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Understanding Lepton Mixing with Discrete Symmetries

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DaMeSyFla in the Higgs Era

Based on: L.L. Everett, T. Garon, and AS, JHEP **1504**, 069 (2015) [arXiv:1501.04336]; L.L. Everett and AS, arXiv:1611.03020 [hep-ph].

The University of Colima

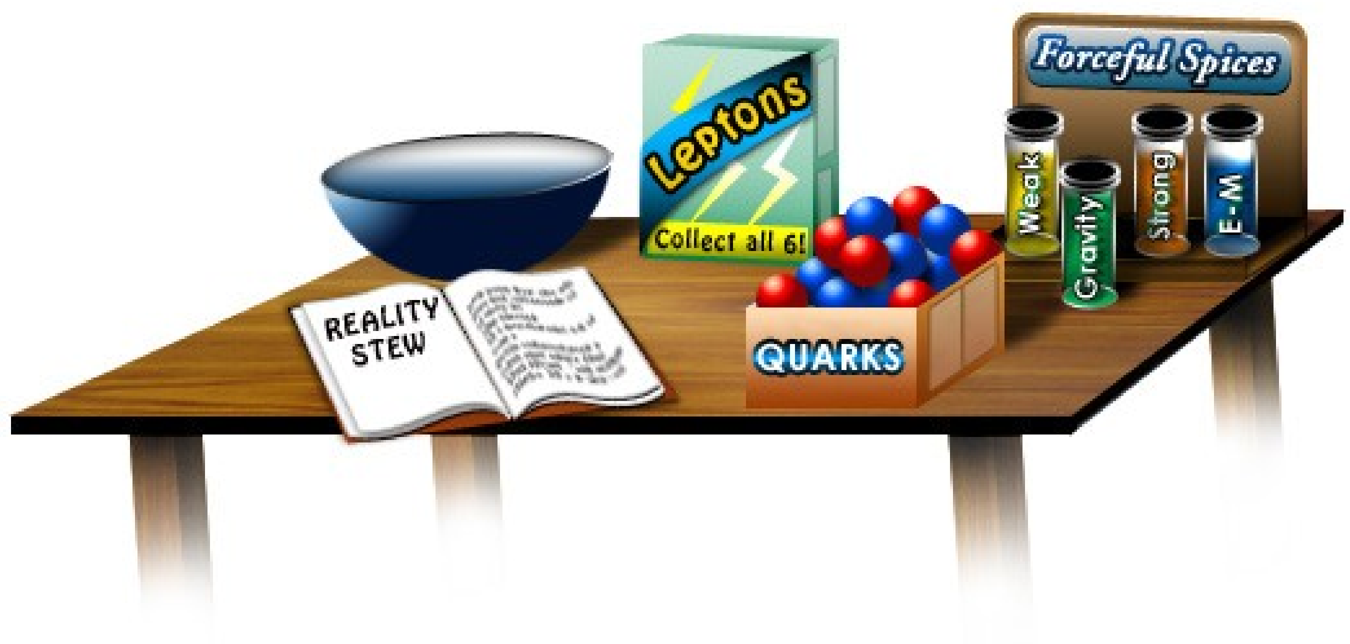
Public University located in Colima, Colima, Mexico. The physics group consists of Paolo Amore, Alfredo Aranda, Elena Cáceres, Christoph Hofmann, Sujoy Modak, Juan Reyes, Cesar Terrero, and myself.

https://watchers.news/2017/02/05/colima-eruption-february-3-2017/

If you are interested in experiencing Colima and Mexico, please join us for the next Dual CP Workshop held at the University of Colima, January 8-19, 2018.

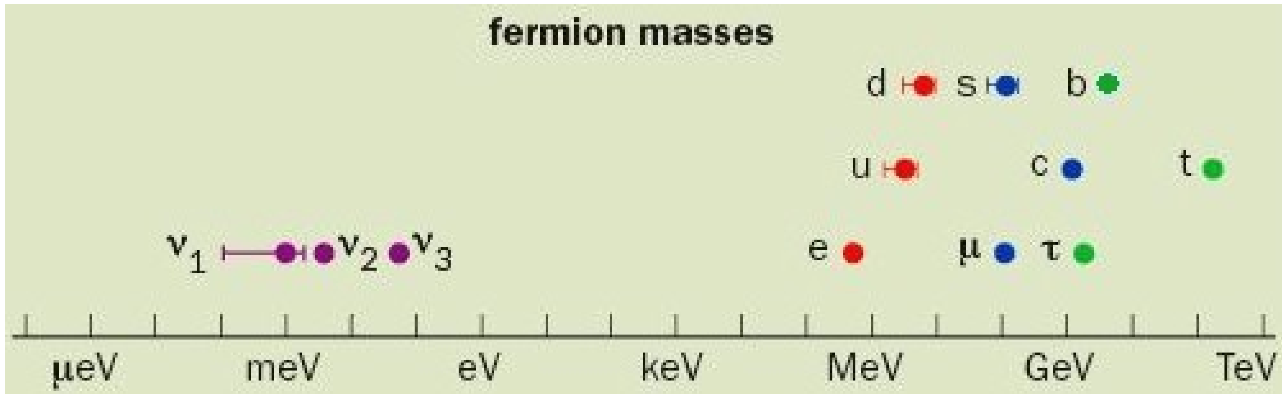
The Standard Model

Triumph of modern science, but incomplete....
Fails to predict the measured fermion masses and mixings.



http://www.particleadventure.org/standard_model.html

What We Taste



Quark Mixing

$$U_{CKM} = R_1(\theta_{23}^{CKM}) R_2(\theta_{13}^{CKM}, \delta_{CKM}) R_3(\theta_{12}^{CKM})$$

$$\theta_{13}^{CKM} = 0.2^\circ \pm 0.1^\circ$$

$$\theta_{23}^{CKM} = 2.4^\circ \pm 0.1^\circ$$

$$\theta_{12}^{CKM} = 13.0^\circ \pm 0.1^\circ$$

$$\delta_{CKM} = 60^\circ \pm 14^\circ$$

Lepton Mixing

$$U_{PMNS} = R_1(\theta_{23}) R_2(\theta_{13}, \delta_{CP}) R_3(\theta_{12}) P$$

$$\theta_{13}^{MNSP} = (8.46^\circ)^{+0.15^\circ}_{-0.15^\circ}$$

$$\theta_{23}^{MNSP} = (41.6^\circ)^{+1.5^\circ}_{-1.2^\circ}$$

$$\theta_{12}^{MNSP} = (33.56^\circ)^{+0.77^\circ}_{-0.75^\circ}$$

$$\delta_{CP} = (261^\circ)^{+51^\circ}_{-59^\circ}$$

M.C. Gonzalez-Garcia et al: 1611.01514

http://lbne.fnal.gov/how-work.shtml

Quarks look like perturbations away from Identity.

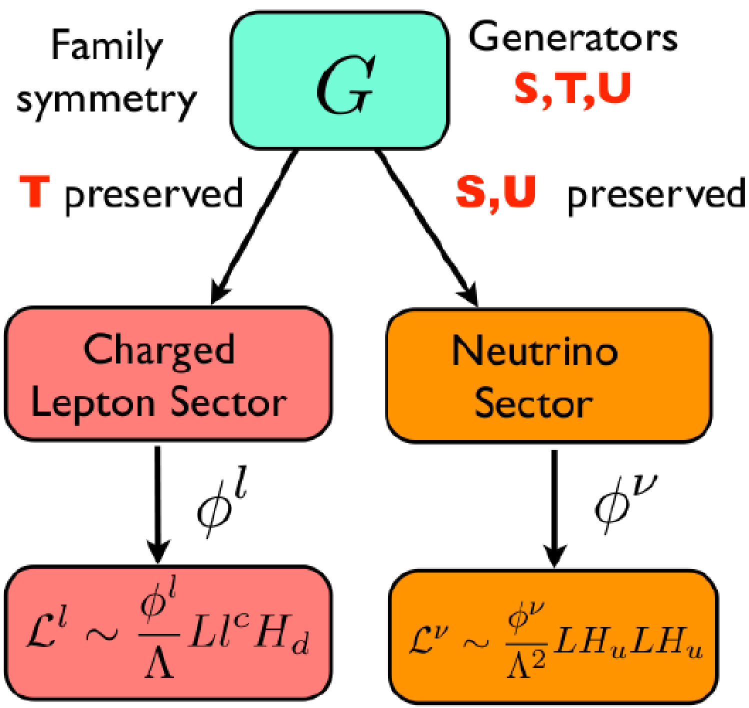
Focus on leptons.

Motivated by Symmetry

Introduce set of flavon fields (e.g. ϕ^ν and ϕ^ℓ) whose vevs break G_f to G_ν in the neutrino sector and G_e in the charged lepton sector.

$$T\langle\phi^\ell\rangle \approx \langle\phi^\ell\rangle \qquad S\langle\phi^\nu\rangle = U\langle\phi^\nu\rangle = \langle\phi^\nu\rangle$$

Non-renormalizable couplings of flavons to mass terms can be used to explain the smallness of Yukawa Couplings.



S.F. King, C. Luhn (2013)

Now that we better understand the framework, what can these symmetries be?

Residual Charged Lepton Symmetry

Since charged leptons are Dirac particles, consider $M_e = m_e m_e^\dagger$.
When **diagonal**, this combination is left invariant by a phase matrix

$$Q_e = \mathrm{Diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3})$$

Because $Q_e^\dagger M_e Q_e = M_e$

$$\mathrm{Det}(Q_e) = +1 \implies \beta_1 = -\beta_2 - \beta_3$$

$$\text{Assume } \beta_{2,3} = 2\pi k_{2,3}/n_{2,3} \text{ with } k_{2,3} = 0, \ldots, n_{2,3} - 1$$

Supposed we keep all $T = Q_e$, then

$$G_e \cong Z_{n_2} \times Z_{n_3} = Z_n \times Z_m$$

Can apply same logic to neutrino sector if neutrinos are Dirac particles, but what if they are Majorana?

Residual Neutrino Flavor Symmetry

Key: Assume neutrinos are Majorana particles

$$U_\nu^T M_\nu U_\nu = M_\nu^{\rm Diag} = {\rm Diag}(m_1, m_2, m_3) = {\rm Diag}(|m_1|e^{-i\alpha_1}, |m_2|e^{-i\alpha_2}, |m_3|e^{-i\alpha_3})$$

Notice $U_\nu \to U_\nu Q_\nu$ with $Q_\nu = {\rm Diag}(\pm 1, \pm 1, \pm 1)$ also diagonalizes the neutrino mass matrix. Restrict to ${\rm Det}(Q_\nu) = 1$ and define $G_0^{\rm Diag} = 1$

$$G_1^{\rm Diag} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad G_2^{\rm Diag} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, G_3^{\rm Diag} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Observe non-trivial relations:

$$(G_i^{\rm Diag})^2 = 1, \text{ for } i\text{=1, 2, and 3,}$$
$$G_i^{\rm Diag} G_j^{\rm Diag} = G_k^{\rm Diag}, \text{ for } i \neq j \neq k$$

Sometimes called *SU, S,* and *U*

Therefore, these form a $Z_2 \times Z_2$ residual Klein symmetry!

In non-diagonal basis: $M_\nu = G_i^T M_\nu G_i$ with $G_i = U_\nu G_i^{\rm Diag} U_\nu^\dagger$

How should we express U_ν to transform to the non-diagonal basis?

Hint: Assume diagonal charged lepton basis. $U_{\rm MNSP} = U_e^\dagger U_\nu$

Guided by the PDG

Choose the 'standard' form but take into account lessons learned from the eigenvectors of existing flavor models, e.g. TBM.

$$U_\nu = \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ -s_{12}s_{23} + c_{12}c_{23}s_{13}e^{i\delta} & c_{12}s_{23} + c_{23}s_{12}s_{13}e^{i\delta} & -c_{13}c_{23} \end{pmatrix}$$

$$U_{MNSP} = U_e^\dagger U_\nu$$

Notice, if charged leptons are diagonal (U_e=1), then the above matrix is the MNSP matrix in the PDG convention up to left-multiplication by P= Diag(1,1,-1).

With this arbitrary form it is now possible to find....

Non-Diagonal Klein Elements

$$G_i = U_\nu G_i^{\rm Diag} U_\nu^\dagger$$

$$G_1 = \begin{pmatrix} (G_1)_{11} & (G_1)_{12} & (G_1)_{13} \\ (G_1)_{12}^* & (G_1)_{22} & (G_1)_{23} \\ (G_1)_{13}^* & (G_1)_{23}^* & (G_1)_{33} \end{pmatrix} \quad G_2 = \begin{pmatrix} (G_2)_{11} & (G_2)_{12} & (G_2)_{13} \\ (G_2)_{12}^* & (G_2)_{22} & (G_2)_{23} \\ (G_2)_{13}^* & (G_2)_{23}^* & (G_2)_{33} \end{pmatrix}$$

$$G_3 = \begin{pmatrix} -c'_{13} & e^{-i\delta} s_{23} s'_{13} & -e^{-i\delta} c_{23} s'_{13} \\ e^{i\delta} s_{23} s'_{13} & s_{23}^2 c'_{13} - c_{23}^2 & -c_{13}^2 s'_{23} \\ -e^{i\delta} c_{23} s'_{13} & -c_{13}^2 s'_{23} & c_{23}^2 c'_{13} - s_{23}^2 \end{pmatrix}$$

$$s_{ij} = \sin(\theta_{ij}) \quad c_{ij} = \cos(\theta_{ij}) \quad s'_{ij} = \sin(2\theta_{ij}) \quad c'_{ij} = \cos(2\theta_{ij})$$

Notice that in general the Klein elements are complex and Hermitian!
Don't depend on Majorana phases because
$U_\nu \to U_\nu P_{\rm Maj}$ leaves transformation invariant.

Non-Diagonal Klein Elements (II)

$$(G_1)_{11} = c_{13}^2 c_{12}' - s_{13}^2 \quad (G_1)_{12} = -2c_{12}c_{13}\left(c_{23}s_{12} + e^{-i\delta}c_{12}s_{13}s_{23}\right)$$

$$(G_1)_{13} = 2c_{12}c_{13}\left(e^{-i\delta}c_{12}c_{23}s_{13} - s_{12}s_{23}\right)$$

$$(G_1)_{22} = -c_{23}^2 c_{12}' + s_{23}^2\left(s_{13}^2 c_{12}' - c_{13}^2\right) + \cos(\delta) s_{13} s_{12}' s_{23}'$$

$$(G_1)_{23} = c_{23}s_{23}c_{13}^2 + s_{13}\left(i\sin(\delta) - \cos(\delta)c_{23}'\right)s_{12}' + \frac{1}{4}c_{12}'\left(c_{13}' - 3\right)s_{23}'$$

$$(G_1)_{33} = \left(s_{13}^2 c_{12}' - c_{13}^2\right)c_{23}^2 - s_{23}^2 c_{12}' - \cos(\delta) s_{13} s_{12}' s_{23}'$$

$$(G_2)_{11} = -c_{12}' c_{13}^2 - s_{13}^2 \quad (G_2)_{12} = 2c_{13}s_{12}\left(c_{12}c_{23} - e^{-i\delta}s_{12}s_{13}s_{23}\right)$$

$$(G_2)_{13} = 2c_{13}s_{12}\left(e^{-i\delta}c_{23}s_{12}s_{13} + c_{12}s_{23}\right)$$

$$(G_2)_{22} = c_{12}' c_{23}^2 - s_{23}^2\left(c_{13}^2 + s_{13}^2 c_{12}'\right) - \cos(\delta) s_{13} s_{12}' s_{23}'$$

$$(G_2)_{23} = e^{-i\delta}s_{13}s_{12}'c_{23}^2 + \frac{1}{4}s_{23}'\left(2c_{13}^2 - c_{12}'\left(c_{13}' - 3\right)\right) - e^{i\delta}s_{12}'s_{13}s_{23}^2$$

$$(G_2)_{33} = -c_{23}^2\left(c_{13}^2 + s_{13}^2 c_{12}'\right) + s_{23}^2 c_{12}' + \cos(\delta) s_{13} s_{12}' s_{23}'$$

There is a Klein symmetry for each choice of mixing angle and CP-violating phase, implying a mass matrix left invariant for each choice.

Invariant Mass Matrix

$$M_\nu = U_\nu^* M_\nu^{\mathrm{Diag}} U_\nu^\dagger$$

$$(M_\nu)_{11} = c_{13}^2 m_2 s_{12}^2 + c_{12}^2 c_{13}^2 m_1 + e^{2i\delta} m_3 s_{13}^2$$

$$(M_\nu)_{12} = c_{13}(c_{12} m_1(-c_{23} s_{12} - c_{12} e^{-i\delta} s_{13} s_{23}) + m_2 s_{12}(c_{12} c_{23} - e^{-i\delta} s_{12} s_{13} s_{23}) + \\ + e^{i\delta} m_3 s_{13} s_{23}),$$

$$(M_\nu)_{13} = c_{13}(-c_{23} m_3 s_{13} e^{i\delta} + m_2 s_{12}(c_{12} s_{23} + c_{23} e^{-i\delta} s_{12} s_{13}) + \\ + c_{12} m_1(-s_{12} s_{23} + c_{12} c_{23} e^{-i\delta} s_{13})),$$

$$(M_\nu)_{22} = m_1(c_{23} s_{12} + c_{12} e^{-i\delta} s_{13} s_{23})^2 + m_2(c_{12} c_{23} - e^{-i\delta} s_{12} s_{13} s_{23})^2 + c_{13}^2 m_3 s_{23}^2$$

$$(M_\nu)_{23} = m_1(s_{12} s_{23} - c_{12} c_{23} e^{-i\delta} s_{13})(c_{23} s_{12} + c_{12} e^{-i\delta} s_{13} s_{23}) + \\ + m_2(c_{12} s_{23} + c_{23} e^{-i\delta} s_{12} s_{13})(c_{12} c_{23} - e^{-i\delta} s_{12} s_{13} s_{23}) - c_{13}^2 c_{23} m_3 s_{23}$$

$$(M_\nu)_{33} = m_2(c_{12} s_{23} + c_{23} e^{-i\delta} s_{12} s_{13})^2 + m_1(-s_{12} s_{23} + c_{12} c_{23} e^{-i\delta} s_{13})^2 + c_{13}^2 c_{23}^2 m_3$$

Recall these masses are complex. How can we predict their phases?

Generalized CP Symmetries

G. Branco, L. Lavoura, M. Rebelo (1986)...

Superficially look similar to flavor symmetries:

$$X_\nu^T M_\nu X_\nu = M_\nu^* \qquad Y_e^\dagger M_e Y_e = M_e^*$$

***X*=*Y*=1 is 'traditional' CP**

Related to automorphism group of flavor symmetry (Holthausen et al. (2012))

Since they act in a similar fashion to flavor symmetries, these two symmetries should be related. (Feruglio et al (2012), Holthausen et al. (2012)):

$$X_\nu G_i^* - G_i X_\nu = 0$$

Can be used to make predictions concerning both Dirac and Majorana CP violating phases, e.g. $X=G_2$

How to understand? Proceed analogously to flavor symmetry.

The Harbingers of Majorana Phases

(S.M. Bilenky, J. Hosek, S.T. Petcov(1980))

Work in diagonal basis. Then it is trivial to see $X = U_\nu X^{\mathrm{Diag}} U_\nu^T$

with $$X^{\mathrm{Diag}} = \begin{pmatrix} \pm e^{i\alpha_1} & 0 & 0 \\ 0 & \pm e^{i\alpha_2} & 0 \\ 0 & 0 & \pm e^{i\alpha_3} \end{pmatrix}$$

where α_i are Majorana phases.

Notice we have freedom to globally re-phase: $M_\nu \to e^{i\theta} M_\nu$

Such a re-phasing will not affect the mixing angles or observable phases.

Now can make the important observation

$$X_i^{\mathrm{Diag}} = G_i^{\mathrm{Diag}} \times \mathrm{Diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3})$$

Therefore, the X_i represent a *complexification* of the Klein symmetry elements!

So, they must inherit an algebra from the Klein elements...

Generalized CP Relations

To eliminate phases, must have one *X* conjugated

$$X_0 X_i^* = G_i \text{ for } i = 1,\ 2,\ 3 \quad X_i X_j^* = G_k \text{ for } i \neq j \neq k \neq 0$$

$$X_i X_i^* = G_0 = 1 \text{ for } i = 0,\ 1,\ 2,\ 3$$

Clearly these imply:

$$(X_0 X_i^*)^2 = 1 \text{ for } i = 1, 2, 3 \qquad (X_i X_j^*)^2 = 1 \text{ for } i \neq j \neq 0$$

$$X_i X_i^* = 1 \text{ for } i = 0,\ 1,\ 2,\ 3$$

Note if $X_i X_j^* = G' \neq G_k$ flavor symmetry is enlarged leading to unphysical predictions because Klein symmetry is largest symmetry to completely fix mixing and masses.

$$X_j^\dagger X_i^T M_\nu X_i X_j^* = M_\nu$$

So what do these generalized CP elements look like in non-diagonal basis?

The Non-Diagonal General CP

$$X_{11} = (-1)^a e^{i\alpha_1} c_{12}^2 c_{13}^2 + (-1)^b e^{i\alpha_2} c_{13}^2 s_{12}^2 + (-1)^c s_{13}^2 e^{i(\alpha_3 - 2\delta)}$$

$$X_{12} = (-1)^{a+1} e^{i\alpha_1} c_{12} c_{13} (c_{23} s_{12} + c_{12} s_{13} s_{23} e^{i\delta}) + (-1)^b e^{i\alpha_2} c_{13} s_{12} (c_{12} c_{23} - \\ - s_{12} s_{13} s_{23} e^{i\delta}) + (-1)^c c_{13} s_{13} s_{23} e^{i(\alpha_3 - \delta)},$$

$$X_{13} = (-1)^{a+1} e^{i\alpha_1} c_{12} c_{13} (s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta}) + (-1)^b e^{i\alpha_2} c_{13} s_{12} (c_{12} s_{23} + \\ + c_{23} s_{12} s_{13} e^{i\delta}) + (-1)^{c+1} c_{13} c_{23} s_{13} e^{i(\alpha_3 - \delta)},$$

$$X_{22} = (-1)^a e^{i\alpha_1} (c_{23} s_{12} + c_{12} s_{13} s_{23} e^{i\delta})^2 + (-1)^b e^{i\alpha_2} (c_{12} c_{23} - s_{12} s_{13} s_{23} e^{i\delta})^2 + \\ + (-1)^c e^{i\alpha_3} c_{13}^2 s_{23}^2,$$

$$X_{23} = (-1)^a e^{i\alpha_1} (s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta})(c_{23} s_{12} + c_{12} s_{13} s_{23} e^{i\delta}) + \\ + (-1)^b e^{i\alpha_2} (c_{12} s_{23} + c_{23} s_{12} s_{13} e^{i\delta})(c_{12} c_{23} - s_{12} s_{13} s_{23} e^{i\delta}) + (-1)^{c+1} e^{i\alpha_3} c_{23} c_{13}^2 s_{23}$$

$$X_{33} = (-1)^a e^{i\alpha_1} (s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta})^2 + (-1)^b e^{i\alpha_2} (c_{12} s_{23} + c_{23} s_{12} s_{13} e^{i\delta})^2 + \\ + (-1)^c e^{i\alpha_3} c_{13}^2 c_{23}^2.$$

$$(-1)^a = (G_i^{\mathrm{Diag}})_{11} \qquad (-1)^b = (G_i^{\mathrm{Diag}})_{22} \qquad (-1)^c = (G_i^{\mathrm{Diag}})_{33}$$

Proofs by Construction

Can use explicit forms for G_j and X_i to easily show

$$X_i G_j^* - G_j X_i = 0 \text{ for } i, j = 0, 1, 2, 3$$

Now when just the Dirac CP violation is trivial, it is easy to see

$$[X_i, G_j]_{\delta=0,\pi} = 0 \text{ for } i, j = 0, 1, 2, 3$$

Can easily be understood from the forms of G_i since $G_i = G_i^*$ implies a trivial Dirac phase.

If just Majorana phases are let to vanish, then

$$(X_i - G_i)_{mn} \propto (e^{2i\delta} - 1) \text{ for } i = 0, 1, 2, 3$$

implying equality if Dirac 'vanishes' as well. Therefore, if one wants commutation between flavor and CP, then this will *always* lead to a trivial Dirac phase. Furthermore, if they are equal then all phases must vanish (**Think** *M*=*M**).

What else can we use this for?

Revisiting Tribimaximal Mixing

P. F. Harrison, D. H. Perkins, W. G. Scott (2002); P. F. Harrison, W. G. Scott (2002); Z. -z. Xing (2002)

$$\theta_{12}^{\mathrm{TBM}} = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \quad \theta_{23}^{\mathrm{TBM}} = \frac{\pi}{4} \quad \theta_{13}^{\mathrm{TBM}} = 0 \quad \delta^{\mathrm{TBM}} = 0$$

Plugging these values into the previous results yield:

$$U^{\mathrm{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad S_4$$

$$G_1^{\mathrm{TBM}} = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{pmatrix} \quad G_2^{\mathrm{TBM}} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \quad G_3^{\mathrm{TBM}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$M_\nu^{\mathrm{TBM}} = \frac{1}{3}\begin{pmatrix} (2m_1 + m_2) & (m_2 - m_1) & (m_2 - m_1) \\ (m_2 - m_1) & \frac{1}{2}(m_1 + 2m_2 + 3m_3) & \frac{1}{2}(m_1 + 2m_2 - 3m_3) \\ (m_2 - m_1) & \frac{1}{2}(m_1 + 2m_2 - 3m_3) & \frac{1}{2}(m_1 + 2m_2 + 3m_3) \end{pmatrix}$$

The well-known mass matrix and Klein elements of TBM.

Tribimaximal Mixing (cont.)

$$X_{11}^{\mathrm{TBM}} = \frac{1}{3}\left(2(-1)^a e^{i\alpha_1} + e^{i\alpha_2}(-1)^b\right) \qquad X_{12}^{\mathrm{TBM}} = \frac{1}{3}\left((-1)^{a+1} e^{i\alpha_1} + e^{i\alpha_2}(-1)^b\right)$$

$$X_{22}^{\mathrm{TBM}} = \frac{1}{6}\left((-1)^a e^{i\alpha_1} + 2e^{i\alpha_2}(-1)^b + 3e^{i\alpha_3}(-1)^c\right)$$

$$X_{13}^{\mathrm{TBM}} = \frac{1}{3}\left((-1)^{a+1} e^{i\alpha_1} + e^{i\alpha_2}(-1)^b\right)$$

$$X_{23}^{\mathrm{TBM}} = \frac{1}{6}\left((-1)^a e^{i\alpha_1} + 2e^{i\alpha_2}(-1)^b - 3e^{i\alpha_3}(-1)^c\right)$$

$$X_{33}^{\mathrm{TBM}} = \frac{1}{6}\left((-1)^a e^{i\alpha_1} + 2e^{i\alpha_2}(-1)^b + 3e^{i\alpha_3}(-1)^c\right)$$

Any generalized CP symmetry consistent with the TBM Klein symmetry will be given by the above results even if TBM is not coming from S_4.

Notice vanishing Majorana phases gives TBM Klein symmetry back.

Golden Ratio Mixing (GR1)

A. Datta, F. Ling, P. Ramond (2003); Y. Kajiyama, M Raidal, A. Strumia (2007); L. Everett, AS (2008)

$$\theta_{12}^{\mathrm{GR1}} = \tan^{-1}\left(\frac{1}{\phi}\right) \quad \theta_{23}^{\mathrm{GR1}} = \frac{\pi}{4} \quad \theta_{13}^{\mathrm{GR1}} = 0 \quad \delta^{\mathrm{GR1}} = 0$$

$$\phi = (1+\sqrt{5})/2 \qquad U^{\mathrm{GR1}} = \begin{pmatrix} \sqrt{\frac{\phi}{\sqrt{5}}} & \sqrt{\frac{1}{\sqrt{5}\phi}} & 0 \\ -\frac{1}{\sqrt{2}}\sqrt{\frac{1}{\sqrt{5}\phi}} & \frac{1}{\sqrt{2}}\sqrt{\frac{\phi}{\sqrt{5}}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}\sqrt{\frac{1}{\sqrt{5}\phi}} & \frac{1}{\sqrt{2}}\sqrt{\frac{\phi}{\sqrt{5}}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad A_5$$

$$G_1^{\mathrm{GR1}} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\phi & \phi-1 \\ -\sqrt{2} & \phi-1 & -\phi \end{pmatrix} \quad G_2^{\mathrm{GR1}} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1-\phi & \phi \\ \sqrt{2} & \phi & 1-\phi \end{pmatrix} \quad G_3^{\mathrm{GR1}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$M_\nu^{\mathrm{GR1}} = \frac{1}{\sqrt{5}}\begin{pmatrix} \frac{m_1\phi^2+m_2}{\phi} & \frac{m_2-m_1}{\sqrt{2}} & \frac{m_2-m_1}{\sqrt{2}} \\ \frac{m_2-m_1}{\sqrt{2}} & \frac{(m_2+m_3)\phi^2+m_1+m_3}{2\phi} & \frac{m_2\phi^2-\sqrt{5}m_3\phi+m_1}{2\phi} \\ \frac{m_2-m_1}{\sqrt{2}} & \frac{m_2\phi^2-\sqrt{5}m_3\phi+m_1}{2\phi} & \frac{(m_2+m_3)\phi^2+m_1+m_3}{2\phi} \end{pmatrix}$$

What about the generalized CP symmetries?

Golden Ratio Mixing (cont.)

$$X_{11}^{\mathrm{GR1}} = \frac{(-1)^a e^{i\alpha_1}\phi^2 + e^{i\alpha_2}(-1)^b}{\sqrt{5}\phi} \qquad X_{12}^{\mathrm{GR1}} = \frac{(-1)^{a+1}e^{i\alpha_1} + e^{i\alpha_2}(-1)^b}{\sqrt{10}}$$

$$X_{13}^{\mathrm{GR1}} = \frac{(-1)^{a+1}e^{i\alpha_1} + e^{i\alpha_2}(-1)^b}{\sqrt{10}} \quad X_{22}^{\mathrm{GR1}} = \frac{(-1)^a e^{i\alpha_1} + e^{i\alpha_2}(-1)^b\phi^2 + \sqrt{5}e^{i\alpha_3}(-1)^c\phi}{2\sqrt{5}\phi}$$

$$X_{23}^{\mathrm{GR1}} = \frac{(-1)^a e^{i\alpha_1} + e^{i\alpha_2}(-1)^b\phi^2 + \sqrt{5}e^{i\alpha_3}(-1)^{c+1}\phi}{2\sqrt{5}\phi}$$

$$X_{33}^{\mathrm{GR1}} = \frac{(-1)^a e^{i\alpha_1} + e^{i\alpha_2}(-1)^b\phi^2 + \sqrt{5}e^{i\alpha_3}(-1)^c\phi}{2\sqrt{5}\phi}$$

Becomes Golden Klein Symmetry when Majorana phases vanish.
Any 'golden' generalized CP symmetry will be given by the above results, even if it does not come from A_5.

Bitrimaximal Mixing

R. Toorop, F. Feruglio, C. Hagedorn (2011); G.J. Ding (2012); S. King, C. Luhn, AS(2013)

$$\theta_{12}^{\rm BTM} = \theta_{23}^{\rm BTM} = \tan^{-1}(\sqrt{3}-1) \;\; \theta_{13}^{\rm BTM} = \sin^{-1}(\frac{1}{6}(3-\sqrt{3})) \;\; \delta^{\rm BTM} = 0$$

Yielding

$$U^{\rm BTM} = \begin{pmatrix} \frac{1}{6}\left(3+\sqrt{3}\right) & \frac{1}{\sqrt{3}} & \frac{1}{6}\left(3-\sqrt{3}\right) \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{6}\left(-3+\sqrt{3}\right) & \frac{1}{\sqrt{3}} & \frac{1}{6}\left(-3-\sqrt{3}\right) \end{pmatrix} \qquad \Delta(96)$$

$$G_1^{\rm BTM} = \begin{pmatrix} \frac{1}{\sqrt{3}}-\frac{1}{3} & -\frac{1}{3}-\frac{1}{\sqrt{3}} & -\frac{1}{3} \\ -\frac{1}{3}-\frac{1}{\sqrt{3}} & -\frac{1}{3} & \frac{1}{\sqrt{3}}-\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{\sqrt{3}}-\frac{1}{3} & -\frac{1}{3}-\frac{1}{\sqrt{3}} \end{pmatrix} \; G_2^{\rm BTM} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} G_3^{\rm BTM} = \begin{pmatrix} -\frac{1}{3}-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}-\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{\sqrt{3}}-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3}-\frac{1}{\sqrt{3}} \\ -\frac{1}{3} & -\frac{1}{3}-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}-\frac{1}{3} \end{pmatrix}$$

And a mass matrix given by

$$(M_\nu^{\rm BTM})_{11} = \frac{1}{6}((2+\sqrt{3})m_1 + 2m_2 - (-2+\sqrt{3})m_3) \qquad (M_\nu^{\rm BTM})_{22} = \frac{1}{3}(m_1+m_2+m_3)$$

$$(M_\nu^{\rm BTM})_{13} = \frac{1}{6}(-m_1+2m_2-m_3) \qquad (M_\nu^{\rm BTM})_{12} = \frac{1}{6}(-(1+\sqrt{3})m_1 + 2m_2 + (-1+\sqrt{3})m_3)$$

$$(M_\nu^{BTM})_{33} = \frac{1}{6}(-(-2+\sqrt{3})m_1 + 2m_2 + (2+\sqrt{3})m_3) \; (M_\nu^{\rm BTM})_{23} = \frac{1}{6}((-1+\sqrt{3})m_1 + 2m_2 - (1+\sqrt{3})m_3)$$

Bitrimaximal Mixing (cont.)

$$X_{11}^{\mathrm{BTM}} = \frac{1}{6}\left((-1)^{c+1}e^{i\alpha_3}\left(-2+\sqrt{3}\right) + (-1)^{a}\left(2+\sqrt{3}\right)e^{i\alpha_1} + 2(-1)^{b}e^{i\alpha_2}\right)$$

$$X_{12}^{\mathrm{BTM}} = \frac{1}{6}\left((-1)^{c}e^{i\alpha_3}\left(-1+\sqrt{3}\right) + (-1)^{a+1}\left(1+\sqrt{3}\right)e^{i\alpha_1} + 2(-1)^{b}e^{i\alpha_2}\right)$$

$$X_{13}^{\mathrm{BTM}} = \frac{1}{6}\left((-1)^{a+1}e^{i\alpha_1} + 2(-1)^{b}e^{i\alpha_2} + (-1)^{c+1}e^{i\alpha_3}\right)$$

$$X_{22}^{\mathrm{BTM}} = \frac{1}{3}\left((-1)^{a}e^{i\alpha_1} + (-1)^{b}e^{i\alpha_2} + (-1)^{c}e^{i\alpha_3}\right)$$

$$X_{23}^{\mathrm{BTM}} = \frac{1}{6}\left((-1)^{a}e^{i\alpha_1}\left(-1+\sqrt{3}\right) + 2(-1)^{b}e^{i\alpha_2} + (-1)^{c+1}\left(1+\sqrt{3}\right)e^{i\alpha_3}\right)$$

$$X_{33}^{\mathrm{BTM}} = \frac{1}{6}\left((-1)^{a+1}e^{i\alpha_1}\left(-2+\sqrt{3}\right) + 2(-1)^{b}e^{i\alpha_2} + (-1)^{c}\left(2+\sqrt{3}\right)e^{i\alpha_3}\right)$$

Non-Trivial Check: $\alpha_1 = \alpha_3 = \frac{\pi}{6} \;\; \alpha_2 = -\frac{\pi}{3} \;\; a = 1, b = 0, c = 1$

Matches known order 4 Δ(96) automorphism group element when unphysical phases redefined.

S. King, T. Neder(2014)
S. King, G. J. Ding (2014)

Bitrimaximal Mixing (cont.)

$$X_{11}^{\mathrm{BTM}} = \frac{1}{6}\left((-1)^{c+1}e^{i\alpha_3}\left(-2+\sqrt{3}\right)+(-1)^{a}\left(2+\sqrt{3}\right)e^{i\alpha_1}+2(-1)^{b}e^{i\alpha_2}\right)$$

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Matches known order 4 Δ(96) automorphism group element when **unphysical phases redefined**.

S. King, T. Neder(2014)
S. King, G. J. Ding (2014)

Bitrimaximal Mixing (cont.)

$$X_{11}^{\rm BTM} = \frac{1}{6}\left((-1)^{c+1}e^{i\alpha_3}\left(-2+\sqrt{3}\right)+(-1)^a\left(2+\sqrt{3}\right)e^{i\alpha_1}+2(-1)^b e^{i\alpha_2}\right)$$

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Non-Trivial Check: $\alpha_1 = \alpha_3 = \frac{\pi}{6}\ \ \alpha_2 = -\frac{\pi}{3}\ \ a = 1, b = 0, c = 1$

Matches known order 4 Δ(96) automorphism group element when **unphysical phases redefined**. What do we mean by this?

S. King, T. Neder(2014)
S. King, G. J. Ding (2014)

Enter Unphysical Phases

Re-Work entire diagonalization logic before and pay special attention to 'unphysical' phases.

$$U_\nu^T M_\nu U_\nu = M_\nu^{\rm Diag} = {\rm Diag}(m_1, m_2, m_3)$$

$$m_1 \neq m_2 \neq m_3 \neq 0$$

Notice

$$U_\nu \to U_\nu Q_\nu$$

$$Q_\nu = {\rm Diag}((-1)^{p_1}, (-1)^{p_2}, (-1)^{p_3})$$

$$p_{1,2,3} = 0, 1$$

The above mapping leaves diagonalization unchanged! The entries in Q_ν are what we will call unphysical phases.

Unphysical Charged Lepton Phases

Apply same logic to charged lepton sector. Then, recall

$$U_e^\dagger M_e U_e = M_e^{\mathrm{Diag}} = \mathrm{Diag}(|m_e|^2, |m_\mu|^2, |m_\tau|^2)$$

$$|m_e| \neq |m_\mu| \neq |m_\tau| \neq 0$$

Then,

$$U_e \to U_e Q_e$$

$$Q_e = \mathrm{Diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}) \qquad \beta_{1,2,3} \in [0, 2\pi)$$

leaves the diagonalization invariant.

One can ask, how can these phases enter the MNSP mixing matrix?

Transforming Leptonic Mixing

Recall,

$$U_{\rm MNSP} = U_e^\dagger U_\nu$$

Then,

$$U_\nu \to U_\nu Q_\nu \qquad\qquad U_e \to U_e Q_e$$

implies

$$U_{\rm MNSP} \to Q_e^\dagger U_{\rm MNSP} Q_\nu = U'_{\rm MNSP}$$

So, $U_{\rm MNSP}$ and (the infinitely many) $U'_{\rm MNSP}$ must all give the same phenomenological predictions!

Really just rephasing invariants. (see, e.g., E. Jenkins and A. Manohar (2008) for discussion on rephasing invariants)

What if these choices are coming from a group whose elements are not diagonal, i.e. whose mass matrix is not diagonal?

Relating Diagonal to Nondiagonal

Nondiagonal mass matrix basis flavor symmetry invariance conditions are

$$S_\nu^T M_\nu S_\nu = M_\nu \qquad T_e^\dagger M_e T_e = M_e$$

These (with diagonalization relationships) imply

$$S_\nu = U_\nu Q_\nu U_\nu^\dagger \qquad T_e = U_e Q_e U_e^\dagger$$

Notice: The above relationships are similarity transformations!
Furthermore, the transformation

$$U_\nu \to U_\nu Q_\nu \qquad U_e \to U_e Q_e$$

leaves the similarity relationships unchanged.

Does something similar happen for generalized CP elements?

Arbitrary Generalized CP Transformations

Recall from our previous discussion of generalized CP:

$$X_\nu^T M_\nu X_\nu = M_\nu^* \qquad Y_e^\dagger M_e Y_e = M_e^*$$

Undiagonalizing these reveals:

$$X_\nu = U_\nu X_\nu^{\mathrm{Diag}} U_\nu^T \qquad Y_e = U_e Y_e^{\mathrm{Diag}} U_e^T$$

(I. Girardi, S. Petcov, A. Titov, AS (2016))

Notice: In general these are not similarity transformations. Yet with all unphysical phases included in the diagonal basis these elements are expressible as:

$$X_\nu^{\mathrm{Diag}} = \mathrm{Diag}(\pm e^{i\alpha_1'}, \pm e^{i\alpha_2'}, \pm e^{i\alpha_3'}) \quad \alpha_i' = \alpha_i + \theta_\nu$$

$$Y_e^{\mathrm{Diag}} = \mathrm{Diag}(e^{i\gamma_1}, e^{i\gamma_2}, e^{i\gamma_3})$$

keeps track of unphysical phase shift.

Making Similarities More Clear

As before we split the neutrino generalized CP transformations as

$$X_\nu^{\mathrm{Diag}} = \mathrm{Diag}(\pm e^{i\alpha'_1}, \pm e^{i\alpha'_2}, \pm e^{i\alpha'_3})$$

$$X_0^{\mathrm{Diag}} = \mathrm{Diag}(e^{i\alpha'_1}, e^{i\alpha'_2}, e^{i\alpha'_3})$$ (flavor symmetry 'removed'!)

Like previous case with unphysical phase=zero: $X_\nu^{\mathrm{Diag}} = Q_\nu \times X_0^{\mathrm{Diag}}$

For charged leptons it is “slightly” trickier because

$$Y_e^{\mathrm{Diag}} = \mathrm{Diag}(e^{i\gamma_1}, e^{i\gamma_2}, e^{i\gamma_3})$$

Make flavor explicit and define: $\beta'_i = \gamma_i - \beta_i$

Then, $Y_0^{\mathrm{Diag}} = \mathrm{Diag}(e^{i\beta'_1}, e^{i\beta'_2}, e^{i\beta'_3})$

$$Y_e^{\mathrm{Diag}} = Q_e \times Y_0^{\mathrm{Diag}}$$

These 2 simple relationships are actually very useful....

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These 2 simple relationships are actually very useful....

Orders of Generalized CP Elements I

Notice the application of two CP transformations looks like:

$$X_j^\dagger X_i^T M_\nu X_i X_j^* = M_\nu$$
$$Y_l^T Y_k^\dagger M_e Y_k Y_l^* = M_e$$

Therefore,

$$G_k = X_i X_j^* \qquad i,j = 0,1,2,3$$

$$T_m = Y_k Y_l^* \qquad m,k,l = 0,\ldots,n_2+n_3-1$$

These relationships with the relationships on the previous slide can be used to find the orders of the generalized CP symmetry transformations, i.e., the smallest integers p, q such that $X^p = Y^q = 1$.

Orders of Generalized CP Elements II

Fair to assume such p and q exist. Thus,

$$(X_\nu^{\mathrm{Diag}})^p = Q_\nu^p \times \mathrm{Diag}(e^{ip\alpha'_1}, e^{ip\alpha'_2}, e^{ip\alpha'_3}) = 1$$

$$(Y_e^{\mathrm{Diag}})^q = Q_e^q \times \mathrm{Diag}(e^{iq\beta'_1}, e^{iq\beta'_2}, e^{iq\beta'_3}) = 1$$

Therefore the order of the generalized CP symmetry elements must be integer multiples of the flavor symmetry elements! Thus, the orders of the *X*'s must be even (C.C. Nishi (2013)).

Clearly only for diagonal CP elements or when $U_\nu = U_\nu^* \text{ and } U_e = U_e^*$

For other cases, can deduce relationships:

$$\mathrm{Det}(U_e^*)^{2q}\,\mathrm{Det}(Y_e)^q = 1 \quad \mathrm{Det}(U_\nu^*)^{2p}\,\mathrm{Det}(X_\nu)^p = 1$$

Relationships invariant under: $U_\nu \to U_\nu Q_\nu \quad U_e \to U_e Q_e$

BT Mixing Revisited

BT mixing known to be generated by Δ(96).
Relevant parts of Δ(96) Character Table:

$\Delta(96)$	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{2}$	$\mathbf{3}$	$\widetilde{\mathbf{3}}$	$\overline{\mathbf{3}}$	$\mathbf{3}'$	$\widetilde{\mathbf{3}}'$	$\overline{\mathbf{3}}'$
$\mathcal{I}$	1	1	2	3	3	3	3	3	3
$3C_4$	1	1	2	$-1+2i$	-1	$-1-2i$	$-1+2i$	-1	$-1-2i$
$3C_2$	1	1	2	-1	3	-1	-1	3	-1
$3C_4'$	1	1	2	$-1-2i$	-1	$-1+2i$	$-1-2i$	-1	$-1+2i$

Restricting to 3-dimensional irreducible representations reveals

$$\mathrm{Tr}(X_\nu^{\mathrm{BTM}}) = e^{i\alpha_1'} + e^{i\alpha_2'} + e^{i\alpha_3'} = -1 \pm 2i$$

Hence

$$\mathrm{Tr}(X_\nu^{\mathrm{BTM}}) = e^{2\pi i/3 + i\theta_\nu}(-1-2i)$$

Thus to match previous results $\theta_\nu = 4\pi/3$

Conclusion

- If neutrinos are Majorana particles, the possibility exists that there is a high scale flavor symmetry spontaneously broken to a residual Klein symmetry in the neutrino sector, completely determining lepton mixing parameters (except Majorana phases).
- To predict Majorana phases, implement a generalized CP symmetry alongside a flavor symmetry.
- In 1501.04336, we have constructed a bottom-up approach that clarifies the interplay between flavor and CP symmetries by expressing the residual, unbroken Klein and generalized CP symmetries in terms of the lepton mixing parameters.
- In 1611.03020, we have expanded this approach to include unphysical phases in any arbitrary basis where the mixing is completely fixed by the symmetries.
- This expanded approach provides further clarifies more statements in the literature as well as yields relationships which must be satisfied in any top-down model, thus serving as a guide for future model-building.

Back-up Slides

Hinting at the Unphysical

Recall each nontrivial Klein element has one +1 eigenvalue.

The eigenvector associated with this eigenvalue will be one column of the MNSP matrix (in the diagonal charged lepton basis).
As an example consider tribimaximal mixing:

$$U^{\text{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

P. F. Harrison, D. H. Perkins, W. G. Scott (2002)
P. F. Harrison, W. G. Scott (2002)
Z. -z. Xing (2002)

Can be shown to originate from the preserve Klein symmetry:

$$G_1^{\text{TBM}} = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{pmatrix} \quad G_2^{\text{TBM}} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \quad G_3^{\text{TBM}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

Notice the eigenvectors are not in the standard MNSP parametrization.

A Caveat

If low energy parameters are not taken as inputs for generating the possible predictions for the Klein symmetry elements, it is possible to generate them by breaking a flavor group G_f *to* $Z_2 \times Z_2$ in the neutrino sector and Z_m in the charged lepton sector, while also consistently breaking H_{CP} *to* $X_{i.}$

Then predictions for parameters can become subject to charged lepton (CL) corrections, renormalization group evolution (RGE), and canonical normalization (CN) considerations.

Although, can expect these corrections to be subleading as RGE and CN effects are expected to be small in realistic models with hierarchical neutrino masses, and CL corrections are typically at most Cabibbo-sized. (J. Casa, J. Espinosa, A Ibarra, I Navarro (2000); S. Antusch, J Kersten, M. Lindner, M. Ratz (2003); S. King I. Peddie (2004); S. Antusch, S. King, M. Malinsky (2009);)