Thermal field theory on the lattice

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Literature

- H. J. Rothe,
  Lattice gauge theories: An Introduction,
  World Sci. Lect. Notes Phys. **43** (1992) 1
- I. Montvay and G. Munster,
  Quantum fields on a lattice,
- T. DeGrand and C. E. Detar,
  Lattice methods for quantum chromodynamics,
  New Jersey, USA: World Scientific (2006) 345 p

Motivation

Ideal hydro:

$$\partial_\mu T^{\mu\nu} = 0 \tag{1}$$

$$T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} = \text{diag}\big[e(T), p(T), p(T), p(T)\big] \tag{2}$$

This lecture will be about numerically computing $p(T)$

Outline and Goals

- Basics of thermal field theory
  - Goal: Thermodynamics of 3+1d field theory from 4d field theory with compact euclidean time
- Lattice discretization of QCD
  - Goal: Gauge invariant formulation of lattice QCD in terms of link matrices
- Computation of equation of state
  - Goal: Practical understanding of lattice simulation on a computer

Basics of thermal field theory

Quantum field theory of small number of particles: vacuum field theory

- Observables: scattering amplitudes
- Amplitudes calculated from *vacuum to vacuum* matrix elements

LSZ reduction

$$\langle \mathbf{p}_1 \mathbf{p}_2 | \mathbf{k}_1 \mathbf{k}_2 \rangle \propto \langle 0 | T\{\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(y_1)\hat{\phi}(y_2)\} | 0 \rangle \tag{3}$$

- Start from vacuum, add a particle though operating with a field operator ...

Basics of thermal field theory

Large number of particles: statistical field theory

- The system may not be a vacuum state to start with

$$\langle 0|\hat{A}|0\rangle \Rightarrow \sum_i p_i \langle i|\hat{A}|i\rangle \tag{4}$$

  - The system starts in state $|i\rangle$ with a probability of p_i
- Define a *density matrix*:

$$\sum_i p_i \langle i|\hat{A}|i\rangle = \mathrm{Tr}\left[\underbrace{|i\rangle p_i \langle i|}_{\hat{\rho}}\hat{A}\right] \equiv \mathrm{Tr}\,\hat{\rho}\hat{A} \tag{5}$$

  - Density matrix defines the state of the system, includes both *quantum* and *statistical* uncertainty

Basics of thermal field theory

- In thermal system, the density matrix the one maximizing the entropy

drop μ from now on

$$\hat{\rho} = e^{-\beta(\hat{H}-\mu\hat{N})} \tag{6}$$

- The *partition function* is given by the trace of the density matrix

$$Z = \mathrm{Tr}\,\hat{\rho} \tag{7}$$

- Thermodynamical properties are related to the derivatives of Z

$$p(T) = \frac{T}{V}\log Z \tag{8}$$

$$\epsilon(T) = \frac{1}{V}\langle\hat{H}\rangle = \frac{T^2}{V}\frac{\partial Z}{\partial T} \tag{9}$$

Basics of thermal field theory

How to compute Z?

- So far our definitions involve *operators*. $\hat{H}, \hat{N}$, *etc.*
- Operators in space of many degrees of freedom are difficult to deal with numerically.
- In order to eventually simulate the system on a computer we want to express the Z in terms of integral over ordinary numbers: Path integral.

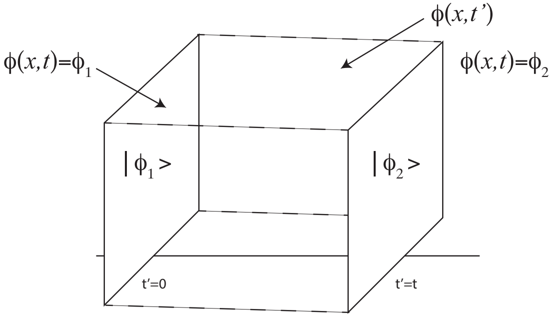
$$\mathrm{Tr}\,\hat{\rho} \Rightarrow \int \underbrace{\prod_{\mathbf{x}} dU_{\mathbf{x}}}_{\mathcal{D}U} \, e^{-S(U(\mathbf{x}))} \tag{10}$$

Basics of thermal field theory

How to compute Z?

- Reminder: expressing matrix elements in terms of path integrals

$$\langle \phi_2 | e^{-it\hat{H}} | \phi_1 \rangle = \int_{\phi(t'=0)=\phi_1}^{\phi(t'=t)=\phi_2} \mathcal{D}\phi \, e^{i\mathcal{S}} \tag{11}$$



Here ϕ stands for any (bosonic) fields in the theory, A_μ for QCD

Basics of thermal field theory

How to compute Z?

- In thermal equilibrium: the density matrix looks exactly like a time translation operator to *imaginary time*!

$$\hat{\rho} = e^{-\beta \hat{H}} = e^{-i \overbrace{(-i\beta)}^{\tau} \hat{H}} \tag{12}$$

- Matrix elements of ρ can be computed by evolving the states in the imaginary time

$$\langle \phi_2 | e^{-it\hat{H}} | \phi_1 \rangle = \int_{\phi(t'=0)=\phi_1}^{\phi(t'=t)=\phi_2} \mathcal{D}\phi \, e^{i\mathcal{S}} \tag{13}$$

$$\langle \phi_2 | e^{-i(-i\beta)\hat{H}} | \phi_1 \rangle = \int_{\phi(t'=0)=\phi_1}^{\phi(t'=-i\beta)=\phi_2} \mathcal{D}\phi \, e^{i\mathcal{S}} \tag{14}$$

Basics of thermal field theory

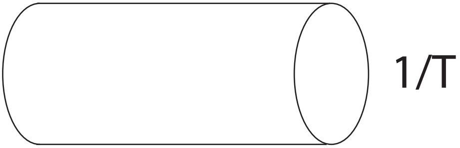
How to compute Z?

- Partition function computed over all periodic field configurations in imaginary time with period of $\frac{1}{T}$

$$Z = \mathrm{Tr}\,\hat{\rho} = \sum_{\phi_i} \langle \phi_i | e^{-i(i\beta)\hat{H}} | \phi_i \rangle \tag{15}$$

$$= \sum_{\phi_i} \int_{\phi(\tau=0)=\phi_i}^{\phi(\tau=i\beta)=\phi_i} \mathcal{D}\phi e^{i\mathcal{S}} \tag{16}$$

$$= \int_{\phi(0)=\phi(i\beta)} \mathcal{D}\phi e^{i\mathcal{S}} \tag{17}$$



Basics of thermal field theory

How to compute Z?

- The action in real time is purely real number, time in special role

$$\mathcal{S} = \int dt\, d^3x \mathcal{L}(\phi, \partial_i\phi) = \int dt\, d^3x \left[-\partial_t^2\phi + \nabla^2\phi - V(\phi)\right] \tag{18}$$

- In imaginary time $t = i\tau$:

$$\mathcal{S} = \int (id\tau)\, d^3x \mathcal{L}(\phi, \partial_i\phi) = i\int_0^\beta d\tau \int d^3x \left[+\partial_\tau^2\phi + \nabla^2\phi - V(\phi)\right]$$
$$\equiv i\mathcal{S}_E \tag{19}$$

- Time direction looks like spatial directions!

Imaginary time = Euclidean time

Basics of thermal field theory

Remarks:

- It is important that we did got rid of the Minkowski signature: Minkowski tricky to discretize on a lattice because distance along light cone $X^2 = 0$.
- It is important that Euclidean action is imaginary for numerical evaluation of the path integral. Instead of rapidly oscillating function of $\mathcal{O}(1)$ one has sharply peaked integrand.

$$\int \mathcal{D}\phi e^{i\mathcal{S}} \quad \text{vs.} \quad \int \mathcal{D}\phi \mathrm{e}^{-\mathcal{S}_\mathrm{E}} \tag{20}$$

- This doesn't happen always. If the integrand oscillates, the theory has a *sign problem*, notable examples:
  - If C symmetry is broken
  - QCD with baryon number chemical potential μ_B
  - "real time" correlation functions
  - Transport coefficients …

Basics of thermal field theory

Remarks:

- For bosons: compute the thermodynamics of 3+1d theory in 4d space with compact euclidean time
- For fermions: the path integral is over anticommuting *grassmann* variables. Boundary conditions are *antiperiodic*.

$$\psi_1 \psi_2 = -\psi_2 \psi_1 \tag{21}$$

- For QCD:

$$\int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\mathcal{S}_E^{QCD}(A_\mu,\bar{\psi},\psi)} \tag{22}$$

Outline and Goals

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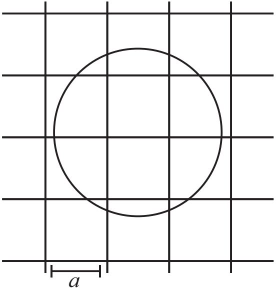
Lattice regularization of QCD

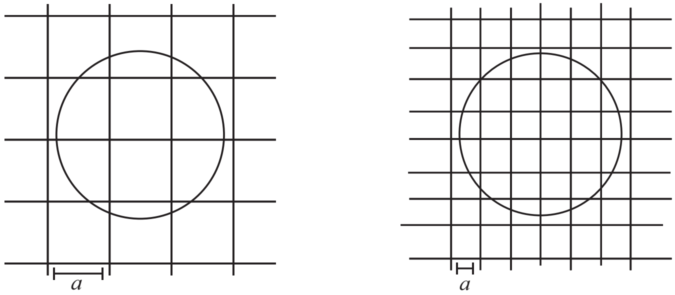
- In order to evaluate the path integral numerically, discretize the space (and euclidean time) coordinates on a finite lattice

$$\mathbf{x} = (\tau, x, y, z) \Rightarrow (an_\tau, an_x, an_y, an_z) \quad \text{with } n \in \text{integers}$$
$$\phi(x) \Rightarrow \phi_{x_i}$$
$$\int \mathcal{D}\phi e^{-\mathcal{S}_E(\phi, \partial_x \phi)} \Rightarrow \int \prod_x d\phi_x e^{-\mathcal{S}_E^L(\phi_x)} \tag{23}$$



Lattice regularization of QCD

- Strategy: Compute at different lattice spacings, take eventually continuum limit $a \to 0$
- Note: Lattice is not an approximation of QFT, but a *non-perturbative regularization*



Lattice regularization of QCD

- Freedom in construction the lattice action:
  - Need to recover the continuum action in the continuum limit
    as fast as possible, $\mathcal{O}(a)$ vs. $\mathcal{O}(a^2)$
  - The continuum limit $a \to 0$ defines a *universality class*: the continuum limit is the same for any valid action.
    At finite a results differ...
  - Caveat: If the lattice action breaks symmetries of the continuum theory, the symmetries may or may not be restored in the continuum limit:
    - if the symmetries are restored, the discretized theory belongs (or may belong) to the same universality class and the continuum limit can be taken
    - if the symmetries are not restored, the continuum limit does not correspond to the continuum theory!

Lattice regularization of QCD

- Easy for scalar theory:

$$\mathcal{L}_E = (\partial_\mu \phi)^2 \tag{24}$$

$$\partial_i \phi \to \frac{1}{a}\left[\phi(x_i + a\hat{e}_i) - \phi(x_i)\right] \tag{25}$$

- Breaks translational invariance of the theory, but symmetry is restored in the continuum limit

Lattice regularization of QCD

- Argument: the long distance physics of the lattice theory described by a *continuum* thy with additional operators

$$\begin{aligned}\mathcal{L}_{eff} &= (\partial_\mu\phi)^2 + V(\phi) + \text{All op. respecting symms. of the lat. thy} \\ &= (\partial_\mu\phi)^2 + V(\phi) + \#a^2\phi\partial_\mu\partial_\mu\partial_\mu\partial_\mu\phi + \ldots \end{aligned} \tag{26}$$

  - All higher order terms have at least dimension 6: come with at least a^{-2}.

- However, doing $\partial_i A_j \to \frac{1}{a}\left(A_j(x_i + a\hat{e}_i) - A_j(x_i)\right)$, breaks *gauge invariance*. This will not be restored in the continuum limit, and the continuum limit is wrong!

Lattice regularization of QCD

Lagrangian of QCD:

$$S = \int_0^{1/T} d\tau \int d^3x \left[\frac{1}{2} \mathrm{Tr}\, F_{\mu\nu} F_{\mu\nu} + \sum_{f=1}^{N_f} \bar{\psi}_f (\gamma_\mu D_\mu + m_f) \psi_f \right] \qquad (27)$$

$$D_\mu = \partial_\mu + igA_\mu \qquad (28)$$

- Gauge fields A_μ belong to *Lie algebra* of the group SU(3):A_μ are 3×3 hermitean matrices.

$$A_\mu^\dagger = A_\mu$$

- Field strength tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \qquad (29)$$

Lattice regularization of QCD

In additional to Lorentz symmetry, the action has the improrant *gauge symmetry.*

- The gauge symmetry is essential part of the theory

  Renormalizability, conserved color current, vanishing gluon mass,2 polarization of gluons, ...

$$S = \int_0^{1/T} d\tau \int d^3x \left[\frac{1}{2} \mathrm{Tr}\, F_{\mu\nu} F_{\mu\nu} + \sum_{f=1}^{N_f} \bar{\psi}_f (\gamma_\mu D_\mu + m_f) \psi_f \right]$$

$$D_\mu = \partial_\mu + igA_\mu \tag{30}$$

- Gauge transformations:

$$\psi_f(x) \longrightarrow G(x)\psi_f(x), \qquad \bar{\psi}_f(x) \longrightarrow \bar{\psi}_f(x) G^\dagger(x),$$

$$A_\mu(x) \longrightarrow G(x)\, A_\mu(x)\, G^\dagger(x) - \frac{i}{g} G(x) \partial_\mu G^\dagger(x) \tag{31}$$

Lattice regularization of QCD

- Consider naively discretizing the action

$$\partial_\mu A_\nu \to \frac{1}{a}\left(A_\nu(x + a\hat{e}_\mu) - A_\nu(x)\right)$$

- The resulting lattice action breaks gauge invariance
- Then the continuum theory describing the long wavelength modes of the lattice theory contain terms like

$$\mathcal{L}_{eff} \in \frac{\#}{a^2} A_\mu A_\mu \tag{32}$$

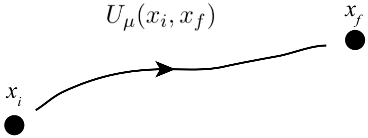
- These terms do not vanish in the continuum limit!
- Can be in principle cancelled by adding *counter terms*. In practice for gauge symmetry not possible.
  - Much more practical to find a lattice action that conserves the symmetry

Lattice regularization of QCD

- In order to discretize the action in a gauge invariant way, consider a path ordered exponential of the field, the *Wilson line*

  The color rotation a color charge gets when moving in chromo-E and -B fields ...

$$U(x_i, x_f) = \mathrm{P}e^{ig \int_{x_i}^{y_f} dy_\nu A_\nu(y)} \tag{33}$$



- Wilson line is an *element* of the group: 3×3 unitary matrix,

$$U^\dagger U = \mathbb{1}$$

.

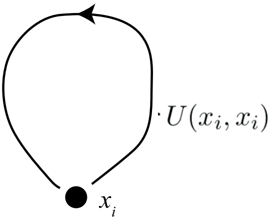
- Wilson line gauge transforms according to its end points:

$$U(x_i, x_f) \longrightarrow G(x_i) U(x_i, x_f) G^\dagger(x_f) \tag{34}$$

Lattice regularization of QCD

- Trace of a closed loop, or *Wilson loop* is gauge invariant

$$\begin{aligned} \mathrm{Tr}\, U(x_i, x_i) \longrightarrow & \mathrm{Tr}\, G(x_i) U(x_i, x_i) G^\dagger(x_i) = \mathrm{Tr}\, G^\dagger(x_i) G(x_i) U(x_i, x_f) \\ = & \mathrm{Tr}\, U(x_i, x_i) \end{aligned} \tag{35}$$



- All gauge invariant quantities can be expressed in terms of closed Wilson loop, including the action $\mathrm{Tr}\, F^{\mu\nu} F^{\mu\nu}$.

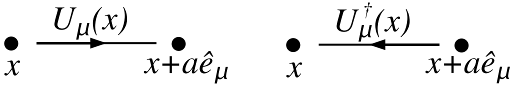
Lattice regularization of QCD

- Strategy:
  - Instead of discretizing the field A_μ, discretize the Wilson lines
  - As traces of all Wilson loops are gauge invariant, such a discretized action is by construction gauge invariant
  - Our next task is to find an expression in terms of discretized Wilson lines on the lattice which goes to the continuum action in the continuum limit

Lattice regularization of QCD

- A short Wilson line connecting two lattice sites is a *link*

$$U_\mu(x) = \mathrm{P} e^{ig\int_x^{x+a\hat{e}_\mu} dy_\nu A_\nu(y)} \approx e^{iagA_\mu(x)}. \tag{36}$$

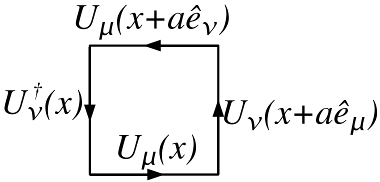


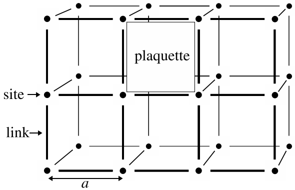
- Wilson line transforms according to its endpoints

$$U_\mu(x) \longrightarrow G(x)U_\mu(x)G^\dagger(x + a\hat{e}_\mu), \tag{37}$$

Lattice regularization of QCD

- The simplest possible Wilson loop constructed from links is the *plaquette*





$$U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+a\hat{e}_\mu)U_\mu^\dagger(x+a\hat{e}_\nu)U_\nu^\dagger(x). \tag{38}$$

- In the continuum limit the plaquette is related to the field strength tensor

$$\begin{aligned} U_{\mu\nu}(x) &= e^{iagA_\mu(x)}e^{iagA_\nu(x+a\hat{e}_\mu)}e^{-iagA_\mu(x+a\hat{e}_\nu)}e^{-iagA_\nu(x)} && (39) \\ &= e^{iga^2F_{\mu\nu}+\mathcal{O}(a^6)} \\ &= \mathbb{1} + iga^2F_{\mu\nu} - g^2a^4F_{\mu\nu}^2 + \mathcal{O}(a^6) && (40) \end{aligned}$$

Lattice regularization of QCD

- Take the combination of plaquettes that reduces to the continuum action in the continuum limit: *Wilson action*

$$S_W = \beta_W \sum_P [1 - \frac{1}{2N_c} \mathrm{Tr}\,(U_{\mu\nu}(x) + U^\dagger_{\mu\nu}(x))] \quad (41)$$

$$= \beta_W \sum_{\mu<\nu,x} \mathrm{Tr}\left[\frac{1}{N_c} - \frac{1}{2N_c}\left(2 - g^2 a^4 F_{\mu\nu}F_{\mu\nu}\right)\right] \quad (42)$$

$$= \beta_W \frac{g^2}{2N_c} \sum_{x,\mu,\nu} a^4 \mathrm{Tr}\,[\frac{1}{2} F_{\mu\nu}F_{\mu\nu}] \quad (43)$$

$$= \int d\tau d^3x \frac{1}{2} \mathrm{Tr}\,[F_{\mu\nu}F_{\mu\nu}] \quad (44)$$

for lattice couping constant $\beta_W = \frac{2N_c}{g^2}$

Lattice regularization of QCD

Remarks:

- Wilson action is the simplest gauge invariant action that reduces to the continuum action in the continuum limit.
- Wilson action is just one of many possible actions. Can add more complicated wilson loops to improve the approach to continuum limit etc.
- Euclidean time formulation of thermal field theory combined with the Wilson action makes it possible to put the thermal quantum field theory on a computer and do practical simulations

$$Z = \int \mathcal{D}U_\mu e^{-S_W} \tag{45}$$

Lattice regularization of QCD

A word about fermions:

- Path integral of fermionic fields is an integral over grassmann variables

$$Z = \int \mathcal{D}U_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S_W - S_F}, \quad S_F \sim \sum_{\mathbf{x},\mathbf{y}} \bar{\psi}_{\mathbf{x}} M_{\mathbf{x},\mathbf{y}}(U_\mu)\psi_{\mathbf{y}} \tag{46}$$

- Grassmann practically impossible to implement on a computer
- However, bilinear fields can always be integrated over

$$Z = \int \mathcal{D}U_\mu \det[M_{\mathbf{x},\mathbf{y}}(U_\mu)] e^{-S_W} \tag{47}$$

  - Resulting expression in terms of ordinary numbers, but non-local

Lattice regularization of QCD

- Discretizing massless fermions (or small mass m_u and m_d) fermions is difficult:
  - for $m_f \to 0$ the continuum theory has an extra symmetry *chiral symmetry*
  - Chiral symmetry necessarily broken by discretization, and not easily recovered in the continuum limit

    Nielsen Ninomiya theorem
  - Many different strategies (actions) to deal with the fermions

    Wilson fermions, staggered fermions, domain wall fermions, overlap fermions

Continuum limit

- At the end of the day one has to take $a \to 0$.
- The lattice spacing a does not appear explicitly anywhere in the expression!

$$\int \mathcal{D}U_\mu \exp\left[-\beta_W \sum_P [1 - \frac{1}{2N_c} \mathrm{Tr}\left(U_{\mu\nu}(x) + U^\dagger_{\mu\nu}(x)\right)]\right] \tag{48}$$

- However, the coupling constant β_W has a hidden scale dependece

The bare unrenormalized g^2 is $\sim g^2(a)$

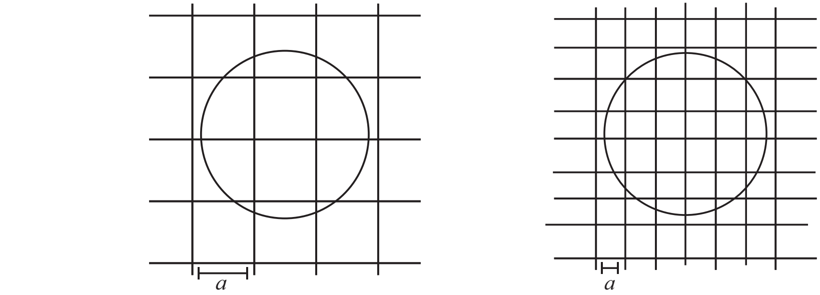
$$\beta_W = \frac{2N_c}{g^2(a)}$$

Continuum limit

- Assume that on a finite lattice there is a finite correlation length.
  Physical scale

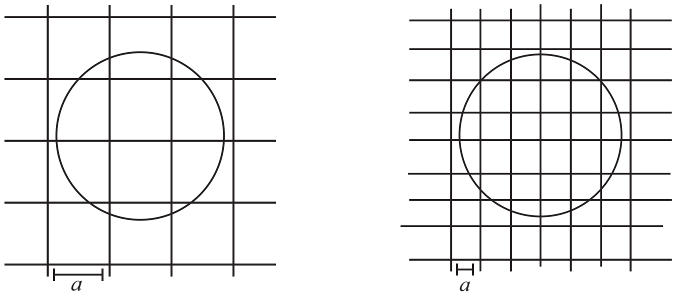
Confinement hypothesis; hadronic scale

$$\xi(\beta_W) \gg a$$



Continuum limit

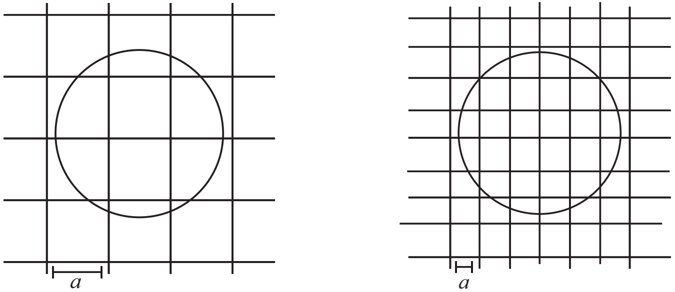
- As the lattice coupling β_W increases ($g(a)$ decreases), correlation length ξ/a grows
    - correlation length grows in lattice units
    - lattice spacing shrinks in units of the correlation length



Continuum limit

- Measure ξ/a as a function of β_W will give $a(\beta)$ in physical units and continuum limit can be taken.

$$\langle \mathcal{O} \rangle_a = \langle \mathcal{O} \rangle_{\text{cont}} + \mathcal{O}(a) \tag{49}$$



Continuum limit

- If the lattice spacing dependence of a quantity is known *Callan-Symanzik* β-function can be defined

Don't confuse $\beta(g)$ to β_W or $1/T$!!

$$\frac{d}{da}\xi(a,g(a)) = 0 \tag{50}$$

$$\frac{1}{a}\left(a\partial_a + \underbrace{\frac{\partial g}{\partial \ln a}}_{-\beta(g)}\partial_g\right)\xi(a,g(a)) = 0 \tag{51}$$

- In an asymptotically free theory, at sufficiently weak coupling (corresponding to small lattice spacing), this can be computed as power series in (lattice) perturbation theory!

Compute any observable as a function of g and a, apply derivatives and solve for $\beta(g)$

$$\beta(g) \equiv \beta_0 g^3 + \mathcal{O}(g^5) \underset{QCD}{=} -\frac{1}{16\pi^2}\left(\frac{11N_c}{3} - \frac{2N_f}{3}\right)g^3 \tag{52}$$

Continuum limit

- Integrating the $\beta(g)$ gives the lattice spacing as a function of the bare lattice coupling β_W

Remember $g^2 = 2N_c/\beta_W$

$$\beta(g) = -\frac{\partial g}{\partial \log a} = -\beta_0 g^2 \quad \Rightarrow \quad a(\beta_W) = \frac{1}{\Lambda_L} e^{-\frac{\beta_W}{4N_c\beta_0}} \tag{53}$$

- Λ_L is integration constant that sets the scale of QCD and is the only input needed.

Remarks:

- If fermions: similar β-functions and renormalization for masses
- *Scaling regime*:

$$\langle \mathcal{O} \rangle_a = \langle \mathcal{O} \rangle_{\text{cont}} + \mathcal{O}(a)$$

- *Asymptotic scaling regime*:

$$a(\beta_W) = \frac{1}{\Lambda_L} e^{-\frac{\beta_W}{4N_c\beta_0}}$$

- In practice almost all simulations performed in scaling regime

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Measurement of the equation of state

- So far we have established that the partition function Z is given over an integral over many, many link matrices

neglect fermions for now...

$$Z = \int \mathcal{D}U_\mu e^{-S_W} = \int \underbrace{\prod_{\mu}}_{\text{directions}} \underbrace{\prod_{\mathbf{x}}}_{\text{lattice sites}} \underbrace{dU_{\mathbf{x},\mu}}_{\text{8 d.o.f in SU(N)}} e^{-S_W(U_{\mathbf{x}},\mu)} \tag{54}$$

  - Consider a 32×8 lattice
  - There are $32 \times 8 \times (4\text{directions}) \times (8\text{d.o.f's}) \sim 10^7$ integral to perform!
  - However: because the exponential form, the integrand is very sharply peaked.

Measurement of the equation of state

For sharply peaked multidimensional integrals there is a powerful method called *importance sampling*.

- Common across many fields of science
- Instead of computing Z directly, consider *expectation values* of operators:

$$\langle \mathcal{O} \rangle \equiv \frac{1}{Z} \int \prod dU_{\mathbf{x},\mu} \, \mathcal{O}[U] e^{-S_W} \tag{55}$$

- *Monte Carlo* algorithms: Approximate the integral by generating an *ensemble* of field configurations $\{U^1, U^2, \ldots\}$ with probability

$$dP(U^j) = \frac{e^{-S_W(U^j)}}{Z}$$

and write the expectation value as an ensemble average

$$\frac{1}{N_{MC}} \sum_{j}^{N_{MC}} \mathcal{O}[U^j] \underset{N_{MC} \to \infty}{\longrightarrow} \langle \mathcal{O} \rangle \tag{56}$$

Measurement of the equation of state

- If we knew what Z is this would be simple. But we don't!
- Need to have an algorithm that generates the ensemble without the knowledge of Z: *Marcov Chain Monte Carlo* algorithm
    - Create a random sequence of configurations $\{U_1, U_2, \ldots\}$ where the configuration U_i depends (only) on the previous configuration U_{i-1}
    - If the *transition probabilities* of going from $U_{i-1} \leftrightarrow U_i$ satisfy *detailed balance*

$$e^{-S_W(U_{i-1})} dP(U_{i-1} \to U_i) = e^{-S_W(U_i)} dP(U_i \to U_{i-1}), \tag{57}$$

the resulting ensemble is the correct one.

Measurement of the equation of state

A particular Markov Chain Monte Carlo algorithm: *Metropolis* algorithm

- Propose a new configuration by somehow freely deforming the previous one.

- Accept it with a probability

$$P_{acc}(U_{i-1} \to U_i) = \min(1, \frac{e^{-S_W(U_i)}}{e^{-S_W(U_{i-1})}})$$

- If reject, then don't change the configuration $U_i = U_{i-1}$

- Lots of development to find an optimal deformation

HMC,RHMC, overrelaxation, ...

Measurement of the equation of state

We are finally (almost) ready to measure the equation of state!

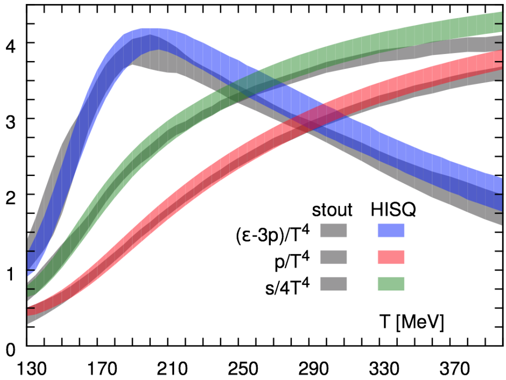
- Still need to reconstruct the Z from operator expectation values
- Many possible ways of doing so, here *integral method*

$$T^{\mu\mu} \equiv \epsilon - 3p = -\frac{T}{V}\frac{\partial \log Z}{\partial \ln a} \tag{58}$$

$$= -\frac{T}{V}\beta(g)\langle\sum_{P}[1 - \frac{1}{2N_c}\mathrm{Tr}\,(U_{\mu\nu}(x) + U^{\dagger}_{\mu\nu}(x))]\rangle \tag{59}$$

$$\Rightarrow \frac{p(T)}{T^4} = \int_0^T dT' \frac{\epsilon - 3p}{T'^5} \tag{60}$$

Measurement of the equation of state



- Continuum limit $a \to 0$
- Thermodynamical limit $N \to \infty$
- Physical quark masses m_u, m_d, m_s

Measurement of the equation of state

Comment of transport coefficients:

- Transport coefficients are also "in principle" measurable on the lattice:

$$G^R_{xy,xy}(\omega, p=0) \equiv \int dt d^3x e^{it\omega}\Theta(t)\langle[T_{x,y}(t), T_{0,0}(0)]\rangle \tag{61}$$

$$\eta = -\lim_{\omega\to 0}\frac{1}{\omega}\Im G_R(\omega, 0) \tag{62}$$

- This correlation function is in *real* time, not in euclidean time!
- However, in thermal equilibrium the correlation functions are *analytically connected.*

Long story, can talk later...

$$G_E(\tau, p) = \int_0^\infty d\omega \Im G_R(\omega, p) K(\omega, \tau) \tag{63}$$

$$K(\omega, \tau) = \frac{\cosh(\omega(\tau - 1/2T))}{\sinh(\omega/2T)} \tag{64}$$

- Inversion with data with limited accuracy not well defined mathematical problem

Outline and Goals

- Basics of thermal field theory
  - Goal: Thermodynamics of 3+1d field theory from 4d field theory with compact euclidean time
- Lattice disceritization of QCD
  - Goal: Gauge invariant formulation of lattice QCD in terms of link matrices
- Computation of equation of state
  - Goal: Practical understanding of lattice simulation on a computer