Borel resummation and Perturbative series in Supersymmetric gauge theories

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References:

[1] M.H., “Borel Summability of Perturbative Series in 4D N=2 and 5D N=1 Supersymmetric Theories”, PRL116, 211601(2016) (arXiv: 1603.06207 [hep-th])

[2] M.H., “How to resum perturbative series in 3d N=2 Chern-Simons matter theories”, PRD94, 025039 (2016) (arXiv:1604.08653 [hep-th])

[3] M.H., to appear

21st, July

Geometry of String and Gauge Theories @CERN

Perturbative expansion in QFT

- Typically non-convergent [Dyson '52]
- Naïve sum → ∞

Then,

what does perturbative series actually know?

General question in this talk

Perturbative series around saddle points:

$$\mathcal{O}(g) \simeq \sum_{\ell=0}^{\infty} c_\ell^{(0)} g^\ell + \sum_{I\in\text{saddles}} e^{-S_I(g)} \sum_{\ell=0}^{\infty} c_\ell^{(I)} g^\ell$$

Can we get the exact result by using the coefficients?

= What is a correct way to resum the perturbative series?

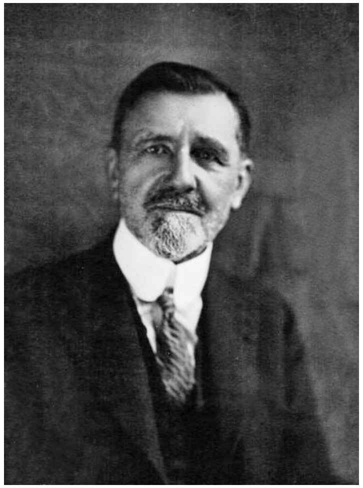
(～continuum definition of QFT?)

This talk = To answer this in SUSY gauge theories

A standard resummation

Borel transformation:

$$\mathcal{O}(g) \simeq \sum_{\ell=0}^{\infty} c_\ell g^{a+\ell} \quad \Rightarrow \quad \boxed{\mathcal{BO}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} t^{a+\ell-1}}$$



(from Wikipedia)

Borel resummation (along θ):

$$\boxed{S_\theta \mathcal{O}(g) = \int_0^{e^{i\theta}\infty} dt \; e^{-\frac{t}{g}} \; \mathcal{BO}(t)}$$

(usually, θ=arg(g)=0)

Why Borel resummation may be nice

(Let us take θ=arg(g))

$$S_\theta \mathcal{O}(g) = \int_0^{e^{i\theta}\infty} dt\ e^{-\frac{t}{g}}\ \mathcal{B}\mathcal{O}(t) \qquad \mathcal{B}\mathcal{O}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} t^{a+\ell-1}$$

① Reproduce original perturbative series:

$$S_\theta \mathcal{O}(g) \simeq \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} \int_0^{e^{i\theta}\infty} dt\ t^{a+\ell-1} e^{-\frac{t}{g}} = \sum_{\ell=0}^{\infty} c_\ell g^{a+\ell}$$

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② Finite for any g if

1. Borel trans. is convergent
2. Its analytic continuation does not have singularities along the contour
3. The integration is finite

"Borel summable (along θ)"

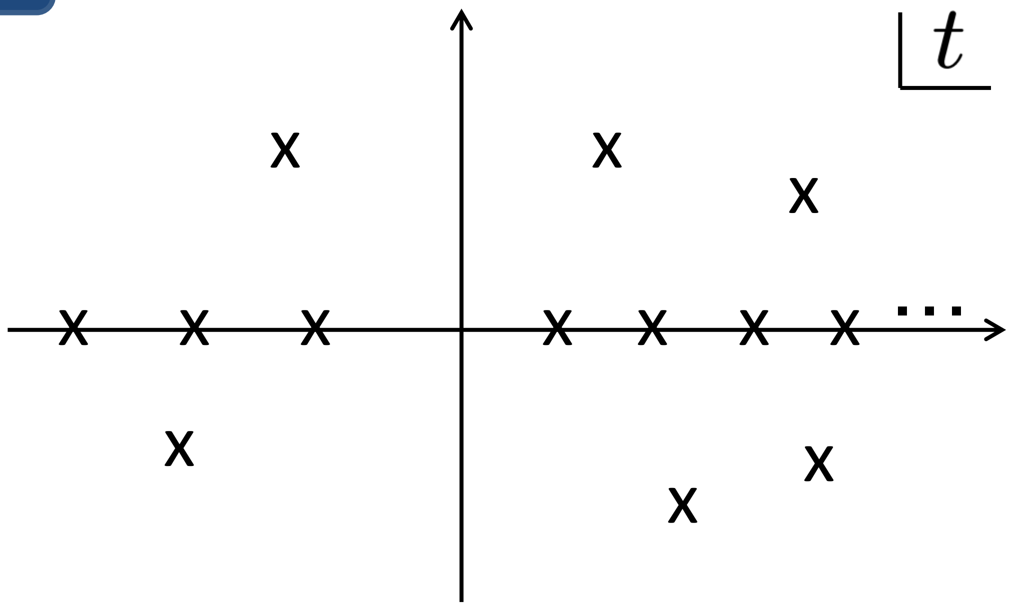
related to exact result?

Expectations in typical QFT

['t Hooft '79]

Non-Borel summable due to singularities along R_+

Borel plane: (singularities of Borel trans.)



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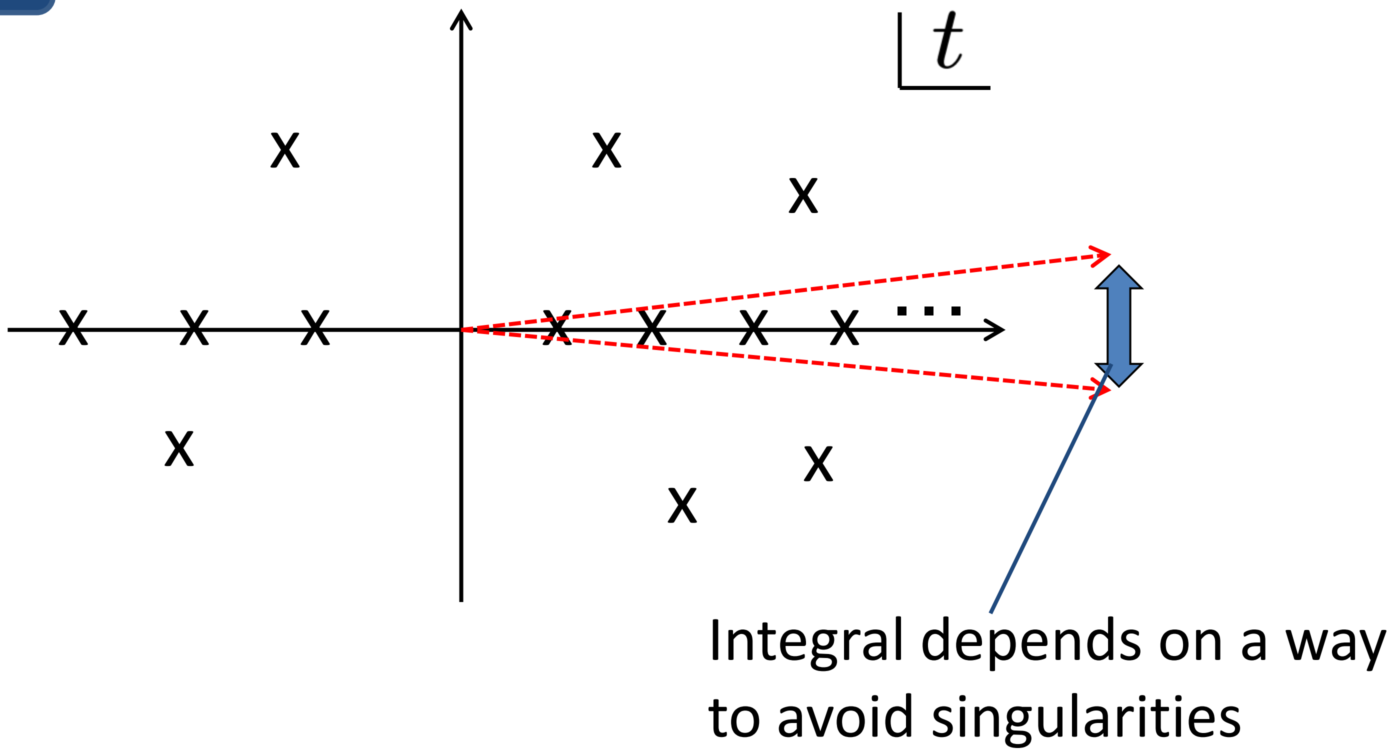

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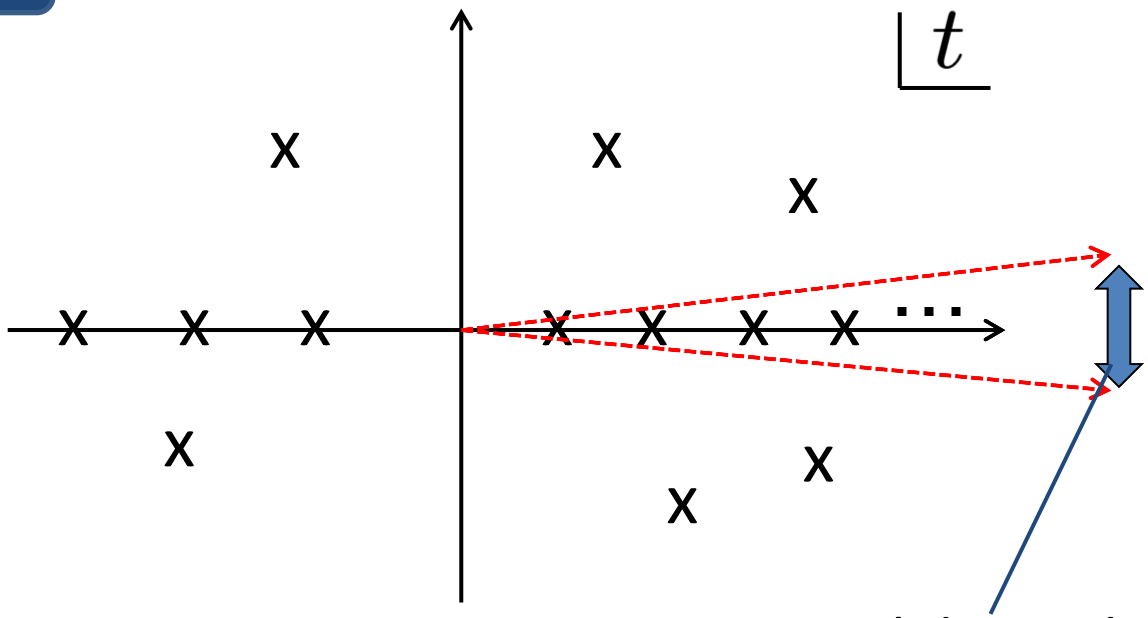


Integral depends on a way to avoid singularities

$$S_{\theta=0}\mathcal{O}(g) = \int_0^\infty dt\ e^{-\frac{t}{g}}\ \mathcal{B}\mathcal{O}(t) \Rightarrow (\text{Residue}) \sim e^{-\frac{\sharp}{g}}$$

Non-perturbative effect?

More concrete questions

- When is perturbative series Borel summable? (along R_+)
- Given a theory,
  what is analytic property of Borel trans.?
- If Borel summable,
  how is Borel resum related to exact result?
- If non-Borel summable,
  what is a correct way to resum perturbative series?

This talk = To answer this in SUSY gauge theories

Setups

- 4d N=2 and 5d N=1 theories on sphere

  —— expansion by g_{YM} around instanton b.g.

- 3d N=2 CS matter theories on sphere & lens sp.

  —— expansion by inverse CS levels

Setups

- 4d N=2 and 5d N=1 theories on sphere

—— expansion by g_{YM} around instanton b.g.

- 3d N=2 CS matter theories on sphere & lens sp.

—— expansion by inverse CS levels

Here we study only localizable quantities.

Motivations:

- We can practically get much perturbative information
- We can also study perturbative series around nontrivial saddles
- We can check relation between resummation and exact results

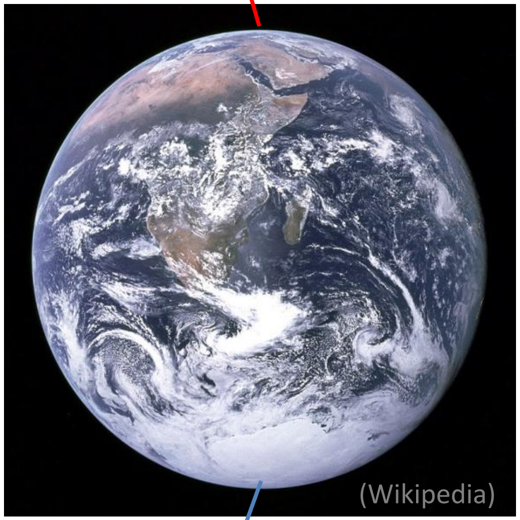
Summary of main results

Results on 4d N=2 SUSY theories (w/ 8 SUSY)

[M.H. '16]

Set up:

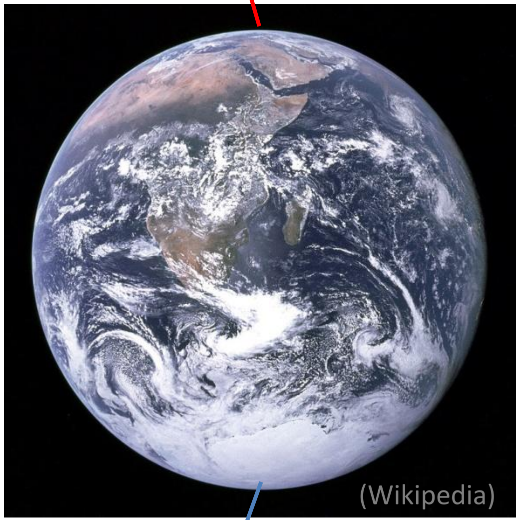
- Theories w/ $\beta \leqq 0$ and Lagrangians $(Z_{S^4} < \infty)$
- Perturbative expansion by g_{YM} around fixed # of instanton/anti-inst.



Results on 4d N=2 SUSY theories (w/ 8 SUSY)

[M.H. '16]

inst.



anti-inst.

Set up:

- Theories w/ $\beta \leqq 0$ and Lagrangians $(Z_{S^4} < \infty)$
- Perturbative expansion by g_{YM} around fixed # of instanton/anti-inst.

Result:

(similar for 5d case)

[cf. some SU(2) theories: Russo, Aniceto-Russo-Schiappa]

- Find explicit finite dimensional integral rep. of Borel trans. for various observables
- $^{\exists}$Singularities only along R- → Borel summable along R+ (for round S^4)
- $(\text{Exact}) = \sum_{\text{instantons}} (\text{Borel resum})$

Examples

For SU(2) case,

$$\mathcal{B}Z_{S^4}^{(k,\bar{k})}(t) \propto a \cdot Z_{1-\mathrm{loop}}(a) Z_{\mathrm{Nek}}^{(k)}(a) Z_{\mathrm{Nek}}^{(\bar{k})}(a) \Big|_{a=\sqrt{t}}$$

(anti-)instanton # Nekrasov partition func.

Ex.1) Pure SYM (trivial b.g.):

$$\mathcal{B}Z_{S^4}^{(0,0)}(t) \propto \sqrt{t} \prod_{n=1}^{\infty} \left(1 + \frac{4t}{n^2}\right)^{2n}$$

No singularities ⟷ Convergent expansion

Ex.2) SQCD (trivial b.g.):

$$\mathcal{B}Z_{S^4}^{(0,0)}(t) \propto \sqrt{t} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{4t}{n^2}\right)^{2n}}{\left(1 + \frac{t}{n^2}\right)^{2N_f n}}$$

Interpretations

Result:

$^{\exists}$Borel singularities only along R- in 4d N=2 SUSY theories

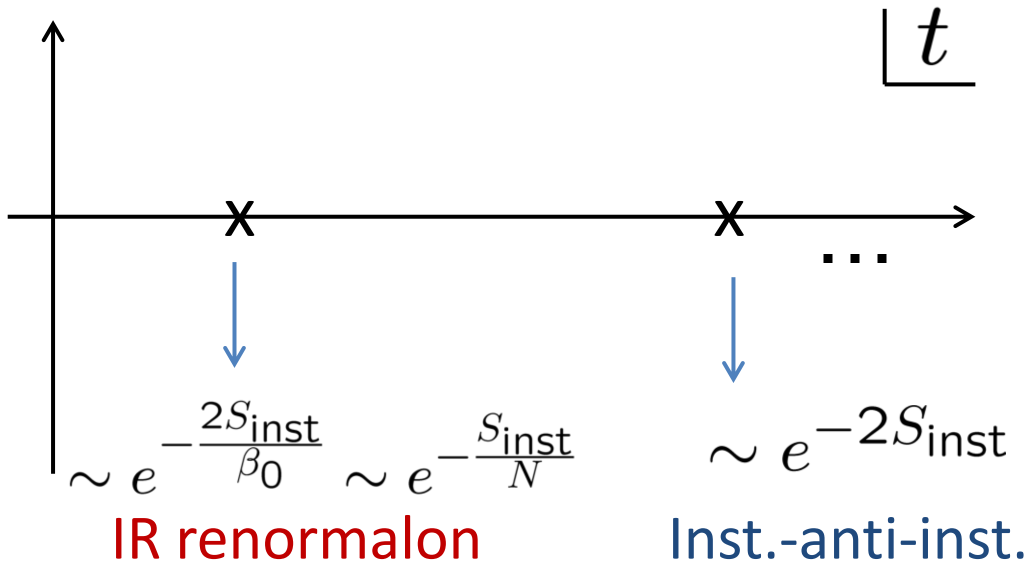
$^{\exists}$Agreement w/ recent conjecture on QCD-like theory

&

Confusion compared w/ usual story of resummation

Nontrivial consistency w/ a conjecture on QCD

Borel plane in typical gauge theory :



Conjecture: (IR renormalon) = (Combination of monopoles)

[Argyres-Unsal '12]

But there is no such solution for $\mathcal{N}=2$ [Popitz-Unsal]

$\longrightarrow$ No IR renormalon type singularities for $\mathcal{N}=2$?

Confusion?

Usually Borel singularities come from nontrivial saddles w/ the same topological numbers [cf. Lipatov '77, Argyres-Unsal '12]

Now we have $\int_{S^4} F \wedge F \propto k - \bar{k}$

Confusion?

Usually Borel singularities come from nontrivial saddles w/ the same topological numbers [cf. Lipatov '77, Argyres-Unsal '12]

Now we have $\int_{S^4} F \wedge F \propto k - \bar{k}$

For example, around trivial saddle, we expect

- No Borel singularities from $k \neq \bar{k}$
- $^{\exists}$ Borel singularities from $k = \bar{k}$ (namely, at t=2k)

But we do not have such singularities.

Results on 3d N=2 SUSY Chern-Simons theories

(w/ 4 SUSY)

Set up:

[M.H. '16]

- General Chern-Simons (CS) theories coupled to matters
  $(Z_{S^3} < \infty)$

- Perturbative expansion by inverse CS levels

Results on 3d N=2 SUSY Chern-Simons theories

(w/ 4 SUSY)

[M.H. '16]

Set up:

- General Chern-Simons (CS) theories coupled to matters $(Z_{S^3} < \infty)$
- Perturbative expansion by inverse CS levels

Result:

$$S_\theta I(g) = \int_0^{e^{i\theta}\infty} dt\ e^{-\frac{t}{g}}\ \mathcal{B}I(t)$$

- Find finite dimensional integral rep. of Borel trans.
- Usually non-Borel summable along R+
- But always Borel summable along (half-)imaginary axis
- (Borel resum. w/ θ=±π/2) = (exact result)

Examples

For SU(2) case,

$$\mathcal{B}Z_{S^3}(t) \propto \sigma \cdot Z_{1-\text{loop}}(\sigma)\Big|_{\sigma=\sqrt{i\,\text{sgn}(k)t}}$$

Ex.1) Pure SUSY CS:

$$\mathcal{B}Z_{S^3}(t) \propto \sigma \cdot \sinh^2(\sigma)\Big|_{\sigma=\sqrt{i\,\text{sgn}(k)t}}$$

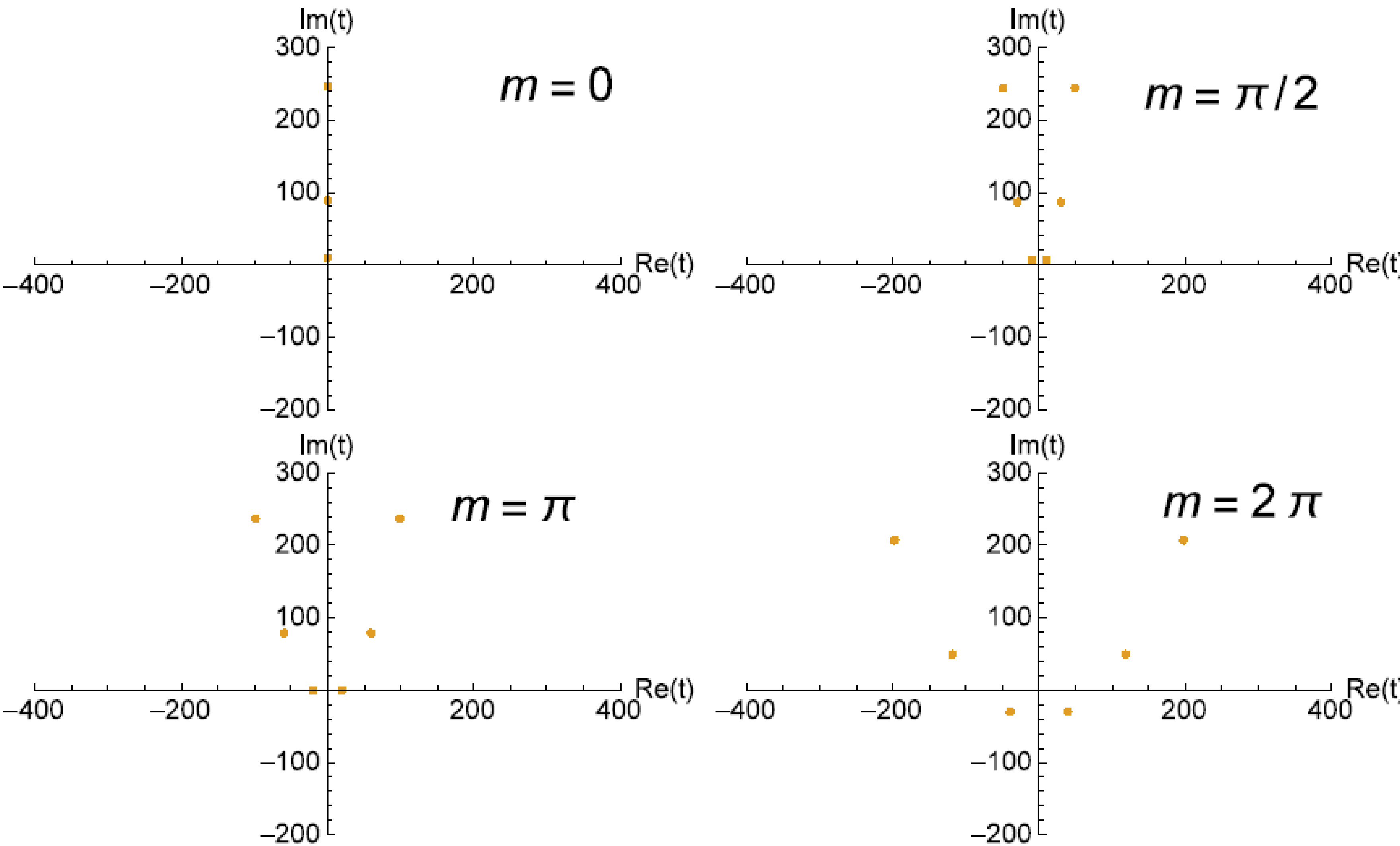
No singularities $\longleftrightarrow$ Convergent expansion

Ex.2) SQCD w/ hypers and real mass:

$$\mathcal{B}Z_{S^3}(t) \propto \frac{\sigma \cdot \sinh^2(\sigma)}{\left(\cosh\frac{\sigma-m}{2}\cosh\frac{\sigma+m}{2}\right)^{N_f}}\Bigg|_{\sigma=\sqrt{i\,\text{sgn}(k)t}}$$

Ex.2) SQCD w/ hypers and real mass

$$\mathcal{B}Z_{S^3}(t) \propto \left. \frac{\sigma \cdot \sinh^2(\sigma)}{\left(\cosh\frac{\sigma-m}{2}\cosh\frac{\sigma+m}{2}\right)^{N_f}} \right|_{\sigma=\sqrt{i\mathrm{sgn}(k)t}}$$



Interpretation of Borel singularities (3d)

[M.H., to appear]

All the singularities can be explained by

complexified SUSY solutions

which are not on original contour of path integral
but formally satisfy SUSY conditions: $Q\lambda = 0, \; Q\psi = 0$

Interpretation of Borel singularities (3d)

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All the singularities can be explained by

complexified SUSY solutions

which are not on original contour of path integral
but formally satisfy SUSY conditions: $Q\lambda = 0, \; Q\psi = 0$

Indeed their actions agree residues:

$$e^{-S} \sim \mathrm{Res}\left[\mathcal{B}\mathcal{O}(t)\right]$$

The numbers also agree if we follow the rule:

1 solution	$\longleftrightarrow$	Simple pole
n sols. w/ the same S	$\longleftrightarrow$	Degree-n pole

Contents

Partition function of Superconformal QCD on S^4

SU(N) SQCD w/ 2N-fundamental hypermultiplets

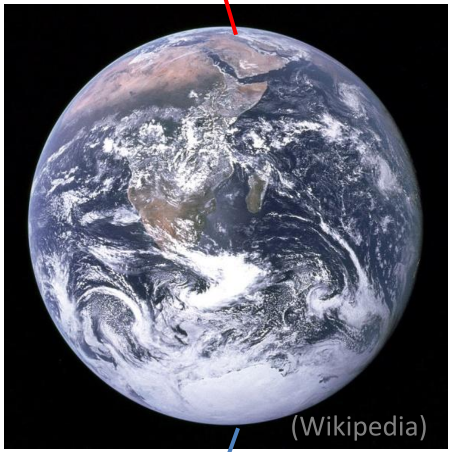
Exact result by localization method:

[Pestun '07]

$$Z_{\text{SQCD}}(g,\theta) = \int_{-\infty}^{\infty} d^N a \, e^{-\frac{1}{g}\sum_{j=1}^{N} a_j^2} \tilde{Z}(a) Z_{\text{inst}}(g,\theta;a)$$

$$Z_{\text{inst}}(g,\theta;a) = \sum_{k,\bar{k}=0}^{\infty} e^{-\frac{k+\bar{k}}{g}+i(k-\bar{k})\theta} Z_{\text{inst}}^{(k,\bar{k})}(a)$$

$$Z_{\text{SQCD}}^{(k,\bar{k})}(g) = \int_{-\infty}^{\infty} d^N a \; e^{-\frac{1}{g}\sum_{j=1}^{N} a_j^2} \tilde{Z}(a) Z_{\text{inst}}^{(k,\bar{k})}(a)$$



We are interested in small-g expansion of this

We would like to study small-g expansion of

$$Z_{\text{SQCD}}^{(k,\bar{k})}(g) = \int_{-\infty}^{\infty} d^N a \ e^{-\frac{1}{g}\sum_{j=1}^{N} a_j^2} \tilde{Z}(a) Z_{\text{inst}}^{(k,\bar{k})}(a)$$

A naïve way:

1. Compute perturbative expansion at all orders
2. Compute Borel transformation
3. Look at its analytic property

Difficult, inappropriate for exploring infinite examples

We would like to study small-g expansion of

$$Z_{\mathrm{SQCD}}^{(k,\bar{k})}(g)=\int_{-\infty}^{\infty} d^N a\ e^{-\frac{1}{g}\sum_{j=1}^N a_j^2}\tilde{Z}(a)Z_{\mathrm{inst}}^{(k,\bar{k})}(a)$$

~~A naïve way:~~ Our method:

1. ~~Compute perturbative expansion at all orders~~
2. ~~Compute Borel transformation~~ Find Borel trans. hidden in localization formula
3. Look at its analytic property

Borel trans. hidden in localization formula

$$Z_{\rm SQCD}^{(k,\bar{k})}(g) = \int_{-\infty}^{\infty} d^N a \ e^{-\frac{1}{g}\sum_{j=1}^N a_j^2} \tilde{Z}(a) Z_{\rm inst}^{(k,\bar{k})}(a)$$

Taking polar coordinate $a_i = \sqrt{t}\hat{x}_i$ w/ $(\hat{x}^i)^2 = 1$,

$$Z_{\rm SQCD}^{(k,\bar{k})}(g) = \int_0^{\infty} dt \ e^{-\frac{t}{g}} f^{(k,\bar{k})}(t)$$

Similar to the Borel resummation formula!

Is this Borel transformation?

$$\left(f^{(k,\bar{k})}(t) = \int_{S^{N-1}} d^{N-1}\hat{x} \ h^{(k,\bar{k})}(t,\hat{x}), \ \ h^{(k,\bar{k})}(t,\hat{x}) = \tilde{Z}(a) Z_{\rm inst}^{(k,\bar{k})}\Big|_{a^i=\sqrt{t}\hat{x}^i} \right)$$

$$Z_{\mathrm{SQCD}}^{(k,\bar{k})}(g) = \int_0^\infty dt\ e^{-\frac{t}{g}} f^{(k,\bar{k})}(t)$$

Is this Borel trans.?

More precisely, given $Z_{\mathrm{SQCD}}^{(k,\bar{k})}(g) \sim \sum_{\ell=0}^{\infty} c_\ell^{(k,\bar{k})} g^{\sharp+\ell}$,

$$f^{(k,\bar{k})}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell^{(k,\bar{k})}}{\Gamma(\sharp+\ell)} t^{\sharp+\ell-1}\ ??$$

(analytic continuation)

We can prove that this is actually true.

Outline of Proof

$$Z^{(k,\bar{k})}_{\mathsf{SQCD}}(g) = \int_0^\infty dt\ e^{-\frac{t}{g}} f^{(k,\bar{k})}(t)$$

$$\boxed{f^{(k,\bar{k})}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell^{(k,\bar{k})}}{\Gamma(\sharp+\ell)} t^{\sharp+\ell-1} \ ??}$$

(1) Show $f^{(k,\bar{k})}(t)$ purely consists of convergent power series:

$$f^{(k,\bar{k})}(t) = \sum_{\ell=0}^{\infty} f_\ell^{(k,\bar{k})} t^{\sharp+\ell-1}$$

(2) Laplace trans. guarantees $f_\ell^{(k,\bar{k})} = \frac{c_\ell^{(k,\bar{k})}}{\Gamma(\sharp+\ell)}$

Outline of Proof

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Proof of (1):

$$f^{(k,\bar{k})}(t) = \int_{S^{N-1}} d^{N-1}\hat{x}\ h^{(k,\bar{k})}(t,\hat{x})$$

(a) Show $h^{(k,\bar{k})}(t,\hat{x})$ consists of convergent power series of t

(b) Small-t expansion of $h^{(k,\bar{k})}(t,\hat{x})$ commutes w/ the integral

(This is true if small-t expansion of $h^{(k,\bar{k})}(t,\hat{x})$ uniform convergent)

Analytic property of Borel trans.

$$Z^{(k,\bar{k})}_{\text{SQCD}}(g) = \int_0^\infty dt\ e^{-\frac{t}{g}} f^{(k,\bar{k})}(t),\ f^{(k,\bar{k})}(t) = \int_{S^{N-1}} d^{N-1}\hat{x}\ h^{(k,\bar{k})}(t,\hat{x})$$

For trivial b.g.,

$$h^{(0,0)}(t,\hat{x}) = \delta\left(\sum_j \hat{x}_j\right) \prod_{i<j} (\hat{x}_i - \hat{x}_j)^2 \prod_{n=1}^\infty \frac{\prod_{i<j}\left(1+\frac{t(\hat{x}_i-\hat{x}_j)^2}{n^2}\right)^{2n}}{\prod_j\left(1+\frac{t(\hat{x}_j)^2}{n^2}\right)^{2Nn}}$$

No singularities for $t \in R_+$ ⇒ Borel summable!!

Non-zero instanton sector

$$Z_{\text{SQCD}}^{(k,\bar{k})}(g) = \int_0^\infty dt\ e^{-\frac{t}{g}} f^{(k,\bar{k})}(t), \quad f^{(k,\bar{k})}(t) = \int d^{N-1}\hat{x}\ h^{(k,\bar{k})}(t,\hat{x})$$

$$h^{(k,\bar{k})}(t,\hat{x}) = h^{(0,0)}(t,\hat{x}) Z_{\text{inst}}^{(k,\bar{k})}(a = \sqrt{t}\hat{x})$$

Rational function of a, whose poles are not in real axis

[cf. Nekrasov '03]

Thus,

Borel trans. is not singular for $t \in R_+$

⇒ Borel summable!!

General theory w/ Lagrangians (&β≦0)

Suppose a theory w/ gauge group: $G = G_1 \times \cdots \times G_n$

$$Z_{S^4}(g,\theta) = \int_{-\infty}^{\infty} d^{|G|}a \; Z_{\rm cl}(g;a)\tilde{Z}(a)Z_{\rm inst}(g,\theta;a)$$

$$Z_{\rm cl}(g;a) = \exp\left[-\sum_{p=1}^{n}\frac{1}{g_p}{\rm tr}(a^{(p)})^2\right]$$

Taking polar coordinate $a_i^{(p)} = \sqrt{t_p}\widehat{x}_i^{(p)}$,

$$Z_{S^4}^{(\{k\},\{\bar{k}\})}(g) = \int_0^{\infty} d^n t \; e^{-\sum_p \frac{t_p}{g_p}} f^{(\{k\},\{\bar{k}\})}(t_1,\cdots,t_n)$$

Borel trans.

⇨ Borel summable!!

Remark on non-conformal case

- g_{YM} is running
- Here g_{YM} is at scale $1/R_{sphere}$

For example, in pure SYM case,

[cf. Pestun '07]

$$e^{-\frac{8\pi^2}{g_{\rm YM}^2}{\rm tr}a^2} \cdot Z_{\rm 1-loop}^{\mathcal{N}=2^*}(a,m) \xrightarrow{mR_{S^4} \gg 1} e^{-\frac{8\pi^2}{\tilde{g}_{\rm YM}^2}{\rm tr}a^2} \cdot Z_{\rm 1-loop}^{{\rm pure}\mathcal{N}=2}(a)$$

$$\frac{1}{\tilde{g}_{\rm YM}^2} = \frac{1}{g_{\rm YM}^2} - \frac{C_2}{8\pi^2}\log\left(mR_{S^4}\right)$$

Relation to the exact result

We have shown

(Borel resum. in sector w/ fixed inst./anti-inst. #)

$$||$$

(Truncation of whole exact result to the same sector)

Thus,

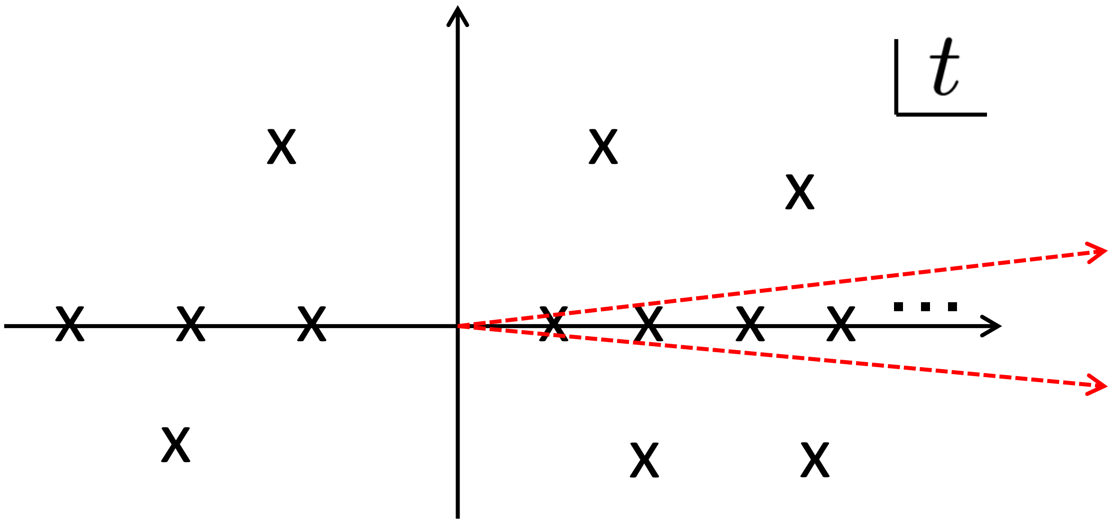
(Exact result including full instanton corrections)

$$||$$

$$\sum_{k,\bar{k}} \text{(Borel resummation)}$$

Relation to (typical) resurgence scenario

Suppose perturbation around trivial saddle is non-Borel summable:



$$S_{\theta=0}\mathcal{O}(g) = \int_0^\infty dt\ e^{-\frac{t}{g}}\ \mathcal{BO}(t) \Rightarrow \text{(Residue)} \sim e^{-\frac{\sharp}{g}}$$

Ambiguity!

But this ambiguity is cancelled by ambiguities of perturbations around different saddles.

Relation to resurgence scenario (Cont'd)

Canonical successful example = Quantum mechanics

[Bogomolny '80, Zinn-Justin '81]

$$\mathcal{O} = [1]$$

$$+[I] + [I^2] + [I^3] + \cdots \quad \text{instantons}$$

$$+[\bar{I}] + [\bar{I}^2] + [\bar{I}^3] + \cdots \quad \text{anti-instantons}$$

$$+[I\bar{I}] + [I^2\bar{I}^2] + [I^3\bar{I}^3] + \cdots \quad \text{inst.-anti-inst.}$$

$$+[I^2\bar{I}] + [I^3\bar{I}^2] + [I^4\bar{I}^3] + \cdots$$

$$+\cdots$$

Relation to resurgence scenario (Cont'd)

Canonical successful example = Quantum mechanics

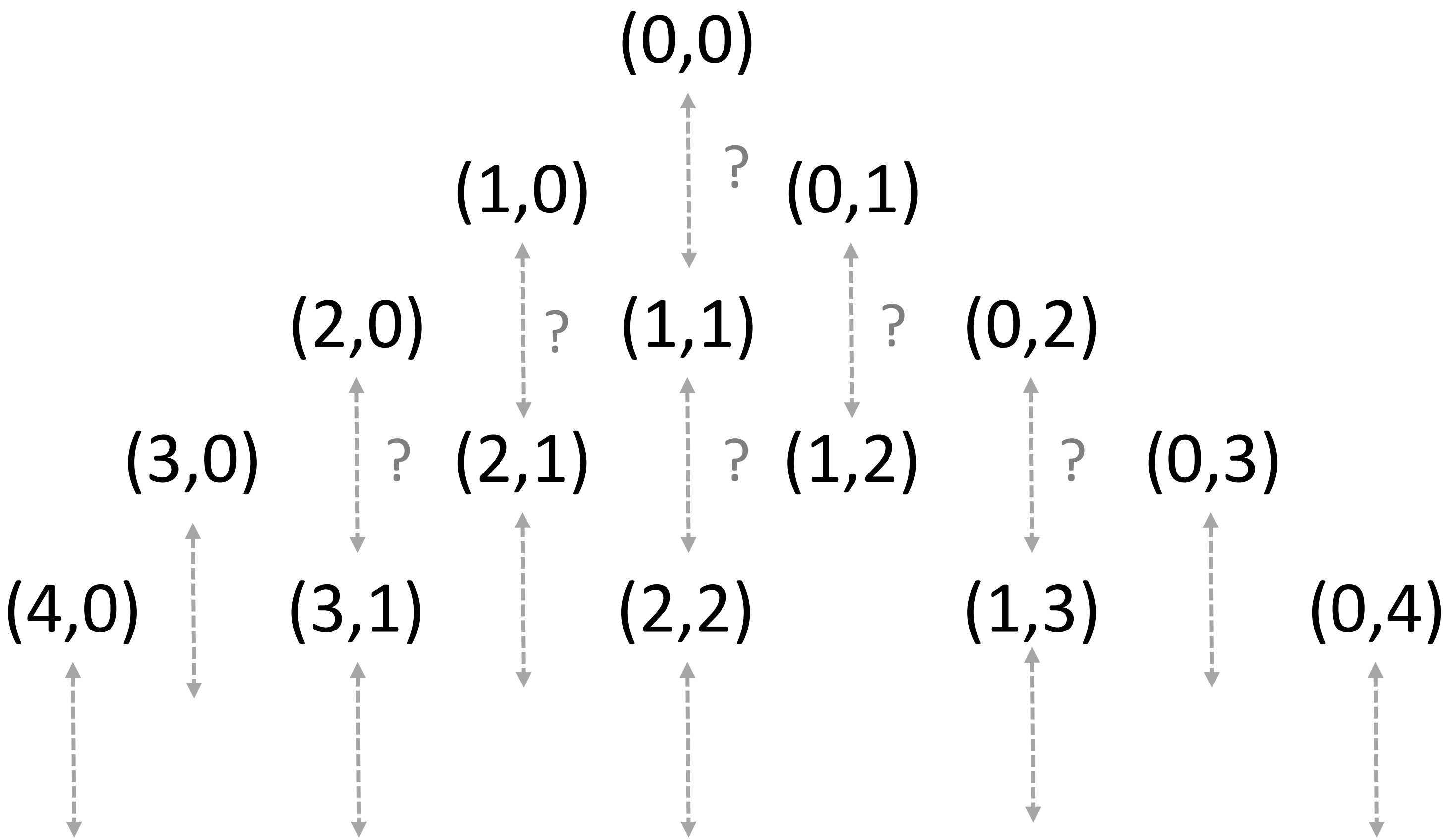
[Bogomolny '80, Zinn-Justin '81]

$$
\begin{aligned}
\mathcal{O} = [1] & \\
&+ [I] + [I^2] + [I^3] + \cdots \quad \text{instantons} \\
&+ [\bar{I}] + [\bar{I}^2] + [\bar{I}^3] + \cdots \quad \text{anti-instantons} \\
&+ [I\bar{I}] + [I^2\bar{I}^2] + [I^3\bar{I}^3] + \cdots \quad \text{inst.-anti-inst.} \\
&+ [I^2\bar{I}] + [I^3\bar{I}^2] + [I^4\bar{I}^3] + \cdots \\
&+ \cdots
\end{aligned}
$$

Ambiguities are cancelled between sectors w/ the same inst. #

Our result

(instanton #, anti-inst.#)



Every sector is Borel summable, unambiguous

Every sector is isolated in some sense (at least from this viewpoint)

Other observables

- Supersymmetric Wilson loop on S^4

$$W = P\,\exp\Big[\oint ds\,(iA_\mu \dot{x}^\mu + \Phi)\Big]$$

- Bremsstrahrung function in SCFT on R^4 [cf. Fiol-Gerchkovitz-Komargodski '15]

$$\text{(Energy of quark)} = B \int dt\ \dot{a}^2$$

- Extremal correlator in SCFT on R^4 [cf. Gerchkovitz-Gomis-Ishtiaque -Karasik-Komargodski-Pufu '16]

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \bar{\mathcal{O}} \rangle$$

- Partition function on squashed S^4～SUSY Renyi entrory

[cf. Hama-Hosomichi, Nosaka-Terashima]

[cf. Nishioka-Yaakov '13, Crossley-Dyer-Sonner, Huang-Zhou]

3d N=2 SUSY CS matter theory

Partition function of CS adjoint SQCD on S^3

$U(N)_k$ SQCD w/ $\left\{ \begin{array}{l} \text{fundamental} \\ \text{anti-fundamental} \\ \text{adjoint} \end{array} \right\}$ chiral multiplets

By localization method,

[Kapustin-Willett-Yaakov, Jafferis
Hama-Hosomichi-Lee]

$$Z_{\mathsf{SQCD}}(g) = \int_{-\infty}^{\infty} d^N\sigma\ e^{\frac{i\cdot \mathsf{sgn}(k)}{g}\sum_{j=1}^{N}\sigma_j^2}\tilde{Z}(\sigma)$$

$$g \propto \frac{1}{|k|}$$

We are interested in small-g (large level) expansion of this

Borel trans. hidden in localization formula

$$Z_{\text{SQCD}}(g) = \int_{-\infty}^{\infty} d^N\sigma \ e^{\frac{i\cdot\text{sgn}(k)}{g}\sum_{j=1}^{N}\sigma_j^2}\tilde{Z}(\sigma)$$

Taking polar coordinate $\sigma_i = \sqrt{\tau}\widehat{x}_i$

$$\begin{aligned} Z_{\text{SQCD}}(g) &= \int_0^{\infty} d\tau \ e^{\frac{i\text{sgn}(k)}{g}\tau} f(\tau) \\ &= i\text{sgn}(k)\int_0^{-i\text{sgn}(k)\infty} dt \ e^{-\frac{t}{g}} f(i\text{sgn}(k)t) \end{aligned}$$

Similar to the Borel resummation formula but w/ different integral contour!

$$\left(f(\tau) = \int d^{N-1}\widehat{x} \ h(\tau,\widehat{x}), \quad h(\tau,\widehat{x}) = \tilde{Z}(\sigma)\Big|_{\sigma^i=\sqrt{\tau}\widehat{x}^i} \right)$$

$$Z_{\mathrm{SQCD}}(g) = i\mathrm{sgn}(k) \int_0^{-i\mathrm{sgn}(k)\infty} dt\ e^{-\frac{t}{g}} f(i\mathrm{sgn}(k)t)$$

Borel transformation?

By using the technique in 4d, we can actually prove

$$i\mathrm{sgn}(k) f(\tau) = \mathcal{B}Z_{\mathrm{SQCD}}(-i\mathrm{sgn}(k)\tau)$$

Namely,

$$Z_{\mathrm{SQCD}}(g) = \int_0^{-i\mathrm{sgn}(k)\infty} dt\ e^{-\frac{t}{g}} \mathcal{B}Z_{\mathrm{SQCD}}(t)$$

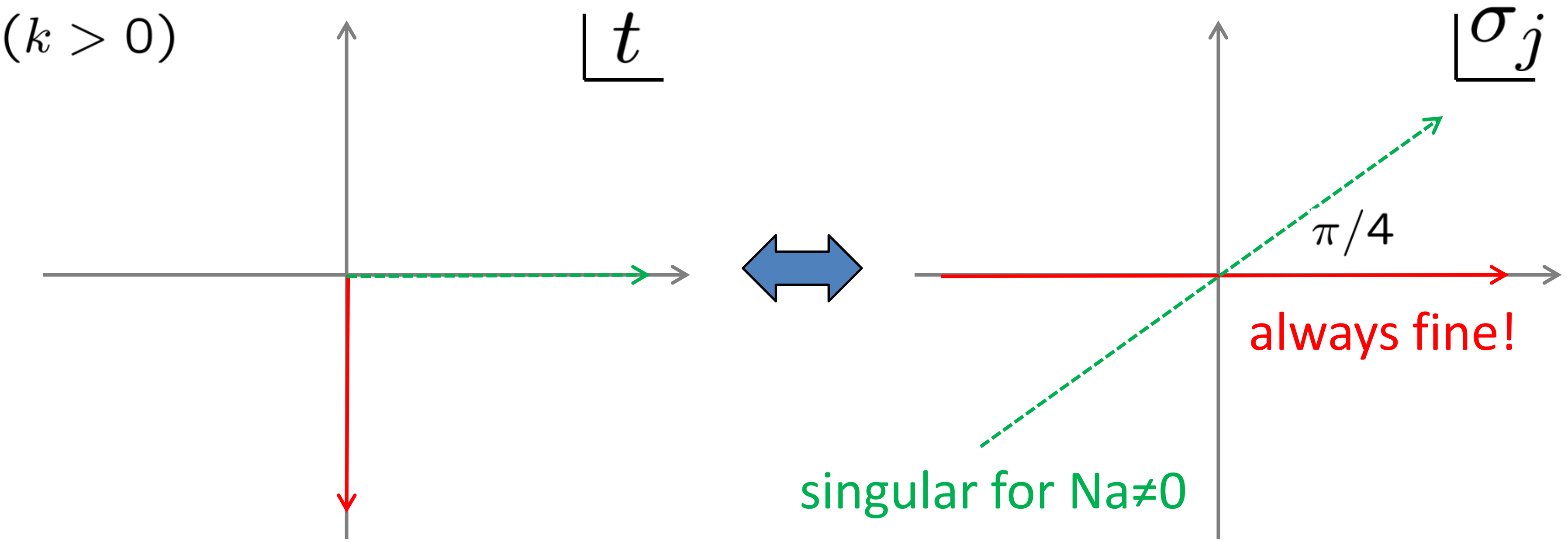
Analytic property of Borel trans.

$$Z_{\rm SQCD}(g) = \int_0^{-i\,{\rm sgn}(k)\infty} dt\ e^{-\frac{t}{g}} \mathcal{B}Z_{\rm SQCD}(t), \quad \mathcal{B}Z_{\rm SQCD}(t) = \int_{S^{N-1}} d^{N-1}\hat{x}\ \tilde{Z}\left(\sigma = \sqrt{i\,{\rm sgn}(k) t}\,\hat{x}\right)$$

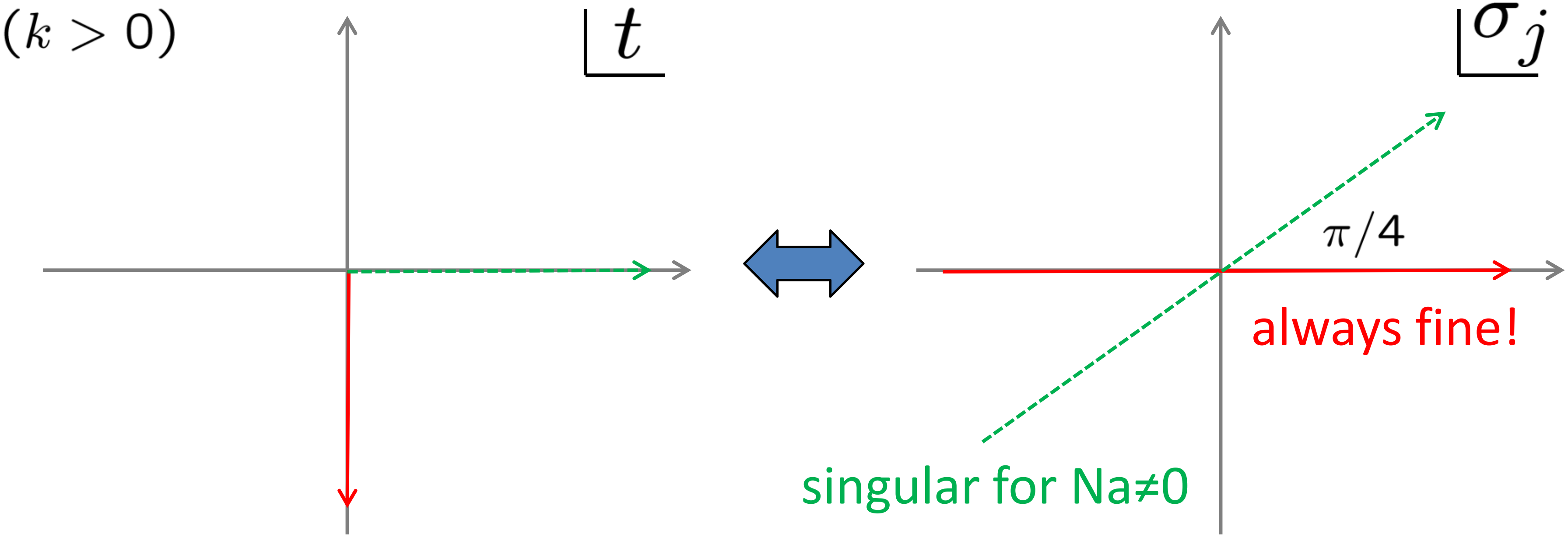
$$\tilde{Z}(\sigma) = \prod_{j=1}^{N} \frac{s_1^{\bar{N}_f}\left(\sigma_j + i(1-\bar{\Delta}_f)\right)}{s_1^{N_f}\left(\sigma_j - i(1-\Delta_f)\right)} \frac{\prod_{i<j} 4\sinh^2\left(\pi(\sigma_i - \sigma_j)\right)}{\prod_{i,j} s_1^{N_a}\left(\sigma_i - \sigma_j - i(1-\Delta_a)\right)}, \qquad s_1(z) = \prod_{n=1}^{\infty}\left(\frac{n - iz}{n + iz}\right)^n$$

Sufficient condition for Borel summability

= Absence of singularities along the contour in $\tilde{Z}(\sigma)$



Analytic property of Borel trans. (Cont'd)



- When we have adjoint matters, it would be non-Borel summable along R+
- But it is always Borel summable along θ=-π/2

General 3d N=2 CS matter theory

Suppose a theory w/ gauge group: $G = G_1 \times \cdots \times G_n$

$$Z_{S^3}(g) = \int_{-\infty}^{\infty} d^{|G|}\sigma \ Z_{\text{cl}}(g;\sigma)\tilde{Z}(\sigma)$$

$$Z_{\text{cl}}(g;a) = \exp\left[\sum_{p=1}^{n} \frac{i \cdot \text{sgn}(k_p)}{g_p} \text{tr}(\sigma^{(p)})^2\right]$$

Taking polar coordinate $\sigma_i^{(p)} = \sqrt{\tau_p}\hat{x}_i^{(p)}$,

$$Z_{S^3}(g) = \left[\prod_{p=1}^{n} \int_0^{-i\text{sgn}(k_p)\infty} d^n t \ e^{-\frac{t_p}{g_p}}\right] \mathcal{B}Z_{S^3}(t)$$

⇨ Borel summable along $\theta_p = -\frac{\text{sgn}(k_p)\pi}{2}$

Relation to the exact result

We have shown

$$Z_{S^3}(g) = \left[\prod_{p=1}^{n} \int_0^{-i\mathrm{sgn}(k_p)\infty} d^n t \; e^{-\frac{t_p}{g_p}} \right] \mathcal{B} Z_{S^3}(t)$$

Thus,

(Exact result)

||

(Borel resummation along the directions)

Other observables

- SUSY Wilson loop on S^3:

$$W = P\,\mathrm{exp}\left[\oint ds\,(iA_\mu \dot{x}^\mu + \Phi)\right]$$

- Bremsstrahrung function in SCFT on R^3 [cf. Lewkowycz-Maldacena '13]

- 2-pt. function of U(1) flavor current in SCFT

- 2-pt. function of stress tensor in SCFT

- Partition function on squashed S^3 ~ SUSY Renyi entropy

- Partition function on squashed lens space

Interpretation of singularities (3d)

[M.H., to appear]

Interpretation of Borel singularities (in general)

$$Z(g) = \int D\Phi e^{-\frac{1}{g}S[\Phi]} \simeq \sum_{\ell} c_\ell g^\ell$$

[Lipatov '77]

Large order coefficient:

$$c_\ell = \frac{1}{2\pi i}\oint \frac{dg}{g^{\ell+1}} Z(g) = \frac{1}{2\pi i}\oint dg \int D\phi e^{-\frac{1}{g}S[\phi]-(\ell+1)\ln g} \qquad (\ell \to \infty)$$

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$$\simeq e^{-\frac{1}{g_*}S[\phi_*]-(\ell+1)\ln g_*} \qquad \left(\left.\frac{\delta S}{\delta \phi}\right|_{\phi=\phi_*} = 0, \ -\frac{1}{g_*^2}S[\phi_*] + \frac{\ell+1}{g_*} = 0 \right)$$

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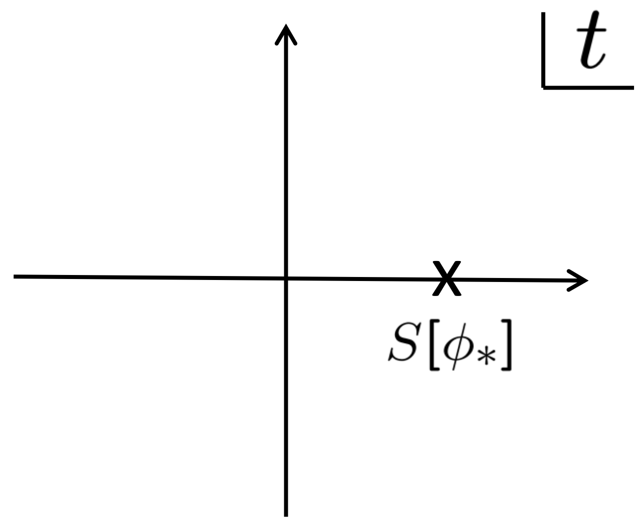
$$\mathcal{B}Z(t) \simeq \sum_\ell \left(S[\phi_*]\right)^{-\ell} = \frac{1}{1-\frac{t}{S[\phi_*]}}$$



Nontrivial saddle point gives Borel singularities

Interpretation of poles (3d ellipsoid case)

All the poles are explained by complexified SUSY solutions:

$$\begin{cases} 0 = Q\lambda = \left(\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho} - D_\mu\sigma\right)\gamma^\mu\epsilon - iD\epsilon - \frac{i}{f(\vartheta)}\sigma\epsilon \\ 0 = Q\psi = -\gamma^\mu\epsilon D_\mu\phi - \epsilon\sigma\phi - \frac{i\Delta}{f(\vartheta)}\epsilon\phi + i\bar{\epsilon}F, \end{cases}$$

Especially, $\sigma \in \mathbf{R}$ on the original path integral contour.

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Especially, $\sigma \in \mathbf{R}$ on the original path integral contour.

If we relax this, we have

$$F_{\mu\nu} = 0, \quad D = -\frac{1}{f(\vartheta)}\sigma, \quad F = 0,$$

$$\sigma = -i\left(mb + nb^{-1} + \frac{b+b^{-1}}{2}\Delta\right), \quad \gamma^\mu\epsilon D_\mu\phi + \epsilon\sigma\phi + \frac{i\Delta}{f(\vartheta)}\epsilon\phi = 0$$

$(m, n \in \mathbf{Z}_{\geq 0},\ b$: squashing parameter$)$

We can show $\boxed{e^{-S} \sim \text{Res}\left[\mathcal{B}Z_{S_b^3}(t)\right]}$

Summary & Outlook

Summary

How to resum perturbative series in SUSY gauge theories

4d N=2 theories:

- $\exists$ Singularities only along R- → Borel summable along R+
  (for round S^4)
- (Exact) = $\sum_{\text{instantons}}$ (Borel resum)

3d N=2 CS matter theories:

- Usually non-Borel summable along R+
- Always Borel summable along (half-)imaginary axis
- (Exact result) =(Borel resummation along the direction)
- (Singularities) = (Complexified SUSY solutions)

Open questions

- Less SUSY case?
- Other observables? [For ‘t Hooft loop, M.H.-D.Yokoyama, in preparation]
- How can we see convergence in planar limit?
- Expansion by other parameters? (such as 1/N)

4d N=2 theories:

- Physical interpretation of poles in complex plane?

3d N=2 CS matter theories:

- Restriction to 3d N=4 case? [cf. Russo ’12]

Thanks!