The Power of Perturbation Theory

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Main result:

All observables in a one-dimensional quantum mechanical system with a bounded potential can be entirely computed from a single perturbative series

Introduction and Motivation

It is known that perturbative expansions in QM and QFT
are only asymptotic with zero radius of convergence

[Dyson, 1952]

In special cases, like the anharmonic oscillator in QM and $\phi^4_{2,3}$ QFT
it has been proved that perturbation theory is Borel resummable
and leads to the exact results (no non-perturbative contributions)

[Loeffel et al, 1969; Simon and Dicke, 1970; Eckmann, Magnen and Seneor, 1975; Magnen and Seneor, 1977]

Most perturbative expansions are **not** Borel resummable

Technically, this is due to the appearance of singularities of the Borel function along an integral needed to perform to get the answer

The singularity can be avoided by deforming the contour
but the result present an ambiguitiy of order $\exp(-a/\lambda^n)$

If one is able to find a semi-classical instanton like configuration (and its whole series) leading to the same factors, one might hope to remove the ambiguity

Alternative geometric picture starts from path integral

Perturbation theory is infinite dimensional generalization of steepest-descent method to evaluate ordinary integrals

Understanding which instantons contribute to a given observable amounts to understand which saddles of the action should be considered in the path integral

We can hope to address this point using generalization to infinite dimensions of known mathematical methods.

[Witten, 2011]

Key question:

Under what conditions only one saddle point (trivial vacuum) contributes so that perturbation theory gives the full answer?

We answered this question in QM.

Plan

One dimensional integrals

Lefschetz thimbles

Borel sums

Exact perturbation theory

Path integral

Examples

Conclusions

One dimensional integrals

$$Z(\lambda) \equiv \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{\infty} dx\, g(x)\, e^{-f(x)/\lambda}$$

Convergent for any $\lambda \geq 0$

We can compute the integral **exactly** using steepest-descent methods:

1. Determine the saddle points contributing to Z
2. Resum the "perturbative" expansion around each saddle

Consider first point 1.

Analytically continue in complex plane

$$Z(\lambda) = \frac{1}{\sqrt{\lambda}} \int_{\mathcal{C}_x} dz\, g(z)\, e^{-f(z)/\lambda}$$

Saddle points z_σ: $f'(z_\sigma) = 0$

For each saddle we call downward and upward flows $\mathcal{J}_\sigma$ and $\mathcal{K}_\sigma$ the lines where Re F decrease and increase, respectively, and Im F is constant

$$F(z) \equiv -f(z)/\lambda$$

$\mathcal{J}_\sigma$ is the path of steepest descent. If it flows to $\mathrm{Re}\ F = -\infty$

it is called a Lefschetz thimble (or just thimble).

By construction, the integral over a thimble is always well defined and convergent.

If $\mathcal{J}_\sigma$ hits another saddle point, the flow splits int two branches and ambiguity arises (Stokes line).

Picard-Lefschetz theory: deform the contour $\mathcal{C}_x$ in a contour $\mathcal{C}$:

$$\mathcal{C} = \sum_\sigma \mathcal{J}_\sigma n_\sigma \qquad n_\sigma = \langle \mathcal{C}_x, \mathcal{K}_\sigma \rangle$$

$\langle \ldots \rangle$ denotes intersection pairings $\langle \mathcal{J}_\sigma, \mathcal{K}_\tau \rangle = \delta_{\sigma\tau}$

[Witten, 1001.2933]

If $\mathcal{J}_\sigma = \mathcal{K}_\tau$ intersection not well defined.
Ambiguity resolved by assigning a small imaginary part to λ

$$Z(\lambda) = \sum_\sigma n_\sigma Z_\sigma(\lambda)$$

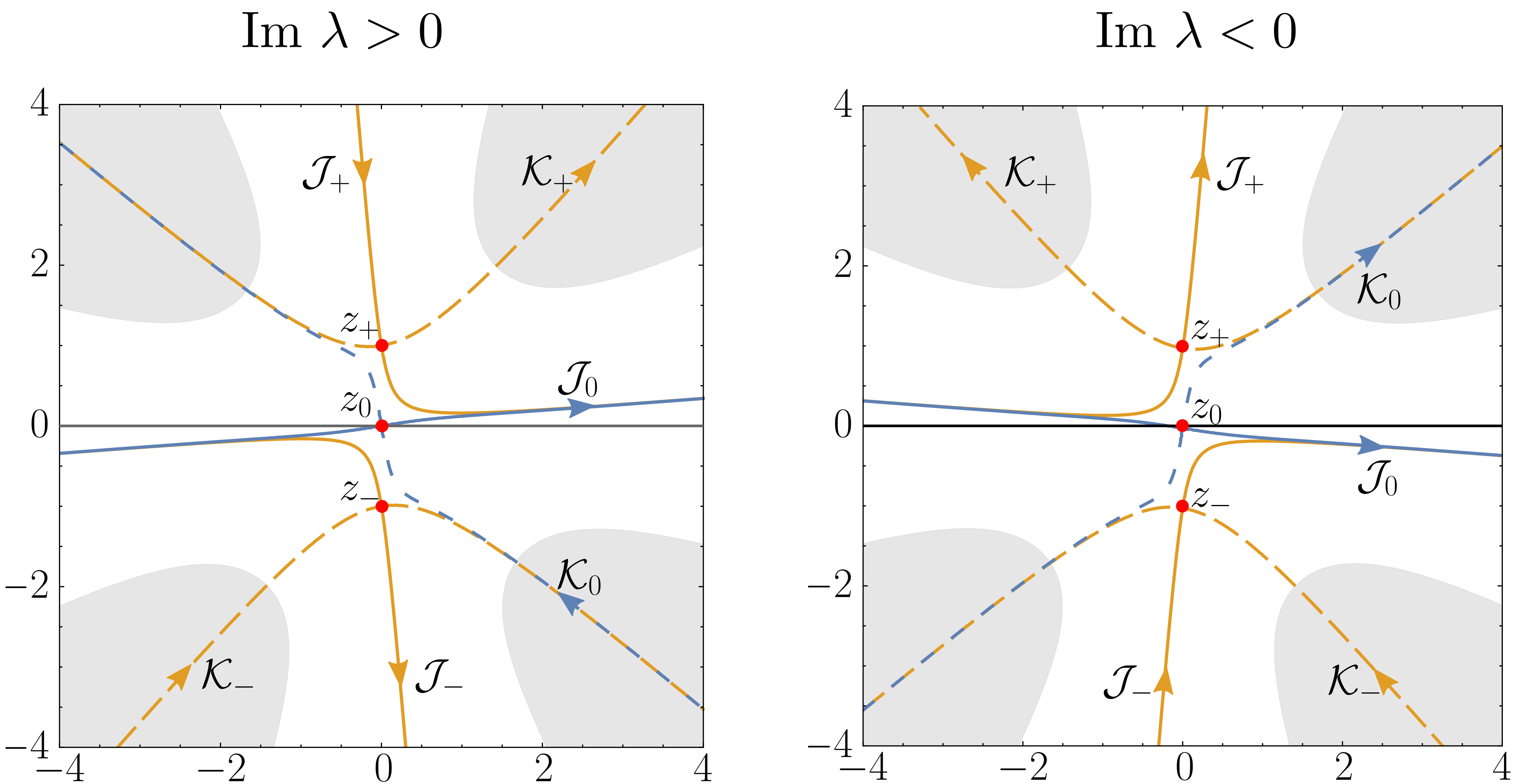
$$Z_\sigma(\lambda) \equiv \frac{1}{\sqrt{\lambda}} \int_{\mathcal{J}_\sigma} dz\, g(z)\, e^{-f(z)/\lambda}$$

Not all saddles contribute to the integral, only those with $n_\sigma \neq 0$

Example: i) $$f(x) = \frac{1}{2} x^2 + \frac{1}{4} x^4 \,, \qquad g(x) = 1$$

Three saddles: $z_0 = 0,\ z_\pm = \pm i$

Degenerate situation. Give a small imaginary part to λ



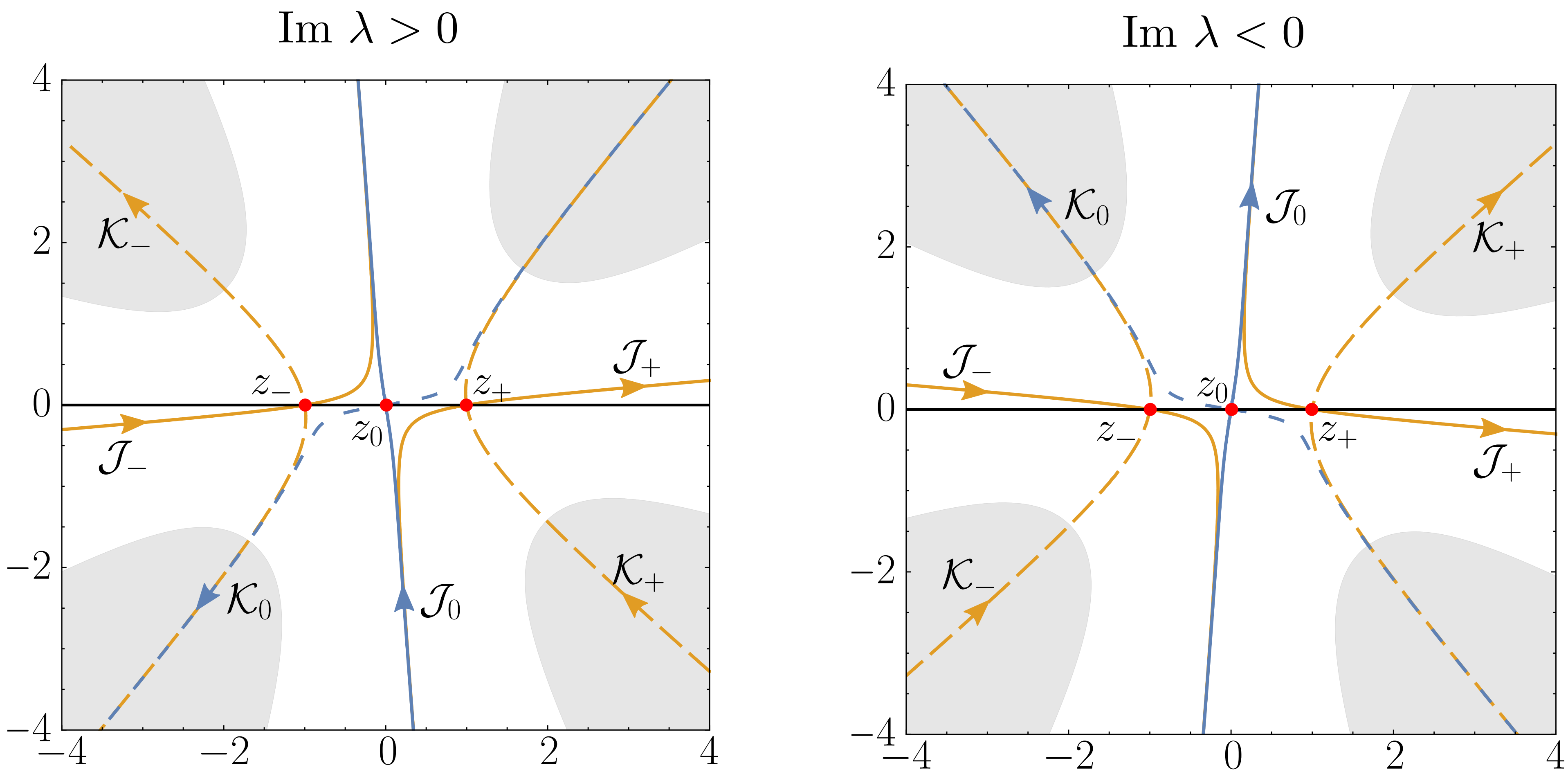
$$n_0 = 1,\ n_\pm = 0$$

Only saddle at the origin contributes to the integral.
No need to deform the original contour of integration (thimble itself)

No “non-perturbative” contributions

Example: ii) $f(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4\,, \qquad g(x) = 1$

Three saddles: $z_0 = 0,\ z_\pm = \pm 1$ Again degenerate situation.



$$\mathcal{C}_+ = \mathcal{J}_- - \mathcal{J}_0 + \mathcal{J}_+ \qquad \mathcal{C}_- = \mathcal{J}_- + \mathcal{J}_0 + \mathcal{J}_+$$

The integral is on a Stokes line and the intersection numbers depend on the deformation. Ending result of the integral will eventually be unambiguous
There are "non-perturbative" contributions

Borel Sums

$$Z_\sigma(\lambda) \equiv \frac{1}{\sqrt{\lambda}} \int_{\mathcal{J}_\sigma} dz\, g(z)\, e^{-f(z)/\lambda}$$

can be computed using a saddle-point expansion. Resulting series is asymptotic

$$Z(\lambda) - \sum_{n=0}^{N} Z_n \lambda^n = \mathcal{O}(\lambda^{N+1})\,, \quad \text{as} \quad \lambda \to 0$$

Borel transform to reconstruct function: assume $Z_n \sim n! a^n n^c$

$$\mathcal{B}_b Z(t) = \sum_{n=0}^{\infty} \frac{Z_n}{\Gamma(n+b+1)} t^n \qquad\qquad Z_B(\lambda) = \int_0^\infty dt\, e^{-t} t^b \mathcal{B}_b Z(t\lambda)$$

(Borel Le Roy function)

If no singularities for t>0 and under certain assumptions

$$\boxed{Z_B(\lambda) = Z(\lambda)}$$

Analytic structure of Borel function close to the origin can be determined by the large-order behaviour of the series coefficients

$$\text{for } Z_n \sim n! a^n,\ \mathcal{B}_0 Z(t) = \sum_n (at)^n \sim \frac{1}{1-at}$$

Singularity dangerous or not depending on the sign of a:

a>0 (same sign series) singularity for t>0 ☹️

a<0 (alternating series) singularity for t < 0 😀

Lateral Borel summation: move the contour off the real positive axis to avoid singularity for t>0.

Result is ambiguous and ambiguity signals presence of non-perturbative contributions to Z not captured by Z_B

Deformation to define the lateral Borel sum corresponds to the one we did to avoid Stokes line

$$Z_\sigma(\lambda) \equiv \frac{1}{\sqrt{\lambda}} \int_{\mathcal{J}_\sigma} dz\, g(z)\, e^{-f(z)/\lambda}$$ is Borel resummable to the exact result.

Proved by changing to variable $t = \dfrac{f(z) - f(z_\sigma)}{\lambda}$

Summarizing

$$Z(\lambda) \equiv \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{\infty} dx\, g(x)\, e^{-f(x)/\lambda} = \sum_\sigma n_\sigma Z_\sigma(\lambda)$$

Each $Z_\sigma(\lambda)$ is Borel resummable to the exact result

1 saddle contributes: $Z(\lambda)$ reconstructable from perturbation theory

More saddles contribute: $Z(\lambda)$ given by multi-series

(non-perturbative corrections)

Example first case: $f(x) = \frac{1}{2}x^2 + \frac{1}{4}x^4$

$$Z_n = \sqrt{2}(-)^n \frac{\Gamma(2n+\frac{1}{2})}{n!} \qquad \mathcal{B}_{-1/2}Z(t) = \sqrt{\frac{1+\sqrt{1+4t}}{1+4t}}$$

$$\int_0^\infty dt\, t^{-\frac{1}{2}} e^{-t} \mathcal{B}_{-1/2}Z(\lambda t) = \frac{1}{\sqrt{2\lambda}} e^{\frac{1}{8\lambda}} K_{\frac{1}{4}}\left(\frac{1}{8\lambda}\right)$$ Exact result

Example second case: $f(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$

Coefficients at minima: $Z_{\pm,n} = \frac{\Gamma\left(2n+\frac{1}{2}\right)}{n!}$ Coefficients at maximum: $iZ_{0,n}$

$$\mathcal{B}_{-1/2}Z_\pm(\lambda t) = \sqrt{\frac{1+\sqrt{1-4\lambda t}}{2(1-4\lambda t)}}$$

For real λ $\mathcal{B}_{-1/2}Z$ not Borel resummable

Recall we need $\mathrm{Im}\,\lambda \neq 0$ to avoid Stokes line

Thimble decomposition tells us how to perform lateral Borel sums and sum contributions:

$$\begin{array}{ll} \mathrm{Im}\,\lambda > 0 & Z_-(\lambda) - Z_0(\lambda) + Z_+(\lambda) \\ \mathrm{Im}\,\lambda < 0 & Z_-(\lambda) + Z_0(\lambda) + Z_+(\lambda) \end{array} = \frac{\pi}{\sqrt{4\lambda}} e^{\frac{1}{8\lambda}} \left[I_{-\frac{1}{4}}\left(\frac{1}{8\lambda}\right) + I_{\frac{1}{4}}\left(\frac{1}{8\lambda}\right)\right]$$ Exact result

Exact Perturbation Theory

When the decomposition in thimbles is non-trivial, the intersection numbers n_σ have to be determined

This is easy for one-dimensional integrals, but very complicated in the path integral case, where in general they will be infinite

Key idea: the thimble decomposition can be modified by a simple trick

The decomposition of the integral Z in terms of thimbles is governed by the function f(z) and not by g(z). Define

$$\hat{Z}(\lambda, \lambda_0) \equiv \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{\infty} dx\, e^{-\hat{f}(x)/\lambda} \hat{g}(x, \lambda_0)$$

$$\hat{f}(x) \equiv f(x) + \delta f(x) \,, \qquad \hat{g}(x, \lambda_0) \equiv g(x) e^{\delta f(x)/\lambda_0} \qquad \lim_{|x| \to \infty} \frac{\delta f(x)}{f(x)} = 0$$

By construction

$$\hat{Z}(\lambda, \lambda_0 = \lambda) = Z(\lambda)$$

Saddle-point expansion in λ at fixed λ_0:

δf in $\hat{f}$: "classical" deformation
δf in $\hat{g}$: "quantum" deformation"

Thimble decomposition is determined by $\hat{f}$

By a proper choice of δf, we can generally trivialize the thimble decomposition

$\Longrightarrow$ one real saddle and no need to deform the contour of integration

Series expansion of $\hat{Z}(\lambda, \lambda_0)$ in λ at fixed λ_0:

Exact Perturbation Theory (EPT)

Non-perturbative contributions of Z are all contained
in the perturbative expansion of $\hat{Z}$

Consider again $$f(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$$

and choose $\delta f(x) = x^2$ so that

$$\hat{f}(x) = \frac{1}{2}x^2 + \frac{1}{4}x^4\,, \qquad \hat{g}(x,\lambda_0) = \exp\Big(-\frac{x^2}{\lambda_0}\Big)$$

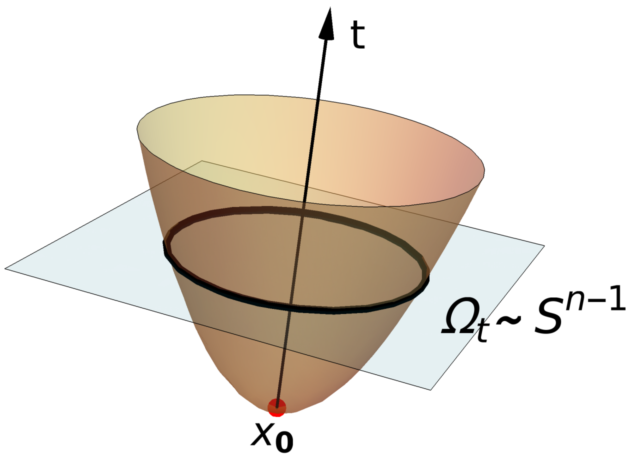
Only saddle at origin contributes. We have

$$\mathcal{B}_{-1/2}\hat{Z}(\lambda t,\lambda_0) = \sqrt{\frac{1+\sqrt{1+4\lambda t}}{1+4\lambda t}}\, e^{\frac{\sqrt{1+4\lambda t}-1}{\lambda_0}}$$

$$\int_0^\infty dt\, t^{-\frac{1}{2}} e^{-t} \mathcal{B}_{-1/2}\hat{Z}(\lambda t,\lambda) = \frac{\pi}{\sqrt{4\lambda}} e^{\frac{1}{8\lambda}} \left[I_{-\frac{1}{4}}\left(\frac{1}{8\lambda}\right) + I_{\frac{1}{4}}\left(\frac{1}{8\lambda}\right)\right]$$

Exact result is now completely perturbative (one series only)

Path Integral



These results can be generalized to higher-dimensional integrals and then to path integrals

Consider quantum mechanics where divergencies are all reabsorbed by the path integral measure (UV limit not expected to be problematic)

$$\mathcal{Z}(\lambda) = \int \mathcal{D}x(\tau)\, G[x(\tau)] e^{-S[x(\tau)]/\lambda}$$

$$S[x] = \int d\tau \left[\frac{1}{2}\dot{x}^2 + V(x)\right]$$

$V(x)$ analytic function of x

$V(x) \to \infty$ for $|x| \to \infty$

Discrete spectrum

Key result

If the action $S[x(\tau)]$ has only one real saddle point $x_0(\tau)$, the formal series expansion of $\mathcal{Z}(\lambda)$ around $\lambda = 0$ is Borel resummable to the exact result.

Remarks:

1. Number of saddles depends on boundary conditions and infinite time limit
2. Depending on the system, not all observables are Borel reconstructable

If V admits only one minimum, the above subtleties do not arise.

Like in the one-dimensional case, we can define an EPT as follows

Suppose $V = V_0 + \Delta V$ such that

1. V_0 has a single non-degenerate critical point (minimum);
2. $\lim_{|x|\to\infty} \Delta V/V_0 = 0\,.$

$$\hat{\mathcal{Z}}(\lambda,\lambda_0) = \int \mathcal{D}x\; G[x]\; e^{\frac{\int d\tau\, \Delta V}{\lambda_0}}\; e^{-\frac{S_0}{\lambda}}\,, \qquad S_0 \equiv \int d\tau \left[\frac{1}{2}\dot{x}^2 + V_0\right]$$

$$\hat{\mathcal{Z}}(\lambda,\lambda) = \mathcal{Z}(\lambda)$$

$\hat{\mathcal{Z}}$ is guaranteed to be Borel resummable to the exact answer

We have then proved the statement made at the beginning:

All observables in a quantum mechanical system with a bounded potential can be entirely computed from a single perturbative series (EPT)

(provided points 1. and 2. above apply)

No need of thimble decomposition and trans-series.

Even observables in systems known to have instanton corrections will be reconstructed by a single perturbative series

Large arbitrariness in the choice of EPT. In principle all choices equally good, although in numerical studies some choices better than others

We will denote by Standard Perturbation Theory (SPT) the usual perturbative computation (instantons included)

Time to show explicit results

Examples

Numerical analysis to test our results. We compute N orders using package

[Sulejmanpasic, Unsal, 1608.08256]

At fixed order N of perturbative terms, we use Padé approximants for the Borel function, that is then numerically integrated.

Results are compared with other methods (such as Rayleigh-Ritz) to get exact answers in QM

Symmetric double well

$$V = \frac{\lambda}{2}\left(x^2 - \frac{1}{4\lambda}\right)^2$$

Prototypical system where instanton occurs. Write

$$V_0 = \Big(\frac{1}{32\lambda} + \frac{\lambda_0}{2}x^2 + \frac{\lambda}{2}x^4\Big)\,, \qquad \Delta V = -\Big(\lambda_0 + \frac{1}{2}\Big)\frac{x^2}{2}$$

Already at small coupling, EPT is able to resolve the instanton contribution to the ground state energy.

At $\lambda = \lambda_0 = 1/25,\ N = 200$

$$\frac{\Delta E_0}{E_0} \approx 10^{-8} \qquad\qquad \frac{\Delta E_1}{E_1} \approx 4\cdot 10^{-14}$$

$$E_0^{\rm inst} \approx \frac{2}{\sqrt{\pi\lambda}}e^{-\frac{1}{6\lambda}} \approx 0.087$$

At larger values of the coupling, instanton computation breaks down

EPT works better and better

No need of many perturbative terms.

With just 2 (using a conformal mapping method) at $\lambda = 1$ we get

$$\frac{\Delta E_0}{E_0} \simeq 3\%$$

We have also computed higher energy states and eigenfunctions. In all cases EPT give excellent agreement with other methods. No signal of missing non-perturbative contributions.

SUSY double well

$$V(x;\lambda) = \frac{\lambda}{2}\left(x^2 - \frac{1}{4\lambda}\right)^2 + \sqrt{\lambda}x$$

Potential one obtains integrating out fermions in SUSY QM

$E_0 = 0$ to all orders in SPT due to SUSY

Yet, $E_0 \neq 0$ by instanton effects that dynamically break supersymmetry

We can approach this system in three ways:

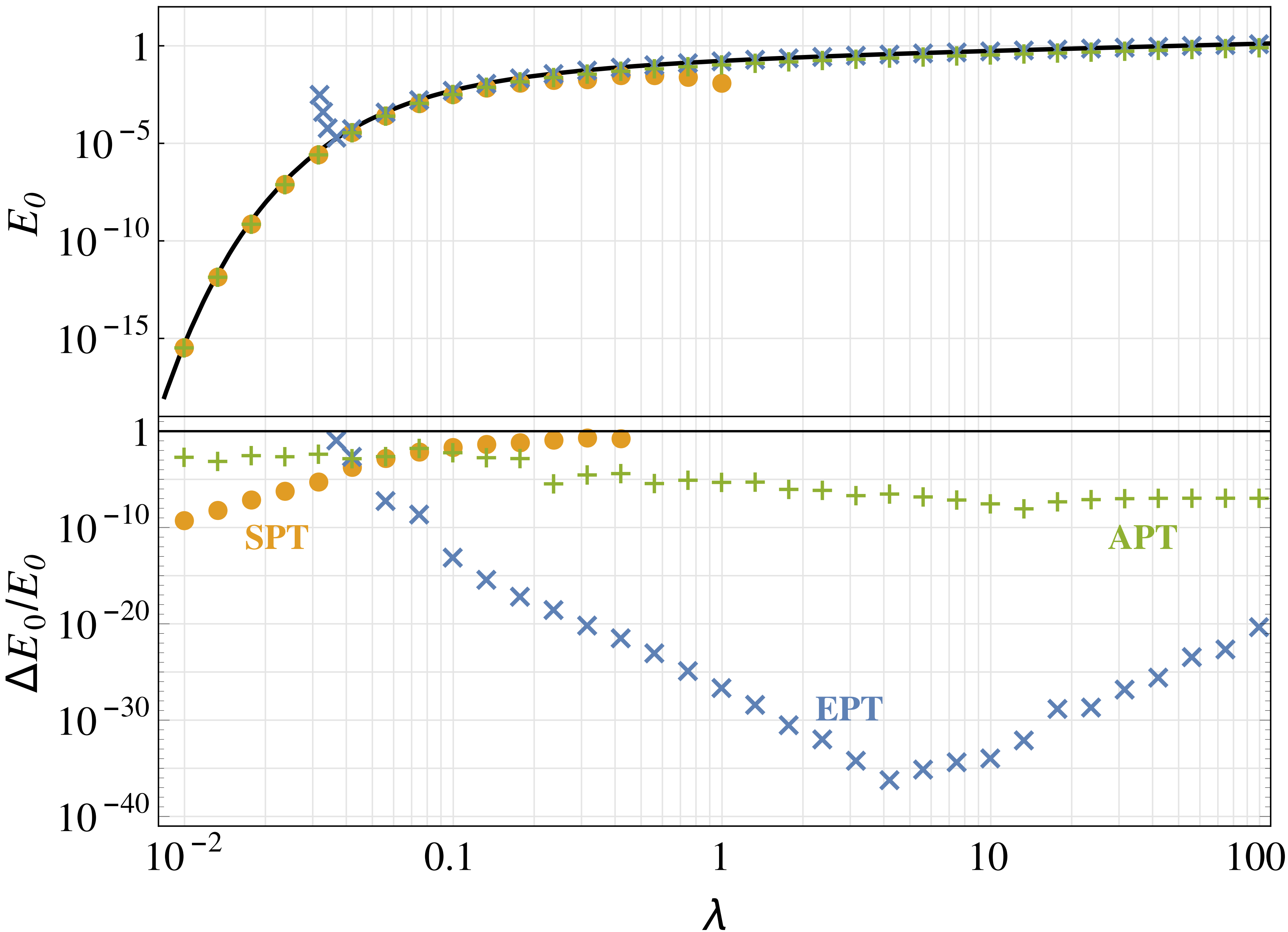
1. ordinary instanton methods, SPT

[Jentschura, Zinn-Justin, hep- ph/0405279]

2. Turn the quantum tilt into classical. Alternative perturbation theory (APT)

$$V_{\mathrm{APT}} = \frac{\lambda}{2}\left(x^2 - \frac{1}{4\lambda}\right)^2 + \frac{\lambda_0}{\sqrt{\lambda}}x$$

3. EPT $\quad V_0 = \left(\frac{1}{32\lambda} + \frac{\lambda_0}{2}x^2 + \frac{\lambda}{2}x^4\right), \qquad \Delta V = \frac{\lambda_0}{\sqrt{\lambda}}x - \left(\lambda_0 + \frac{1}{2}\right)\frac{x^2}{2}$


E_0
1
10^{-5}
10^{-10}
10^{-15}
$\Delta E_0/E_0$
1
10^{-10}
10^{-20}
10^{-30}
10^{-40}
SPT
APT
EPT
10^{-2}
0.1
1
10
100
λ

N=200

Pure anharmonic potentials:

$$H = p^2 + x^{2l}$$

Usual perturbation theory breaks down

k	$E_k^{(4)}$	$\Delta E_k^{(4)}/E_k^{(4)}$	$E_k^{(6)}$	$\Delta E_k^{(6)}/E_k^{(6)}$
0	1.0603620904	$3 \cdot 10^{-45}$	0.5724012268	$2 \cdot 10^{-19}$
1	3.7996730298	$2 \cdot 10^{-44}$	2.1692993557	$3 \cdot 10^{-19}$
2	7.4556979379	$9 \cdot 10^{-37}$	4.5365422804	$9 \cdot 10^{-17}$
3	11.6447455113	$4 \cdot 10^{-36}$	7.4675848174	$7 \cdot 10^{-16}$
4	16.2618260188	$4 \cdot 10^{-36}$	10.8570827110	$2 \cdot 10^{-16}$

Table 2: Energy eigenvalues $E_k^{(2\ell)}$ and the corresponding accuracies $\Delta E_k^{(2\ell)}/E_k^{(2\ell)}$ of the first five levels of the pure anharmonic x^4 and x^6 potentials, computed using EPT with $N = 200$. Only the first ten digits after the comma are shown (no rounding on the last digit).

Conclusions and future perspectives

We have introduced a new perturbation theory in quantum mechanics (EPT) that allows us to compute observables perturbatively

No need of trans-series and to worry for possibly ambiguous results (possible addressed by resurgence or by thimble decomposition)

Is it possible to get the same results directly in Minkowski space?

Generalize to higher dimensional QM systems

What about QFT?

Is renormalization an issue? UV problem

Infinite volume limit, spontaneous symmetry breaking? IR problem

Borel summability of $\phi^4_{2,3}$ theories suggest a promising development in QFT and is the next thing to do

A new approach to strongly coupled physics has begun …

Thank You

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