Field Theory & EW Standard Model

Lecture I

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Outline

Lecture 1: Introduction and QFT

- What is the Standard Model?
- Particle (field) content of the SM
- Principles of the SM
- Brief notes on Quantum Field Theory

Lecture 2: Construction of the SM

Lecture 3: Phenomenology of the SM

What is the Standard Model?

The SM is

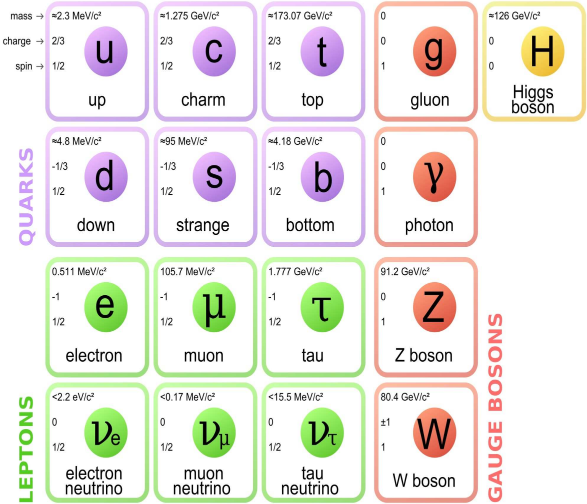
- — the most successful physical model ever
- — constructed within the Quantum Field Theory (QFT)
- — based on symmetry principles
- — minimal
- — a model with an enormous predictive power

But we do not understand why it works so well…

Questions to the SM:

- Is the SM a fundamental theory?
- If not, where is the limit of its applicability?
- Are the fields and interactions of the SM fundamental?
- Can it be joined with gravity?
- Does it give hints where to look for anything beyond it?

Particle (field) content of the SM (I)



Courtesy to Wikipedia: "Standard Model of Elementary Particles" by MissMJ - Own work by uploader, PBS NOVA [1], Fermilab, Office of Science, United States Department of Energy, Particle Data Group.

Particle (field) content of the SM (fermions)

So we have (i.e. observe)
fermions (spin = 1/2) and bosons (with spin = 0 or 1)

Fermions are of two types: leptons and quarks. They are:

— 3 charged leptons (e, μ, τ);

— 3 neutrinos ν_e, ν_μ, ν_τ (or ν_1, ν_2, ν_3, see lect. by S. Davidson)

— 6 quarks of different flavors, see lect. by M. Beneke;

Each quark can have one of three colors, see lect. by K. Melnikov;

Each fermion has 2 degrees of freedom, e.g. can be left or right

Each fermion particle has an anti-particle, $f \neq \bar{f}$

N.B.1. The later was not yet verified for neutrinos

N.B.2. Traditionally fermions are called *matter fields*

Particle (field) content of the SM (bosons)

In the SM we have a few boson fields:

— 8 vector (spin=1) gluons

— 4 vector (spin=1) electroweak bosons: γ, Z, W^+, W^-

— 1 scalar (spin=0) Higgs boson

Gluons and photon are massless* and have 2 degrees of freedom (polarizations)

Each gluon has one color and one anti-color

Z and W bosons are massive** and have 3 degrees of freedom (polarizations)

N.B.1. Gluons and EW bosons are gauge bosons which transmit*** interactions between matter fields

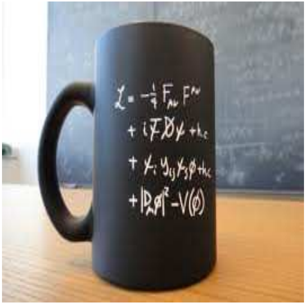
N.B.2. Electrically neutral bosons (H, γ, Z, and gluons) coincide with their anti-particles, e.g. $\gamma \equiv \bar{\gamma}$

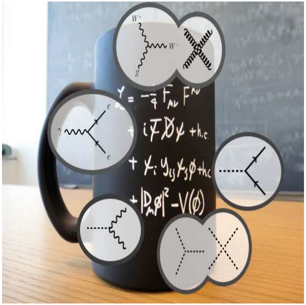
Interactions in the SM (I)

How many fundamental interactions are there in Nature?

How many interactions are there in the Standard Model?

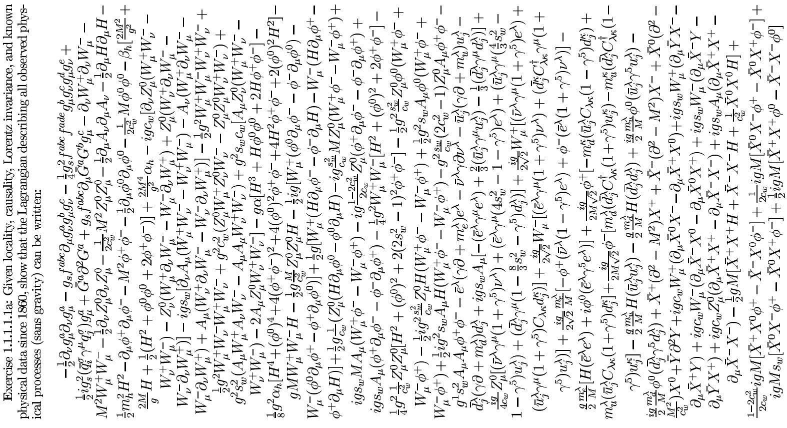
To answer the last question we have to look at the SM Lagrangian.





Interactions in the SM (II)

The complete SM Lagrangian look quite long and cumbersome:



$+\ldots$

Our task is to derive the long expression and realize that it is nothing else but the short one. Question: But why can it be so?

Principles and symmetries of the SM

Principles (keep in mind SM $\subset$ QFT):

- correspondence: to QM, QED, Fermi model etc.
- minimality: only observed and/or unavoidable objects
- unitarity: $0 \leq P \leq 1$ and $P(\Omega) = 1$
- renormalizability: finite predictions for observables
- gauge interactions between fermions and vector fields
- SYMMETRIES

Symmetries:

- the Lorentz symmetry
- the CPT symmetry
- the gauge symmetries: $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$
- Global symm. in the Higgs sector (spontaneously broken)
- unrevealed symmetries (between generations, for anomaly cancellation, conformal, etc.)

Elements of Quantum Field Theory

Assume that we remember the basics of Quantum Mechanics. But QFT can be constructed on its own. We just have to check the correspondence.

Let us fix the notation: $\boxed{\hbar = 1 \text{ and } c = 1}$

$\mu = \{0, 1, 2, 3\}$ is a Lorentz index (Greek letters)
p_μ is a four-momentum, $\mathbf{p} = (p_1, p_2, p_3)$ is a three-momentum
$pq = p_\mu q_\mu = p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3$ is a scalar product of two vectors, which is a relativistic invariant

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad g_{\mu\nu} p_\nu = p_\mu, \quad g_{\mu\mu} = 4$$

$$\frac{\partial}{\partial x_\mu} = \partial_\mu = (\partial_0, -\partial_1, -\partial_2, -\partial_3)$$

$x_0 = t$ is time, $p_0 = E$ is energy
$p^2 = pp = p_0^2 - \mathbf{p}^2 = E^2 - \mathbf{p}^2 = m^2$ is the on-mass-shell condition (valid for any free particle)

QFT: scalar field

Postulate a scalar quantum field as

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2p_0}} \left(e^{-ipx} a^-(\mathbf{p}) + e^{+ipx} a^+(\mathbf{p})\right)$$

$$[a^-(\mathbf{p}), a^+(\mathbf{p}')] = \delta(\mathbf{p} - \mathbf{p}'), \quad [a^-(\mathbf{p}), a^-(\mathbf{p}')] = [a^+(\mathbf{p}), a^+(\mathbf{p}')] = 0$$

Its Lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi\partial_\mu\varphi - m^2\varphi^2)$$

Variation ($\varphi \to \varphi + \delta\varphi$) of the corresponding action should be equal to zero in accord with the least action principle:

$$\delta \int dx \mathcal{L} = \int dx \left(\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi) \right) = 0$$

EXERCISE: Check that this gives the Klein-(Fock)-Gordon equation of motion

$$(m^2 + \partial_\mu^2)\varphi(x) = 0$$

Check that the postulated above field $\varphi(x)$ satisfies it

QFT: the Fock space

$a^-(\mathbf{p})$ and $a^+(\mathbf{p}')$ are annihilation and creation operators. They act in the Fock space which consists of vacuum $|0\rangle$

$$a^-(\mathbf{p})|0\rangle = 0, \quad \langle 0|a^+(\mathbf{p}) = 0, \quad \langle 0|0\rangle = 1$$

and field excitations, i.e. states of the form

$$|f\rangle = \int d\mathbf{p}\, f(\mathbf{p}) a^+(\mathbf{p})|0\rangle, \quad |g\rangle = \int d\mathbf{p} d\mathbf{q}\, g(\mathbf{p},\mathbf{q}) a^+(\mathbf{p}) a^+(\mathbf{q})|0\rangle, \ldots$$

The most simple excitation $a^+(\mathbf{p})|0\rangle \equiv |p\rangle$ is used to describe a single on-mass-shell particle with momentum $\mathbf{p}$. Then $a^+(\mathbf{p})a^+(\mathbf{q})|0\rangle$ is a two-particle state etc.

N.B. The Fock space is ∞-dimensional

EXERCISES: 1) Find the norm $\langle p|p\rangle$, 2) check that the operator $N = \int d\mathbf{p}\, a^+(\mathbf{p})a^-(\mathbf{p})$ acts as a particle number operator.

QFT: charged scalar field

A charged scalar field is defined as

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2p_0}} \left(e^{-ipx} a^-(\mathbf{p}) + e^{+ipx} b^+(\mathbf{p})\right)$$

$$\varphi^*(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2p_0}} \left(e^{-ipx} b^-(\mathbf{p}) + e^{+ipx} a^+(\mathbf{p})\right)$$

$$[a^-(\mathbf{p}), a^+(\mathbf{p}')] = [b^-(\mathbf{p}), b^+(\mathbf{p}')] = \delta(\mathbf{p} - \mathbf{p}'), \quad [a^\pm, b^\pm] = 0$$

$a^\pm$ and $b^\pm$ are creation (and annihilation) operators of particles and anti-particles, respectively (or vice versa)

The Lagrangian reads

$$\mathcal{L} = \partial_\mu \varphi^* \partial_\mu \varphi - m^2 \varphi^* \varphi$$

N.B.1. φ and φ^* are related by generalized conjugation

N.B.2. φ and φ^* are NOT particle and anti-particle

QFT: massive vector fields (I)

A massive charged vector field (remind W boson*) is defined as

$$U_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2p_0}} \sum_{n=1,2,3} e^n_\mu(\mathbf{p}) \left(e^{-ipx} a^-_n(\mathbf{p}) + e^{+ipx} b^+_n(\mathbf{p}) \right)$$

$$U^*_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2p_0}} \sum_{n=1,2,3} e^n_\mu(\mathbf{p}) \left(e^{-ipx} b^-_n(\mathbf{p}) + e^{+ipx} a^+_n(\mathbf{p}) \right)$$

$$[a^-_n(\mathbf{p}), a^+_l(\mathbf{p}')] = [b^-_n(\mathbf{p}), b^+_l(\mathbf{p}')] = \delta_{nl}\delta(\mathbf{p}-\mathbf{p}'), \quad [a^\pm, b^\pm] = 0$$

$e^n_\mu(\mathbf{p})$ are polarization vectors

$$e^n_\mu(\mathbf{p}) e^l_\mu(\mathbf{p}) = -\delta_{nl}, \quad p_\mu e^n_\mu(\mathbf{p}) = 0$$

EXERCISE: Using the above orthogonality conditions, show that

$$\sum_{n=1,2,3} e^n_\mu(\mathbf{p}) e^n_\nu(\mathbf{p}) = -\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2} \right)$$

QFT: massive vector fields (II)

For a massive charged vector field

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu U^*_\nu - \partial_\nu U^*_\mu)(\partial_\mu U_\nu - \partial_\nu U_\mu) + m^2 U^*_\mu U_\mu$$

The corresponding Euler-Lagrange equation reads

$$\partial_\nu(\partial_\mu U_\nu - \partial_\nu U_\mu) + m^2 U_\mu = 0$$

EXERCISE: Using the above equation, show that $\partial_\nu U_\nu = 0$, i.e. the Lorentz condition

N.B. The Lorentz condition removes one of four independent field components

QUESTION: How can it be that the signs before the mass terms in the scalar and vector field Lagrangians are different?

QFT: massless vector fields

A massless neutral vector field (photon) is defined as

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2p_0}} e^\lambda_\mu(\mathbf{p}) \left(e^{-ipx} a^-_\lambda(\mathbf{p}) + e^{+ipx} a^+_\lambda(\mathbf{p})\right)$$

$$[a^-_\lambda(\mathbf{p}), a^+_\nu(\mathbf{p}')] = -g_{\lambda\nu}\delta(\mathbf{p} - \mathbf{p}')$$

$$e^\lambda_\mu(\mathbf{p}) e^\lambda_\nu(\mathbf{p}) = g_{\mu\nu}, \quad e^\lambda_\mu(\mathbf{p}) e^\nu_\mu(\mathbf{p}) = g_{\lambda\nu}$$

N.B. Formally this field has four polarizations, but only two of them correspond to physical degrees of freedom

The corresponding Lagrangian reads

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu}, \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

QFT: spinor fields (I)

A Dirac fermion field is defined as

$$\Psi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2p_0}} \sum_{r=1,2} \left(e^{-ipx} a_r^-(\mathbf{p}) u_r(\mathbf{p}) + e^{+ipx} b_r^+(\mathbf{p}) v_r(\mathbf{p}) \right)$$

$$\overline{\Psi}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2p_0}} \sum_{r=1,2} \left(e^{-ipx} b_r^-(\mathbf{p}) \bar{v}_r(\mathbf{p}) + e^{+ipx} a_r^+(\mathbf{p}) \bar{u}_r(\mathbf{p}) \right)$$

$$[a_r^-(\mathbf{p}), a_s^+(\mathbf{p}')]_+ = [b_r^-(\mathbf{p}), b_s^+(\mathbf{p}')]_+ = \delta_{rs}\delta(\mathbf{p} - \mathbf{p}')$$

$$[a_r^+(\mathbf{p}), a_s^+(\mathbf{p}')]_+ = [a_r^-(\mathbf{p}), b_s^+(\mathbf{p}')]_+ = \ldots = 0$$

EXERCISE: Show that $a_r^+(\mathbf{p}) a_r^+(\mathbf{p}) = 0$, i.e. the Pauli principle

EXERCISE: Define a Majorana fermion field

QFT: spinor fields (II)

u_r, u_r, $\bar{u}_r$, and $\bar{v}_r$ are four-component spinors, so
$\Psi(x) \equiv \{\Psi_\alpha(x)\}$ is a four-vector column, $\alpha = \{1, 2, 3, 4\}$
and $\overline{\Psi}(x)$ is a four-vector row

$$\bar{u}u = \sum_{\alpha=1}^{4} \bar{u}_\alpha u_\alpha = \sum_{\alpha=1}^{4} u_\alpha \bar{u}_\alpha = \mathrm{Tr}(u\bar{u})$$

Spinors are solutions of the (Dirac) equations:

$$\begin{aligned} &(\hat{p} - m)u_r(\mathbf{p}) = 0, \qquad \bar{u}_r(\mathbf{p})(\hat{p} - m) = 0 \\ &(\hat{p} + m)v_r(\mathbf{p}) = 0, \qquad \bar{v}_r(\mathbf{p})(\hat{p} + m) = 0 \\ &\hat{p} \equiv p_\mu\gamma_\mu = p_0\gamma_0 - p_1\gamma_1 - p_2\gamma_2 - p_3\gamma_3, \quad m \equiv m \cdot \mathbf{1} \end{aligned}$$

with normalization conditions

$$\bar{u}_r(\mathbf{p})u_s(\mathbf{p}) = -\bar{v}_r(\mathbf{p})v_s(\mathbf{p}) = 2m\,\delta_{rs}$$

QFT: Dirac's gamma matrixes

The gamma matrixes (should) satisfy the commutation condition

$$[\gamma_\mu, \gamma_\nu]_+ = 2g_{\mu\nu}\mathbf{1} \quad \Rightarrow \quad \gamma_0^2 = \mathbf{1}, \quad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -\mathbf{1}$$

and the condition of Hermitian conjugation

$$\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0$$

The latter leads to the rule of the Dirac conjugation:

$$\overline{\Psi} = \Psi^\dagger \gamma_0, \quad \bar{u} = u^\dagger \gamma_0, \quad \bar{v} = v^\dagger \gamma_0$$

N.B. Explicit expressions for gamma matrixes are not unique, but they are not necessary for construction of observables. Why?

EXERCISE: Show that the Dirac conjugation rule is consistent with the set of Dirac equations given on the previous slide

QFT: Left and Right spinors

One can show that the Dirac equations have two independent solutions $u_{1,2}$ which correspond to different polarization states. But it is useful to consider also another choice. We introduce

$$\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3 \quad \Rightarrow \quad [\gamma_\mu, \gamma_5]_+ = 0, \quad \gamma_5^2 = \mathbf{1}, \quad \gamma_5^\dagger = \gamma_5$$

By definition left and right spinors are

$$\Psi_L \equiv P_L \Psi, \quad \Psi_R \equiv P_R \Psi, \quad P_{L,R} \equiv \frac{\mathbf{1} -,+ \gamma_5}{2}, \quad \Psi = \Psi_L + \Psi_R$$

The Dirac conjugation gives $\overline{\Psi}_L \equiv \overline{\Psi}\frac{1+\gamma_5}{2}, \quad \overline{\Psi}_R \equiv \overline{\Psi}\frac{1-\gamma_5}{2}$

N.B.1. In Weyl's representation of γ matrixes $\Psi_L = \frac{1+\gamma_5}{2}\Psi$.
N.B.2. Definition of left and right spinors as polarization sates $\mathbf{s} \downarrow\uparrow \mathbf{p}$ and $\mathbf{s} \uparrow\uparrow \mathbf{p}$ is wrong. It is just an approximation working in ultra-relativistic kinematics $|\mathbf{p}| \gg m$

EXERCISE: Prove the $P_{L,R}$ is a basis of orthogonal projection operators is the space of spinors:

$$P_L + P_R = \mathbf{1}, \quad P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_R P_L = P_L P_R = 0$$

QFT: Lagrangian for spinor fields

Remind some properties of gamma matrixes

$$\mathrm{Tr}\gamma_\mu = \mathrm{Tr}\gamma_5 = 0, \quad \mathrm{Tr}\gamma_\mu\gamma_\nu = 4g_{\mu\nu}, \quad \mathrm{Tr}\gamma_5\gamma_\mu\gamma_\nu = 0,$$
$$\mathrm{Tr}\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta = 4(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}),$$
$$\mathrm{Tr}\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta = -4i\varepsilon_{\mu\nu\alpha\beta}$$

The equations for u and v are chosen so that we get the conventional Dirac equations

$$(i\gamma_\mu\partial_\mu - m)\Psi(x) = 0, \qquad i\partial_\mu\overline{\Psi}(x)\gamma_\mu + m\overline{\Psi}(x) = 0$$

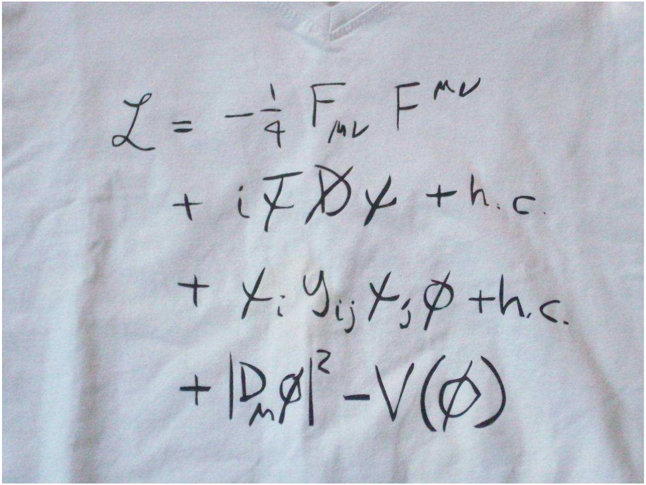
These equations follow also from the Lagrangian

$$\mathcal{L} = \frac{i}{2}\left[\overline{\Psi}\gamma_\mu(\partial_\mu\Psi) - (\partial_\mu\overline{\Psi})\gamma_\mu\Psi\right] - m\overline{\Psi}\Psi \equiv i\overline{\Psi}\gamma_\mu\partial_\mu\Psi - m\overline{\Psi}\Psi$$

N.B. QFT Lagrangians (Hamiltonians) should be Hermitian: $\mathcal{L}^\dagger = \mathcal{L}$. QUESTION: Why?

EXERCISE: Find two bugs on the CERN mugs

The SM Lagrangian (on a T-shirt)



QFT: Evolution of states

Up to now we considered only free non-interacting fields. Studies of transitions between free states is the main task of QFT. Collective, nonperturbative effects, bound states etc. are also of interest of course.

Let us postulate the transition amplitude $\mathcal{M}$ of a physical process:

$$\mathcal{M} \equiv \langle out|S|in\rangle, \qquad S \equiv T\exp\left(i\int dx \mathcal{L}_I(\varphi(x))\right)$$

The initial and final states are

$$|in\rangle = a^+(\mathbf{p}_1)\ldots a^+(\mathbf{p}_s)|0\rangle, \qquad |out\rangle = a^+(\mathbf{p}'_1)\ldots a^+(\mathbf{p}'_r)|0\rangle$$

The differential probability to evolve from $|in\rangle$ to $|out\rangle$ is

$$dw = (2\pi)^4\delta(\sum p'_i)\frac{n_1\ldots n_s}{2E_1\ldots E_s}|\mathcal{M}|^2\prod_{j=1}^{r}\frac{d\mathbf{p}'_j}{(2\pi)^3 2E'_j}$$

here n_i is the particle number density of $i^{\rm th}$ particle beam

QFT: Interaction Lagrangians

Nontrivial transitions happen due to interactions of fields. QFT prefers* dealing with local interactions $\Rightarrow \quad \mathcal{L}_I = \mathcal{L}_I\big(\varphi(x)\big)$
Examples of interaction Lagrangians:

$$g\varphi^3(x), \qquad h\varphi^4(x), \qquad y\varphi(x)\overline{\Psi}(x)\Psi(x)$$
$$e\overline{\Psi}(x)\gamma_\mu\Psi(x)A_\mu(x), \qquad G\,\overline{\Psi}_1(x)\gamma_\mu\Psi_1(x)\cdot\overline{\Psi}_2(x)\gamma_\mu\Psi_2(x)$$

IMPORTANT: Always keep in mind the dimension of your objects! The reference unit is the dimension of energy (mass):

$$[E] = [m] = 1 \quad \Rightarrow \quad [p] = 1, \qquad [x] = -1$$

An action should be dimensionless $\left[\int dx \mathcal{L}(x)\right] = 0 \;\Rightarrow\; [\mathcal{L}] = 4$

EXERCISE: Show that $[\varphi] = [A_\mu] = 1$ and $[\Psi] = 3/2$. Find the dimensions of the coupling constants g, h, y, e, and G in the examples of $\mathcal{L}_I$ above

QFT: Time ordering

By definition

$$T\, A_1(x_1)\ldots A_n(x_n) = (-1)^k A_{i_1}(x_{i_1})\ldots A_{i_n}(x_{i_n}) \text{ with } x^0_{i_1} > \ldots > x^0_{i_n}$$

where k is the number of fermion field permutations

N.B. Objects like $A_1(x)A_2(y)$ with $x = y$ are not well defined, they lead to divergences

Perturbative expansion of S matrix exponent leads to terms like

$$\frac{i^n g^n}{n!}\langle 0|a^-(\mathbf{p'}_1)\ldots a^-(\mathbf{p'}_r)\int dx_1\ldots dx_n T\varphi^3(x_1)\ldots\varphi^3(x_n)a^+(\mathbf{p}_1)\ldots a^+(\mathbf{p}_s)|0\rangle$$

Remind that fields φ also contain creation and annihilation operators. By permutation of operators $a^-(\mathbf{p})a^+(\mathbf{p}') = a^+(\mathbf{p}')a^-(\mathbf{p}) + \delta(\mathbf{p}-\mathbf{p}')$ we move a^- to the right and a^+ to the left. At the end we get either 0 because $a^-|0\rangle = 0$ or some terms proportional to $\langle 0|0\rangle = 1$

EXERCISE: Show that $[a^-(\mathbf{p}), \varphi(x)] = \frac{e^{ipx}}{(2\pi)^{3/2}\sqrt{2p_0}}$ and

$$[a_r^-(\mathbf{p}), \overline{\Psi}(x)]_+ = \frac{e^{ipx}\bar{u}_r(\mathbf{p})}{(2\pi)^{3/2}\sqrt{2p_0}}$$

QFT: the Green functions

By definition the causal Green function is given by

$$\langle 0|T\varphi(x)\varphi(y)|0\rangle \equiv -iD^c(x-y)$$

It is a building block for construction of amplitudes

One can show (see textbooks) that

$$(\partial^2 + m^2)D^c(x) = \delta(x)$$

so that D^c is the Green function of the Klein-Gordon operator,

$$D^c(x) = \frac{-1}{(2\pi)^4}\int \frac{dp\; e^{-ipx}}{p^2 - m^2 + i0}$$

For other fields we have

$$\langle 0|T\;\Psi(x)\overline{\Psi}(y)|0\rangle = \frac{i}{(2\pi)^4}\int \frac{dp\; e^{-ip(x-y)}(\hat{p}+m)}{p^2 - m^2 + i0}$$

$$\langle 0|T\;U_\mu(x)U^*_\nu(y)|0\rangle = \frac{-i}{(2\pi)^4}\int \frac{dp\; e^{-ip(x-y)}(g_{\mu\nu} - p_\mu p_\nu/m^2)}{p^2 - m^2 + i0}$$

$$\langle 0|T\;A_\mu(x)A_\nu(y)|0\rangle = \frac{-i}{(2\pi)^4}\int \frac{dp\; e^{-ip(x-y)}g_{\mu\nu}}{p^2 + i0}$$

QFT: the Wick theorem

The Wick theorem states that for any combinations of fields

$$T\,A_1 \dots A_n \equiv \sum (-1)^k \langle 0|TA_{i_1}A_{i_2}|0\rangle \dots \langle 0|TA_{i_{m-1}}A_{i_m}|0\rangle \ :A_{i_{m+1}} \dots A_{i_n}:$$

The sum is taken over all possible ways to pair the fields

The normal ordering operation acts as

$$:a_1^- a_2^+ a_3^- a_4^- a_5^+ a_6^- a_7^+ : = (-1)^k a_2^+ a_5^+ a_7^+ a_1^- a_3^- a_4^- a_6^-$$

Using the Wick theorem we construct the Feynman rules for simple $g\phi^3$ and $h\varphi^4$ interactions. But for the case of gauge interactions we need something more…

QFT: the Noether theorems (I)

There are two major types of symmetries in the SM: global and local ones

The 1st Noether theorem:
If the action is invariant with respect to the global Lie group G_r with r parameters, then there are r linearly independent combinations of Lagrange derivatives which become complete divergences; and vice versa.

If the field satisfies the Euler-Lagrange equations, then $\mathrm{div}J = \nabla J = 0$, i.e. the Noether currents are conserved.

Integration of those divergences over 3-dim space (with certain boundary conditions) leads to r conserved charges.

EXERCISE: Remind examples from QED and Poincaré global symmetries

QFT: the Noether theorems (II)

Much more important for us is the 2nd Noether theorem:

If the action is invariant with respect to the infinite-dimensional r-parametric group $G_{\infty,r}$ with derivatives up to the $k^{\rm th}$ order, then there are r independent relations between Lagrange derivatives and derivatives of them up to the $k^{\rm th}$ order; and vice versa.

N.B. The 2nd Noether theorem provides r conditions on the fields which are additional to the standard Euler-Lagrange equations. These conditions should be used to exclude double counting of physically equivalent field configurations.

Examples of infinite-dimensional groups are local gauge transformations (see below) and the general coordinate transformations in General Relativity

QFT: Gauge symmetry (I)

The free Lagrangians for electrons and photons

$$\mathcal{L}_0(\Psi) = i\overline{\Psi}\gamma_\mu\partial_\mu\Psi - m\overline{\Psi}\Psi, \quad \mathcal{L}_0(A) = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu}$$

are invariant with respect to the global $U(1)$ transformations

$$\Psi(x) \to \exp(ie\theta)\Psi(x), \quad \overline{\Psi}(x) \to \exp(-ie\theta)\overline{\Psi}(x), \quad A_\mu(x) \to A_\mu(x)$$

One can note that $F_{\mu\nu}$ is invariant also with respect to local transformations $A_\mu(x) \to A_\mu(x) + \partial_\mu\omega(x)$. For fermions the corresponding transformations are

$$\Psi(x) \to \exp(ie\omega(x))\Psi(x), \quad \overline{\Psi}(x) \to \exp(-ie\omega(x))\overline{\Psi}(x)$$

How to make the fermion Lagrangian being also invariant?

The answer is to introduce the covariant derivative:

$$\partial_\mu \to D_\mu, \qquad D_\mu\Psi \equiv (\partial_\mu - ieA_\mu)\Psi, \quad D_\mu\overline{\Psi} \equiv (\partial_\mu + ieA_\mu)\overline{\Psi}$$

Then we get the QED Lagrangian:

$$\begin{aligned}\mathcal{L}_{\rm QED} &= -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + i\overline{\Psi}\gamma_\mu D_\mu\Psi - m\overline{\Psi}\Psi \\ &= -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + i\overline{\Psi}\gamma_\mu\partial_\mu\Psi - m\overline{\Psi}\Psi + e\overline{\Psi}\gamma_\mu\Psi A_\mu\end{aligned}$$

QFT: Gauge symmetry (II)

EXERCISES: 1) Check the covariance: $D_\mu \Psi \to e^{ie\omega(x)}(D_\mu \Psi)$,
2) construct the Lagrangian of scalar QED

Let's look at the photon free Lagrangian

$$\mathcal{L}_0(A) = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 = -\frac{1}{2} A_\nu K_{\mu\nu} A_\nu,$$
$$K_{\mu\nu} = g_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu \quad \Rightarrow K_{\mu\nu}(p) = p_\mu p_\nu - g_{\mu\nu} p^2$$

Operator $K_{\mu\nu}(p)$ has zero modes (since $p_\mu K_{\mu\nu} = 0$), so it is not invertable. Definition of the photon propagator within the functional integral formalism becomes impossible. The reason is the unresolved symmetry.

The solution is to introduce a gauge fixing term into the Lagrangian:

$$\mathcal{L}_0(A) = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A_\mu)^2 \quad \Rightarrow$$
$$\langle 0 | T\, A_\mu(x) A_\nu(y) | 0 \rangle = \frac{-i}{(2\pi)^4} \int dp\, e^{-ip(x-y)} \frac{g_{\mu\nu} + (\xi - 1) p_\mu p_\nu / p^2}{p^2 + i0}$$

N.B. Physical (observable) quantities do not depend on the value of ξ

QFT: Non-abelian Gauge symmetry

Transformations for a non-abelian case read

$$\Psi_i \to e^{ig\omega^a t^a_{ij}}\Psi_j, \qquad [t^a, t^b] = if^{abc}t^c$$
$$B^a_\mu \to B^a_\mu + \partial_\mu \omega^a + gf^{abc}B^b_\mu \omega^c, \quad F_{\mu\nu} \equiv \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + gf^{abc}B^b_\mu B^c_\nu$$

where t^a are the group generators, f^{abc} are the structure constants

Introduce the covariant derivative

$$\partial_\mu \Psi \to D_\mu \Psi \equiv (\partial_\mu - igB^a_\mu t^a)\Psi$$

and we get

$$\mathcal{L}(\Psi, B) = i\overline{\Psi}\gamma_\mu D_\mu \Psi + \mathcal{L}(B),$$
$$\mathcal{L}(B) = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu B^a_\mu)^2 = -\frac{1}{4}\left(\partial_\mu B^a_\nu - \partial_\nu B^a_\mu\right)^2$$
$$-\frac{1}{2\xi}(\partial_\mu B^a_\mu)^2 - \frac{g}{2}f^{abc}\left(\partial_\mu B^a_\nu - \partial_\nu B^a_\mu\right)B^b_\mu B^c_\nu - \frac{g^2}{4}f^{abc}f^{ade}B^b_\mu B^c_\nu B^d_\mu B^e_\nu$$

N.B.1. $\mathcal{L}(B)$ contains self-interactions. N.B.2. $m_B \equiv 0$, why?

N.B.3. Non-abelian charge g is universal

QFT: Faddeev-Popov ghosts

Exclusion of double-counting due to the physical equivalence of the field configurations related to each other by non-abelian gauge transformations is nontrivial. Functional integration over those identical configurations (or application of the BRST method) leads to the appearance of the so-called Faddeev-Popov ghosts:

$$\mathcal{L}(\Psi, B) \to \mathcal{L}(\Psi, B) + \mathcal{L}_{gh}$$
$$\mathcal{L}_{gh} = -\partial_\mu \bar{c}^a \partial_\mu c^a + g f^{acb} \bar{c}^a B_\mu^c \partial_\mu c^a = -\partial_\mu \bar{c}^a \partial_\mu c^a - g f^{acb} \partial_\mu \bar{c}^a B_\mu^c c^a$$

where c and $\bar{c}$ are ghost fields, they are fermions with a boson-like kinetic term.

IMPORTANT: Ghosts are fictitious particles. In the Feynman rules they (should) appear only as virtual states in propagators

N.B. Ghosts in QED are non-interacting since $f^{abc} = 0$ there

QFT: Regularization of UV divergences

Higher-order terms in perturbative series contain loop integrals, e.g.

$$I_2 \equiv \int \frac{d^4 p}{(p^2 + i0)((k-p)^2 + i0)} \sim \int_0^\infty \frac{|p|^3 \, d|p|}{|p|^4} \sim \ln \infty$$

Introduction of a cut-off M leads to a finite, i.e. regularized value of the integral:

$$I_2^{\text{cut-off}} = i\pi^2 \left(\ln \frac{M^2}{k^2} + 1 \right) + \mathcal{O}\left(\frac{k^2}{M^2} \right) = i\pi^2 \left(\ln \frac{M^2}{\mu^2} - \ln \frac{k^2}{\mu^2} + 1 \right) + \mathcal{O}\left(\frac{k^2}{M^2} \right)$$

Another possibility is the dimensional regularization where $dim = 4 \to dim = 4 - 2\varepsilon$

$$I_2^{\text{dim.reg.}} = \mu^{2\varepsilon} \int \frac{d^{4-2\varepsilon} p}{(p^2 + i0)((k-p)^2 + i0)} = i\pi^2 \left(\frac{1}{\varepsilon} - \ln \frac{k^2}{\mu^2} + 2 \right) + \mathcal{O}(\varepsilon)$$

N.B. The origin of UV divergences is the locality of interactions

QFT: Renormalization

Let's consider a three-point (vertex) function in the $g\phi^3$ model

$$G = \int dx\,dy\,dz\;\varphi(x)\varphi(y)\varphi(z)F(x,y,z,)$$

$$F^{\text{dim.reg.}} = \frac{A}{\varepsilon}\delta(y-x)\delta(z-x) + \dots \quad \Rightarrow \quad G = \frac{A}{\varepsilon}\int dx\varphi^3(x) + \dots$$

IMPORTANT: Divergent terms are local.

A QFT model is called renormalizable if all UV-divergent terms are of the type of the ones existing in the (semi)classical Lagrangian. Otherwise the model is called nonrenormalizable.

EXAMPLES:

a) renormalizable models: QED, QCD, SM ['t Hooft & Veltman], $h\varphi^4$, $g\varphi^3$

b) nonrenormalizable models: $G(\overline{\Psi}\gamma_\mu\Psi)^2$, General Relativity

N.B. Models with dimensionful ($[G] < 0$) coupling constants are nonrenormalizable

QFT: Subtractions and counter terms

In renormalizable models all UV divergences can be subtracted from amplitudes and shifted into counter terms in $\mathcal{L}$. Each* term in $\mathcal{L}$ gets a renormalization constant:

$$\mathcal{L} = \frac{Z_2}{2}(\partial\varphi)^2 - \frac{Z_m m^2}{2}\varphi^2 + Z_4 h\varphi^4 = \frac{1}{2}(\partial\varphi_B)^2 - \frac{m_B^2}{2}\varphi^2 + h_B\varphi^4$$

where $\varphi_B = \sqrt{Z_2}\varphi$, $m_B^2 = m^2 Z_M Z_2^{-1}$, $h_B = hZ_4 Z_2^{-2}$ are bare field, mass, and charge,

$$Z_i(h,\varepsilon) = 1 + \frac{Ah}{\varepsilon} + \frac{Bh^2}{\varepsilon^2} + \frac{Ch^2}{\varepsilon} + \mathcal{O}(h^3)$$

Remonrmalization constants are chosen in such a way that divergences in amplitudes are cancelled out with divergences in Z_i. That happens order by order.

N.B. R. Feynman: *"I think that the renormalization theory is simply a way to sweep the difficulties of the divergences of electrodynamics under the rug."*

QFT: Renormalization group

Physical results should not depend on the auxiliary scale μ:

$$F(k,g,m) \xrightarrow{\infty} F_{\rm reg}(k,M,g,m) \xrightarrow{M\to\infty} F_{\rm ren}(k,\mu,g,m) \xrightarrow{RG} F_{\rm phys}(k,\Lambda,m)$$

where Λ is some dimensionful scale

Charge (and mass) become running, i.e. energy-dependent:

$$g \to g\left(g, \frac{\mu'}{\mu}\right), \qquad \beta(g) \equiv \left.\frac{dg}{d\ln\mu}\right|_{g_B=const}$$

N.B.1. Renormalization scale unavoidably appears in any scheme

N.B.2. Scheme and scale dependencies are reduced after including higher orders of the perturbation theory

QFT: Dimensional Transmutation

Resummation of multi-leg Feynman diagrams (with loops) in a QFT model provides the so called effective potential $V(\phi)$

S. Coleman and E. Weinberg (PRD'1973) have shown that starting from a mass less (conformal invariant) semi-classical Lagrangian (e.g. for a scalar field) one can get an effective potential which is infra-red divergent:

$$\begin{aligned} V_{\text{classical}}(\phi) &= \frac{\lambda}{4!}\phi^4 \to \left[\text{resummation} + \text{renormalization}\right] \to \\ V_{\text{quantum}}(\phi) &= \frac{\lambda}{4!}\phi^4 + \frac{\lambda^2\phi^4}{256\pi^2}\left(\ln\frac{\phi^2}{\Lambda^2} - \frac{25}{6}\right) \end{aligned}$$

The stability condition $V'(\langle\phi\rangle) = 0$ provides a relation between $\langle\phi\rangle \neq 0$, M and λ. The dimensionless coupling constant is traded for a dimensionful scale Λ

Conformal anomaly generates non-zero condensate and mass

N.B. Relation $\alpha_{\text{QCD}}(Q^2) = \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda_{\text{QCD}})}$ is another example of dimensional transmutation