Anosov-Kolmogorov CK systems and MIXMAX PRNG for MC simulations

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Integration of the MIXMAX engine into the CERN Scientific Software ROOT and Geant4

8 May 2017
CERN

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On the Monte Carlo simulation of physical systems
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2.K.Savvidy, The MIXMAX random number generator
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5.G. Savvidy and K. Savvidy,
Hyperbolic Anosov C-systems.
Exponential Decay of Correlation Functions,
arXiv:1702.03574 [math-ph]

- HEPFORGE.ORG, `http://mixmax.hepforge.org`;
  by Konstantin Savvidy
  http://www.inp.demokritos.gr/ savvidy/mixmax.php

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- ROOT, Release 6.04/06 on 2015-10-13,
  by Lorenzo Moneta
  https://root.cern/doc/master/classROOT_1_1Math_1_1MixMaxEngine.html

- HEPFORGE.ORG, `http://mixmax.hepforge.org`;
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- ROOT, Release 6.04/06 on 2015-10-13,
  by Lorenzo Moneta
  `https://root.cern/doc/master/classROOT_1_1Math_1_1MixMaxEngine.html`

- CLHEP, Release 2.3.1.1, on 2015-11-10,
  by Gabriele Cosmo
  `http://proj-clhep.web.cern.ch/proj-clhep/`

MIXMAX core developed by Konstantin Savvidy
http://mixmax.hepforge.org

$$mixmax\ realese\ 110 \qquad speed = 4 \cdot 10^{-9} second$$

$$mixmax.c$$

$$mixmax.h$$

$$driver\ main.c$$

$$driver\ verification.c$$

$$mixmax\ skip\ N240.c$$

$$mixmax\ skip\ N17.c$$

$$- - - - - > README.pdf$$

$$mixmax\ gsl\ driver.c$$

$$driver\ testU01.c$$

Size N	Magic m	Magic s	Entropy	Period $\approx \log_{10}(q)$
8	$m = 2^{53} + 1$	s=0	220.4	129
17	$m = 2^{36} + 1$	s=0	374.3	294
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240	$m = 2^{51} + 1$	s=487013230256099140	8418.8	4389

Table : Table of three-parameter MIXMAX generators A(N,s,m). These generators have an advantage of having a very high quality sequence for moderate and small N. In particular, the smallest generator we tested, $N = 8$, passes all tests in the BigCrush suite.

Size N	Magic s	Entropy (lower bound)	Period $\approx \log_{10}(q)$
7307	0	4502.1	134158
20693	0	12749.5	379963
25087	0	15456.9	460649
28883	1	17795.7	530355
40045	-3	24673.0	735321
44851	-3	27634.1	823572

Table : Table of properties of the operator $A(N,s)$ for large matrix size N. The third column is the value of the Kolmogorov entropy. All these generators passes tests in the BigCrush suite. For the largest of them the period approaches a *million digits*.

Dmitri Anosov, in his fundamental work on *hyperbolic dynamical C-systems* pointed out that the basic property of the geodesic flow on closed Riemannian manifolds V^n of negative curvature is a *uniform instability of all its trajectories*.

In physical terms that means that *in the neighbourhood of every fixed trajectory the trajectories behave similarly to the trajectories in the neighbourhood of a saddle point*.

The hyperbolic instability of the dynamical system $\{T^t\}$ which is defined by the equations $(w \in W^m)$

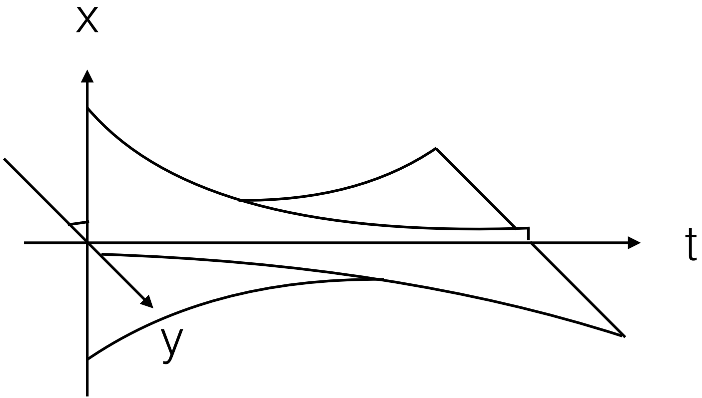
$$\dot{w} = f(w) \tag{1}$$

takes place for all solutions $\delta w \equiv \omega$ of the deviation equation

$$\dot{\omega} = \frac{\partial f}{\partial w}\bigg|_{w(t)=T^t w} \omega \tag{2}$$

in the neighbourhood of each phase trajectory $w(t) = T^t w$.

The Anosov C-systems are genuine hyperbolic systems



the behaviour of all nearby trajectories is exponentially unstable

The contracting and expanding foliations Σ_w^k and Σ_w^l

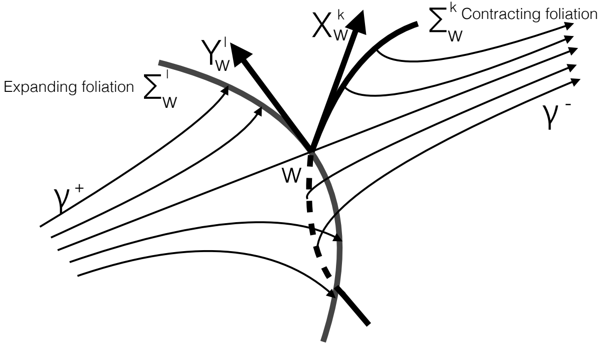


Figure : At each point w of the C-system the tangent space R_w^m is decomposable into a direct sum of two linear spaces Y_w^l and X_w^k. The expanding and contracting geodesic flows are γ^+ and γ^-. The expanding and contracting invariant foliations Σ_w^l and Σ_w^k *are transversal to the geodesic flows* and their corresponding tangent spaces are Y_w^l and X_w^k.

Important Example of C-system: *Torus Automorphisms*
Consider linear automorphisms of the unit hypercube in Euclidean space R^N with coordinates $(u_1, ..., u_N)$ where $u \in [0, 1)$

$$u_i^{(k+1)} = \sum_{j=1}^{N} A_{ij}\, u_j^{(k)}, \quad mod\ 1, \quad k = 0, 1, 2...... \tag{3}$$

- The dynamical system defined by the integer matrix A has determinant equal to one $DetA = 1$.

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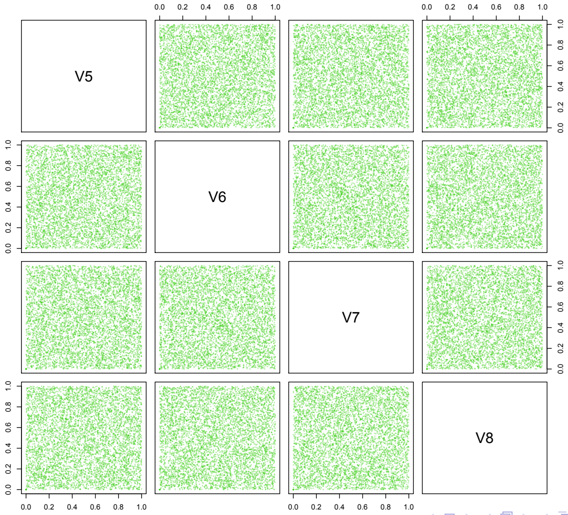
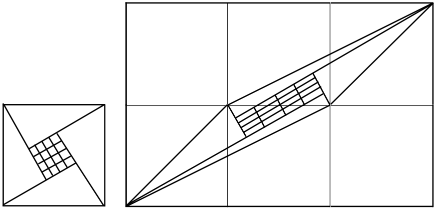
- The dynamical system defined by the integer matrix A has determinant equal to one $Det A = 1$.
- The Anosov hyperbolicity C-condition: the matrix A has no eigenvalues on the unit circle. Thus the spectrum $\Lambda = \lambda_1, ..., \lambda_N$ fulfils the two conditions:

$$1)\ Det A = \lambda_1 \lambda_2 .... \lambda_N = 1, \qquad 2)\ \ |\lambda_i| \neq 1. \qquad (4)$$

Plot of the coordinate pairs (v_i, v_j), $i \neq j$





a) b)

- The Kolmogorov entropy of a Anosov C-system is:

$$h(A) = \sum_{|\lambda_\beta|>1} \ln |\lambda_\beta|. \tag{5}$$

The entropy h(A) depends on the spectrum of the operator A.

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- *This allows to characterise and compare the chaotic properties of dynamical C-systems quantitatively →*

  computing and comparing their entropies.

- The decorrelation time τ_0 of the dynamical system A can be expressed in terms of its entropy

$$\tau_0 = 1/h(A) \tag{6}$$

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- These important characteristic time scales should fulfil the following fundamental relation

$$\tau_0 \leq T \leq \tau \tag{8}$$

where $T = 1$ is a time of one iteration.

Thus there are three characteristic time scales associated with the C-system:

$$\begin{pmatrix} Decorrelation \\ time \\ \tau_0 = \frac{\pi}{4pN^2} \end{pmatrix} < \begin{pmatrix} Interaction \\ time \\ t_{int} = n = 1 \end{pmatrix} < \begin{pmatrix} Stationary \\ distribution\ time \\ \tau = \frac{1}{h(T)} \ln \frac{1}{\delta v_0} \end{pmatrix}.$$

The generator $N = 256$ has the entropy $h(T) = 194$, therefore the characteristic time scales for this generator are

$$\begin{pmatrix} Decorrelation \\ time \\ \tau_0 = 0.000012 \end{pmatrix} < \begin{pmatrix} Interaction \\ time \\ t_{int} = 1 \end{pmatrix} < \begin{pmatrix} Stationary \\ distribution\ time \\ \tau = 95 \end{pmatrix}.$$

The MIXMAX generator $N = 240$ has the entropy $h(T) = 8679$, therefore the characteristic time scales for this generator are

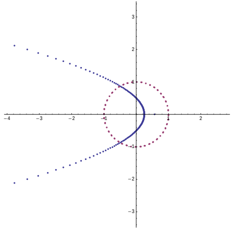
$$\begin{pmatrix} Decorrelation \\ time \\ \tau_0 = 0.000004 \end{pmatrix} < \begin{pmatrix} Interaction \\ time \\ t_{int} = 1 \end{pmatrix} < \begin{pmatrix} Stationary \\ distribution\ time \\ \tau = 1.17 \end{pmatrix}.$$

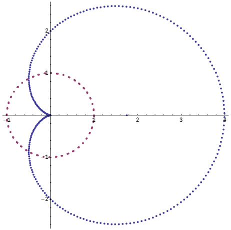
Both generators have very short decorrelation time. The second generator $N = 240$ has much bigger entropy and therefore its relaxation time τ is much smaller, of order 1.17, and is close to the interaction time.

Family of operators A(N,s) parametrised by the integers N and s

$$A(N,s) = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & 1 & 1 & \dots & 1 & 1 \\ 1 & 3+s & 2 & 1 & \dots & 1 & 1 \\ 1 & 4 & 3 & 2 & \dots & 1 & 1 \\ & & & \dots & & & \\ 1 & N & N-1 & N-2 & \dots & 3 & 2 \end{pmatrix} \qquad (9)$$

The matrix is of the size $N \times N$
Its entries are all integers $A_{ij} \in \mathbb{Z}$
Det A $=1$
The spectrum and the value of the Kolmogorov entropy?





Eigenvalue Distribution of A(N,s) and of $A^{-1}(N,s)$
all of them are lying outside of the unit circle

A(N,s,m)

A three-parameter family of C-operators $A(N, s, m)$, where m is some integer:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & 1 & 1 & \dots & 1 & 1 \\ 1 & m+2+s & 2 & 1 & \dots & 1 & 1 \\ 1 & 2m+2 & m+2 & 2 & \dots & 1 & 1 \\ 1 & 3m+2 & 2m+2 & m+2 & \dots & 1 & 1 \\ & & & \dots & & & \\ 1 & (N-2)m+2 & (N-3)m+2 & (N-4)m+2 & \dots & m+2 & 2 \end{pmatrix}$$

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Conclusion

Use MIXMAX for your Monte-Carlo simulations !

it will provide a fast convergence!

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Thank you!