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Wilson lines and webs in higher order QCD

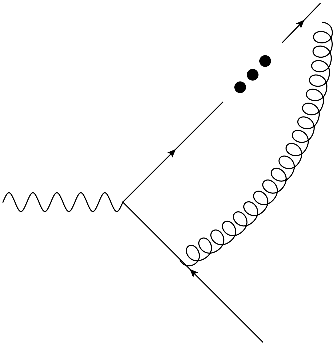
Light Cone 2017, Mumbai

Overview

- Introduction to infrared singularities.
- Webs in QCD for multiparticle scattering.
- Calculating webs: the bootstrap approach.
- Outlook.

Infrared divergences

- In scattering amplitudes, get singularities due to gluon emission at large distances.



- Due to integrals over gluon positions:

$$\int d^n x$$

- Uncertainty principle $\Rightarrow$ equivalent to emission of zero energy gluons.

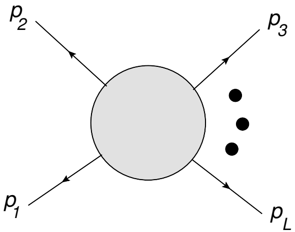
- Common to abelian / non-abelian gauge theories, including gravity.

Why study IR singularities?

- Singularities cancel for suitably inclusive observables, once real and virtual diagrams are combined (Block, Nordsieck).
- However, large kinematic contributions remain, which need to be resummed to all orders.
- Also more formal applications of IR singularities e.g. all-order insights in (S)YM theory.
- IR singularities related to Wilson lines, which occur in many contexts (e.g. TMDs).

Soft-collinear factorisation

- The general schematic form of an amplitude is (Mueller, Collins, Sen, Korchemsky, Magnea, Sterman):



$$\mathcal{A} = \mathcal{H} \cdot \mathcal{S} \cdot \prod_{i=1}^{L} \frac{J_i}{\mathcal{J}_i}.$$

- Soft function is a VEV of Wilson lines:

$$\mathcal{S} = \langle 0 | \Phi_1 \ldots \Phi_n | 0 \rangle \,,$$

where

$$\Phi_i = \mathcal{P} \exp \left[i \mathbf{T}^a \int dx^\mu \, A_\mu \right] .$$

Exponentiation

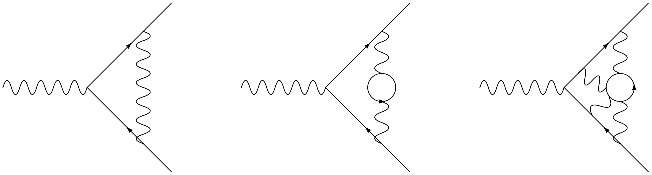
- Calculating IR singularities then amounts to calculating Feynman diagrams for multiple Wilson lines meeting at a point.
- Furthermore, one may show that the soft function has an exponential form. Schematically:

$$\mathcal{S} \sim \exp\left[\sum_W W\right].$$

- Here W are certain special diagrams called *webs*.
- Precisely what these look like depends on the theory.

Webs in QED

- In QED, one may show that the exponent of the soft function contains only connected subdiagrams ("QED webs"):



- Originally derived using combinatoric methods (Yennie, Frautschi, Suura), and recently rederived using path integral methods (Laenen, Stavenga, White).
- An example is shown for two Wilson lines, but the definition of a QED web generalises to the multiline case.

Webs in QED - Comments

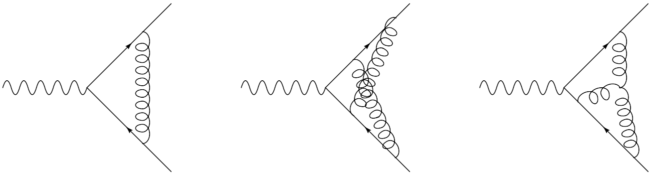
- Exponentiation implies that IR singularities get summed up to all orders in perturbation theory.
- We need only calculate a subset of diagrams at each order.
- Note also that any large logs in perturbation theory which are related to IR singularities will also be summed up to all orders in perturbation theory ("resummation").
- In QED with no propagating fermions, the exponent is *one-loop exact*.

Webs in QCD

- Webs in QED are relatively simple (connected subdiagrams).
- In QCD, things are complicated, as emission vertices for soft gluons carry non-commuting colour matrices.
- One may still show that the soft function exponentiates, albeit with a more complicated form.
- Furthermore, there is a distinction between the case of two Wilson lines meeting at a point, and more than two.
- Let's look at the former case first...

Webs in QCD

- For two line processes, webs can be classified as the set of *irreducible diagrams* (Gatheral, Frenkel, Taylor, Sterman).



- Includes the diagrams from the QED case, but also additional (non-connected) subgraphs.
- Diagrams have modified colour weights, so that the soft function has the form

$$\mathcal{S} \sim \exp\left\{\sum_W \tilde{C}(W)\mathcal{F}(W)\right\},$$

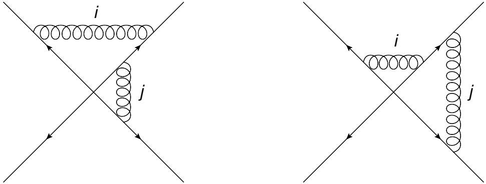
with $\tilde{C}(W)$ and $\mathcal{F}(W)$ the colour / kinematic parts of W.

Webs in QCD - Comments

- More complicated than abelian case, but IR singularities still predicted to all orders in perturbation theory, from a subset of diagrams.
- We can think of the *exponentiated colour factors* $\tilde{C}$ as picking out which diagrams contribute.
- For two Wilson lines, webs are single irreducible diagrams.
- The full multiline case has only recently been considered (Mitov, Sterman, Sung; Gardi, Laenen, Stavenga, White; Vladimirov).
- Results indicate that, in general, webs are *closed sets* of diagrams, which can in fact be *reducible*.

Multiparton webs in QCD

- Consider the following two diagrams:



(a) (b)

related by gluon permutations.

- Each of these has a kinematic factor $\mathcal{F}(D)$ ($D = a, b$) and a colour factor $C(D)$.
- The contribution to the exponent of the soft function turns out to be

$$\begin{pmatrix} \mathcal{F}(a) \\ \mathcal{F}(b) \end{pmatrix}^T \begin{pmatrix} \tilde{C}(a) \\ \tilde{C}(b) \end{pmatrix} = \begin{pmatrix} \mathcal{F}(a) \\ \mathcal{F}(b) \end{pmatrix}^T \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} C(a) \\ C(b) \end{pmatrix}.$$

Multiparton webs in QCD

- The set of diagrams mixes in the exponent.
- Colour and kinematic information is entangled in a non-trivial way.
- It thus makes sense to consider the set of two diagrams as a *single web*.
- Associated with the web is a *web mixing matrix*, that tells us how the kinematic and colour information gets mixed up.
- The interpretation of multiparton webs as sets of diagrams is backed up by further study of renormalisation properties (Gardi, Smillie, White).

Multiparton webs - General structure

- In general, webs are closed sets of diagrams, related by permutations of gluons on the external lines.
- The contribution of each set to the exponent of the soft function is

$$\sum_{D,D'} \mathcal{F}_D R_{DD'} C_{D'},$$

  where $R_{DD'}$ is a web-mixing matrix.
- The study of webs (and thus IR singularities) in multiparton scattering is equivalent to the study of these matrices.
- They encode a huge amount of physics!
- They also have interesting properties.

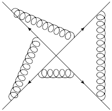
Web mixing matrices

- We observe the following interesting properties:
  1. Rows of web mixing matrices sum to zero i.e.

$$\sum_{D'} R_{DD'} = 0.$$

  2. The matrices are idempotent i.e. $R^2 = R$. This implies they have eigenvalues 0 and 1.
- We are starting to understand the physics of these results.
- Proofs use statistical physics methods ("the replica trick"), and enumerative combinatorics (Gardi, White).
- Combinatorics related to partially ordered sets (*posets*): Dukes, Gardi, McAslan, Scott, Steingrimsson, White.

A four loop example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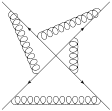
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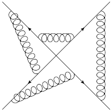
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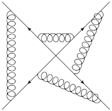
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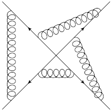
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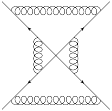
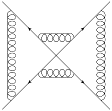

[[1,2],[1,3],[4,3],[4,2]]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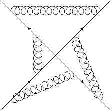
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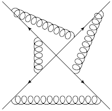
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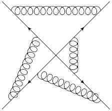
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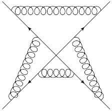
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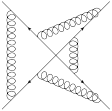
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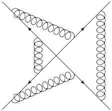
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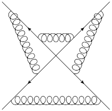
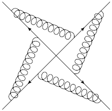

[[1,2],[3,2],[4,3],[4,1]]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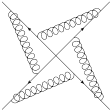

[[1,2],[2,3],[3,4],[4,1]]

Four loop mixing matrix ($\times 24$)

$$\begin{pmatrix}
6 & -6 & 2 & 2 & -2 & 4 & -4 & 2 & -2 & -2 & -4 & 4 & -4 & 4 & 0 & 0 \\
-6 & 6 & -2 & -2 & 2 & -4 & 4 & -2 & 2 & 2 & 4 & -4 & 4 & -4 & 0 & 0 \\
2 & -2 & 6 & -2 & 2 & 4 & -4 & -2 & 2 & -6 & 4 & 4 & -4 & -4 & 0 & 0 \\
2 & -2 & -2 & 6 & 2 & 4 & -4 & -2 & -6 & 2 & -4 & -4 & 4 & 4 & 0 & 0 \\
-2 & 2 & 2 & 2 & 6 & 4 & -4 & -6 & -2 & -2 & 4 & -4 & 4 & -4 & 0 & 0 \\
2 & -2 & 2 & 2 & 2 & 4 & -4 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 2 & -2 & -2 & -2 & -4 & 4 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & -2 & -2 & -6 & -4 & 4 & 6 & 2 & 2 & -4 & 4 & -4 & 4 & 0 & 0 \\
-2 & 2 & 2 & -6 & -2 & -4 & 4 & 2 & 6 & -2 & 4 & 4 & -4 & -4 & 0 & 0 \\
-2 & 2 & -6 & 2 & -2 & -4 & 4 & 2 & -2 & 6 & -4 & -4 & 4 & 4 & 0 & 0 \\
-2 & 2 & 2 & -2 & 2 & 0 & 0 & -2 & 2 & -2 & 4 & 0 & 0 & -4 & 0 & 0 \\
2 & -2 & 2 & -2 & -2 & 0 & 0 & 2 & 2 & -2 & 0 & 4 & -4 & 0 & 0 & 0 \\
-2 & 2 & -2 & 2 & 2 & 0 & 0 & -2 & -2 & 2 & 0 & -4 & 4 & 0 & 0 & 0 \\
2 & -2 & -2 & 2 & -2 & 0 & 0 & 2 & -2 & 2 & -4 & 0 & 0 & 4 & 0 & 0 \\
-18 & -6 & -6 & -6 & -18 & 12 & 12 & -6 & -18 & -18 & 12 & 12 & 12 & 12 & 24 & 0 \\
-6 & -18 & -18 & -18 & -6 & 12 & 12 & -18 & -6 & -6 & 12 & 12 & 12 & 12 & 0 & 24
\end{pmatrix}$$

Connected colour factors

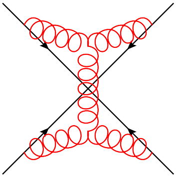
- Web mixing matrices imply that only certain combinations of web diagrams survive in the exponent.
- Each is associated with a specific combination of colour factors.
- We now know that these colour factors correspond to those of *connected* soft gluon diagrams (Gardi, Smillie, White).
- Generalises a similar result (the non-Abelian exponentiation theorem) known for the two-line case (Gatheral, Frenkel, Taylor).

Calculating web diagrams

- Previously, IR singularities have been known to two-loop order in the exponent (Sterman, Aybat, Dixon, Kidonakis, Mitov, Sung, Becher, Neubert, Beneke, Falgari, Schwinn, Ferroglia, Pecjak, Yang).
- The web language has allowed us to extend this to three-loop order (Gardi, Laenen, Stavenga, Smillie, White, Almelid, Duhr).
- See also calculations by Korchemsky, Henn, Huber, Grozin, Marquard, Correa, Maldacena, Sever.
- The key quantity to calculate is the *soft anomalous dimension* Γ_S, which controls the UV singularities of Wilson line products.
- It is this quantity that enters resummation formulae etc.

Soft anomalous dimension for massless particles

- Recently, Γ_S was calculated for the special case of Wilson lines corresponding to massless particles (Almelid, Duhr, Gardi).



- Most complicated diagram took several years to complete.
- Final result has a very simple form, if expressed in the right way!

- The m-loop massless soft anomalous dimension for n particles can be written (Becher, Neubert; Gardi, Magnea)

$$\Gamma_S = \Gamma_S^{\mathrm{dip.}} + \Delta_n^{(m)},$$

where the first term depends only on pairs of particles.
- The correction term starts at three-loop order.

The correction function $\Delta_n^{(m)}$

- If β_i be the 4-velocity of the $i^{\rm th}$ Wilson line, and $\mathbf{T}_i$ a colour generator on line i, then

$$\Delta_n^{(3)}\left(\{\rho_{ijkl}\},\{\mathbf{T}_i\}\right) = 16\, f_{abe} f_{cde} \Big\{ -C \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n \\ j,k\neq i}} \left\{\mathbf{T}_i^a, \mathbf{T}_i^d\right\} \mathbf{T}_j^b \mathbf{T}_k^c$$

$$+ \sum_{1\le i<j<k<l\le n} \Big[\mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d\, \mathcal{F}(\rho_{ikjl}, \rho_{iljk}) + \mathbf{T}_i^a \mathbf{T}_k^b \mathbf{T}_j^c \mathbf{T}_l^d\, \mathcal{F}(\rho_{ijkl}, \rho_{ilkj})$$

$$+ \mathbf{T}_i^a \mathbf{T}_l^b \mathbf{T}_j^c \mathbf{T}_k^d\, \mathcal{F}(\rho_{ijlk}, \rho_{iklj}) \Big] \Big\},$$

where $C = \zeta_5 + 2\zeta_2\zeta_3$, and $\mathcal{F}$ is a function of *conformally invariant cross-ratios*

$$\rho_{ijkl} = \frac{(\beta_i\cdot\beta_j)(\beta_k\cdot\beta_l)}{(\beta_i\cdot\beta_k)(\beta_j\cdot\beta_l)}.$$

The function $\mathcal{F}$

- The non-trivial kinematic dependence is simplified by introducing the (in general complex) variables

$$z_{ijkl}\bar{z}_{ijkl} = \rho_{ijkl}, \quad (1 - z_{ijkl})(1 - \bar{z}_{ijkl}) = \rho_{ilkj}.$$

- Then one has

$$\mathcal{F}(\rho_{ijkl}, \rho_{ilkj}) = F(1 - z_{ijkl}) - F(z_{ijkl}),$$

where

$$F(z) = \mathcal{L}_{10101}(z) + 2\zeta_2[\mathcal{L}_{001}(z) + \mathcal{L}_{100}(z)],$$

and $\mathcal{L}_w(z)$ is a *single-valued harmonic polylogarithm* (SVHPL), introduced by Brown.

The bootstrap approach

- The simplicity of the three-loop soft anomalous dimension suggests an alternative way to calculate it (Almelid, Duhr, Gardi, McLeod, White).
- By mapping Wilson lines to the Riemann sphere, one can show that the correction function $\Delta_n^{(3)}$ can only depend on SVHPLs.
- One can then write a general ansatz for C and $F(z)$, and constrain the coefficients using:
  1. Bose symmetry.
  2. Colour conservation.
  3. Uniform transcendental weight.
  4. Collinear limits (Dixon, Gardi, Magnea; Almelid, Duhr, Gardi).
  5. Regge limits (Caron-Huot, Gardi, Vernazza).
- $\Delta_n^{(3)}$ is then fixed up to an overall constant!
- This is called a *bootstrap approach*. Previously used in $\mathcal{N} = 4$ SYM theory, but works for QCD too!

Conclusion

- Infrared singularities are important for both `hep-ph` and `hep-th` reasons.
- Much recent progress in calculating the *soft anomalous dimension* that controls them.
- Webs provide a highly efficient language for higher order calculations in QCD and related theories.
- New powerful techniques (e.g. bootstrap approach) greatly simplify calculations at three loops and beyond!

Open Problems

- What is the general structure of web mixing matrices?
- Are their combinatoric properties useful for something else?
- Can we calculate all 3-loop Wilson line diagrams for massive particles?
- Can we use the bootstrap at four loops and beyond?
- Can we use these results to extend resummation?
- Can we use webs beyond the soft approximation?
- Are webs useful for TMDs, GPDs, or other LC2017-friendly subjects?