Q_1 = p\eta + G_5(x)\bar{\eta} \qquad \{\eta, \bar{\eta}\} = 1$$

- Example 2: Quintic

$$Q_2 = \sum_{i=1}^{5} x_i\eta_i + \frac{p}{5}\frac{\partial G_5}{\partial x_i}\bar{\eta}_i \quad \{\eta_i, \bar{\eta}_j\} = \delta_{ij}$$

- Example 3: Rødland

$$Q_3 = \sum_{i=1}^{7} p^i\eta_i + \frac{\partial W}{\partial p^i}\bar{\eta}_i$$

- Studied mostly in the context of B-twisted Landau-Ginzburg models.

Hemisphere Partition Function

- SUSY localization in the GLSM yields the hemisphere partition function. [Sugishita-Terashima,Honda-Okuda,Hori-Romo '13]

$$Z_{D^2}(\mathcal{B}) = C \int_\gamma d^{\mathrm{rk}_G}\sigma \prod_{\alpha>0} \alpha(\sigma)\,\mathrm{sinh}(\pi\alpha(\sigma)) \prod_i \Gamma\left(iQ_i(\sigma) + \frac{R_i}{2}\right) e^{it(\sigma)} f_{\mathcal{B}}(\sigma)$$

  - $\alpha > 0$ positive roots
  - $\sigma \in t_{\mathbb{C}}$ twisted chirals
  - R_i ... R-charges, Q_i ... gauge charges
  - $t = \zeta - i\theta$ ... complexified FI (Kähler) parameter(s)
  - γ ... integration contour (s.t. integral is convergent)
- Brane factor

$$f_{\mathcal{B}}(\sigma) = \mathrm{tr}_M\left(e^{i\pi\mathbf{r}_*} e^{2\pi\rho(\sigma)}\right)$$

  - M ... Chan-Paton space
  - The brane input is obtained by restricting the matrices $\rho(g)$ and $\lambda^{\mathbf{r}_*}$ to the maximal torus.

What does Z_{D^2} compute?

- Z_{D^2} computes the fully quantum corrected D-brane central charge.
- In large volume of $U(1)$ GLSMs (CY hypersurface X in toric ambient space), it reduces to the Gamma class: [Hori-Romo '13]

$$Z_{D^2}^{LV} = C \sum_{n=0}^{\infty} e^{-nt} \int_X \hat{\Gamma}_X(n) e^{B+\frac{1}{2\pi}\omega} \mathrm{ch}(\mathcal{B}_{LV})$$

- $\hat{\Gamma}(x) = \Gamma\left(1 - \frac{x}{2\pi i}\right)$ . . . Gamma class: $\hat{\Gamma}\hat{\Gamma}^* = \hat{A}_X$
- ω . . . Kähler class
- $\mathrm{ch}(\mathcal{B}_{LV})$ . . . Chern character of the LV brane
- $B \in H^2(X, \mathbb{Z})$ . . . B-field

Mirror Symmetry

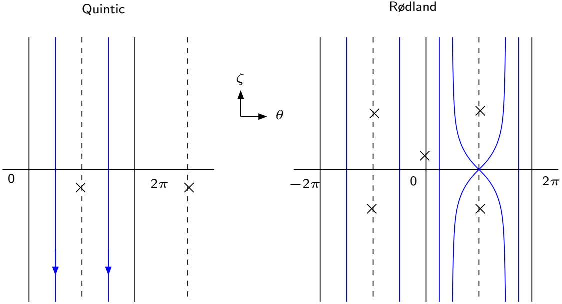
- Instanton corrections can be expressed in terms of the periods of the mirror CY.
- Example: Structure sheaf $\mathcal{O}_X$

$$
\begin{aligned}
Z(\mathcal{O}_X) &= \frac{H^3}{3!}\left(\frac{it}{2\pi}\right)^3 + \left(\frac{it}{2\pi}\right)\frac{c_2 H}{24} + i\frac{\zeta(3)}{(2\pi)^3}\chi(X) + O(e^{-t}) \\
&\overset{mirror}{=} \frac{H^3}{3!}\varpi_3 + \frac{c_2 H}{24}\varpi_2 + i\frac{\zeta(3)\chi(X)}{(2\pi)^3}\varpi_0
\end{aligned}
$$

- This is exactly what one gets when one evaluates Z_{D^2} in the $\zeta \gg 0$-phase for Q_1 and Q_3.
- However, no mirror symmetry required!

D-brane transport

- One can use the GLSM to transport D-branes from one phase to another.
    - Solved for abelian GLSMs. [Herbst-Hori-Page '08]
- Transport is non-trivial due to singularities in the moduli space.



Paths and grade restriction

- Not every GLSM brane can be transported along a given path in a well-defined way.
  - Brane may become unstable/break SUSY.
- Which branes are "allowed" is determined by the grade restriction rule. [Herbst-Hori-Page '08]
  - Given a path (i.e. choice of θ-angle) only certain gauge charges of the brane are allowed.
  - Given a brane in a phase, one can always find a GLSM brane that is in the charge window.
  - Understood in the abelian case.
- New approach: The hemisphere partition function knows about the grade-restriction rule. [Eager-Hori-Romo-JK in progress]

Z_{D^2} and grade restriction

- We can obtain the grade restriction rule from the asymptotic behavior of the hemisphere partition function.

$$
\begin{aligned}
Z^q_{D^2} &= \int_\gamma d^{l_G}\sigma\, e^{-A_q(\sigma)}, \quad \sigma = \sigma_1 + i\sigma_2 \\
A_q(\sigma) &= \zeta(\sigma_2) - (\theta - 2\pi q)(\sigma_1) + \\
&\quad + \sum_i \left\{ Q_i(\sigma_2)\left(\log|Q_i(\sigma)| - 1\right) + |Q_i(\sigma_1)| \left(\frac{\pi}{2} + \arctan \frac{Q_i(\sigma_2)}{|Q_i(\sigma_1)|} \right) \right\}
\end{aligned}
$$

  - Condition: Z_{D_2} has to be convergent for a given contour γ.
  - Depending on the path, this restricts the allowed charges q of the brane.
  - A_q is the effective boundary potential on the Coulomb branch.
  - Difficult to analyze because we have to find a parametrization for γ.

Results on grade restriction

- For the quintic, the known grade restriction rule is reproduced. [Hori-Romo '13]

$$A_q(\sigma) = \underbrace{(\zeta - 5\log 5)}_{\zeta^{eff}}\sigma_2 + (5\pi - \mathrm{sgn}(\sigma_1)(\theta + 2\pi q))|\sigma_1|$$

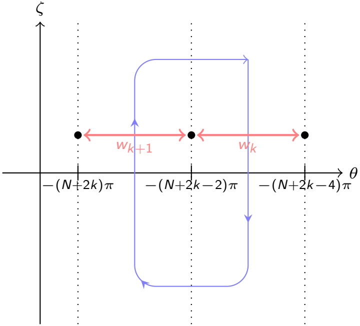
  - $\zeta^{eff} \gtrless 0$: $\sigma_2 \gtrless 0$ is admissible.
  - $\zeta^{eff} = 0$: $-\frac{5}{2} < \frac{\theta}{2\pi} + q < \frac{5}{2}$
- Rødland CY:
  - We find four different windows corresponding to the four inequivalent paths.
  - We can verify the correct monodromy behavior for the paths around the three conifold points.
  - In the strongly coupled Pfaffian phase, there is a grade restriction rule deep inside the phase.

How to grade restrict?

- Paths in the moduli space $\Longleftrightarrow$ window of allowed charges
- Grade restriction: replace a brane $\widehat{\mathcal{B}}$ outside the window by an equivalent one $\mathcal{B}$ that is the same in a given phase A
- Empty branes: branes $\mathcal{B}_E^A$ that reduce to "nothing" in a specific phase A
  - $Z_{D^2}^A(B_E^A) = 0$
  - $Q_1, Q_3 \ldots$ empty branes in the LG/Pfaffian, $Q_2 \ldots$ empty brane in the geometric phase
- $\widehat{\mathcal{B}}$ and a bound state $\mathcal{B} = \widehat{\mathcal{B}} \xrightarrow{\Psi} \mathcal{B}_E^A$ (tachyon condensation) reduce to the same brane in phase A
  - Binding empty branes changes the gauge charges
  - Use empty branes to grade restrict to a given charge window

GLSM and Monodromy

- We can compute the monodromy of a D-brane by transporting it along a non-contractible loop in the moduli space [Herbst-Hori-Page '08]
- Monodromy inside a phase: θ-angle shift by $2\pi \leftrightarrow$ shift of gauge charges of the brane by $+1$
- Monodromy across a phase boundary: grade restrict with respect to adjacent window



Z_{D^2} and Monodromy

- We can read off monodromy matrices from the brane factors of the hemisphere partition function in the GLSM

  [Romo-Scheidegger-JK '16][Erkinger-JK '17]

- Recipe:
  1. Choose a reference path/window
  2. Take a brane grade $\mathcal{B}$ restricted to a path
  3. Perform the monodromy operation with image brane $\mathcal{B}'$
  4. Grade restrict back to the reference window to get a brane $\widetilde{\mathcal{B}}'$
  5. Read off the the monodromy matrix M_{ij} from the brane factors

$$f_{\widetilde{\mathcal{B}}'_i} = \sum_j M_{ij} f_{\mathcal{B}_i}$$

- Explicit examples for one-parameter hypersurfaces
- Also works for the Rødland example.

Z_{D^2} in non-geometric phases

- In geometric phases Z_{D^2} computes the quantum corrected central charge of the D-brane.
- What about Landau-Ginzburg orbifold phases?

[Lerche-Vafa-Warner '89][Intriligator-Vafa '90][Fan-Jarvis-Ruan-Witten][Chiodo-Iritani-Ruan]

- In one-parameter models with a LG-phase the hemisphere partition function computes [Romo-Scheidegger-JK]

$$Z_{D^2}^{LG} = \left(I_{FJRW}, \hat{\Gamma}_{FJRW} \circ \mathrm{ch}(Q) \right)$$

- I_{FJRW} . . . I-Function (periods)
- $\hat{\Gamma}_{FJRW}$ . . . Gamma-class
- $\mathrm{ch}(Q)$ . . . Chern character [Walcher '04]
- $(\cdot, \cdot)$ . . . pairing

Landau-Ginzburg Central Charge (ctd.)

- Show that Z_{D^2} reduces to this in the LG phase.
- Works for a basis of branes/periods on the quintic.
- Generalization to two-parameter models seems to work.
- Does it define a stability condition?

  [Douglas-Fiol-Römelsberger '00][Aspinwall-Douglas '01][Douglas '02][Bridgeland]

- At the Landau-Ginzburg point this should reduce to R-stability [Walcher '04]
  - Compare stable vs. unstable configurations of branes and their bound states.

Summary

- GLSMs are useful physics tools to study CYs and their moduli spaces.
- Non-abelian GLSMs lead to exotic CYs that are not complete intersections in toric ambient spaces.
- Supersymmetric localization gave us new tools to calculate quantum corrections in CY compactifications.
    - Without mirror symmetry
    - Applies to "exotic" CYs: determinantal varieties, etc.
    - Applies also to "exotic" phases: strongly coupled, hybrid, etc.
- The hemisphere partition function computes the quantum corrected central charge of a D-brane
    - D-brane transport and monodromies
    - Landau-Ginzburg central charge

Open Problems

- Systematic construction of exotic CYs via non-abelian GLSMs
  - Higher rank gauge groups
  - Exotic matter [Galkin's talk]
- Central charge in non-geometric phases.
  - Hemisphere partition function as ("global") stability condition?
  - Analytic continuation of periods of CYs with more than one conifold point [Romo-Scheidegger-JK][Klemm's talk]
- Better understanding of hybrid phases
- D-branes and D-brane transport on exotic CYs
- Mirror symmetry for exotic CYs
- Relation to localization results in higher dimensions and implications for CY physics