Search and Discovery Statistics in HEP
LECTURE 2

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This presentation would have not been possible without the tremendous help of the following people throughout many years

Louis Lyons, Alex Read, Glen Cowan ,Kyle Cranmer
Ofer Vitells & Bob Cousins

Confidence Interval and Confidence Level (CL)

CL & CI - Wikipedia

$$\mu = 1.1 \pm 0.3$$

$$\mu = [0.8, 1.4] \ @ \ 68\% \ CL$$

CI=[0.8,1.4]

what does it mean?

- A **confidence interval** (**CI**) is a particular kind of interval estimate of a population parameter.
- Instead of estimating the parameter by a single value, an interval likely to include the parameter is given.
- How likely the interval is to contain the parameter is determined by the **confidence level** or confidence coefficient.
- Increasing the desired confidence level will widen the confidence interval.

Confidence Interval & Coverage

- Say you have a measurement μ_{meas} of μ with μ_{true} being the unknown true value of μ
- Assume you know the probability distribution function $p(\mu_{meas}|\mu)$
- Given the measurement you deduce somehow (based on your statistical model) that there is a 95% Confidence interval $[\mu_1,\mu_2]$.

(it is 95% likely that the μ_{true} is in the quoted interval)

The correct statement:

- In an ensemble of experiments 95% of the obtained confidence intervals will contain the true value of μ.

Upper limit

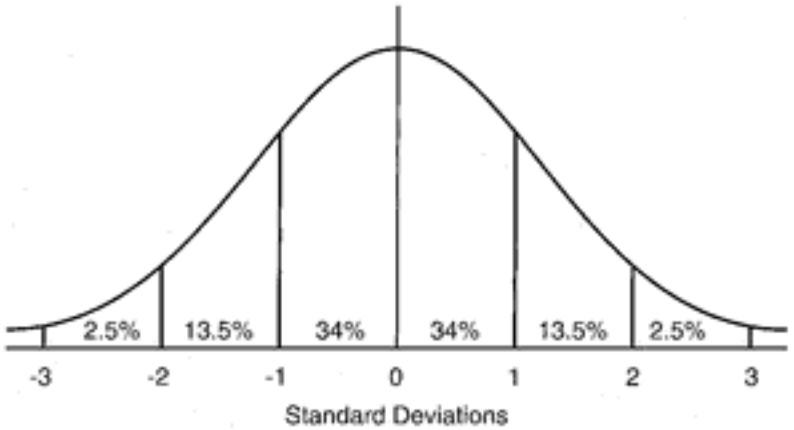
- Given the measurement you deduce somehow (based on your statistical model) that there is a 95% Confidence interval $[0,\mu_{up}]$.
- This means: our interval contains $\mu=0$ (no Higgs)
- We therefore deduce that $\mu<\mu_{up}$ at the 95% Confidence Level (CL)
- μ_{up} is therefore an upper limit on μ
- If $\mu_{up}<1 \rightarrow$
  $\sigma(m_H)<\sigma_{SM}(m_H) \rightarrow$
  a SM Higgs with a mass m_H is excluded at the 95% CL

Confidence Interval & Coverage

- Confidence Level: A CL of (e.g.) 95% means that in an ensemble of experiments, each producing a confidence interval, 95% of the confidence intervals will contain the true value of μ
- Normally, we make one experiment and try to estimate from this one experiment the confidence interval at a specified CL
- If in an ensemble of (MC) experiments our estimated Confidence Interval fail to contain the true value of μ 95% of the cases (for every possible μ) we claim that our method undercover
- If in an ensemble of (MC) experiments our estimated Confidence Interval contains the true value of μ more than 95% of the cases (for every possible μ) we claim that our method overcover (being conservative)
- If in an ensemble of (MC) experiments the true value of μ is covered within the estimated confidence interval , we claim a coverage

How to deduce a CI?

- One can show that if the data is distributed normal around the average i.e. P(data | μ)=normal



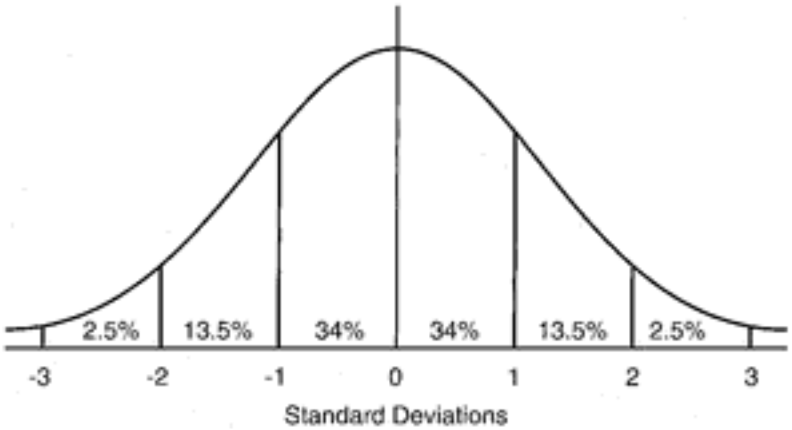
$$f(x \mid \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

then one can construct a 68% CI around the estimator of μ to be

$$\hat{x} \pm \sigma$$

However, not all distributions are normal, many distributions are even unknown and coverage might be a real issue

How to deduce a CI?



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However, not all distributions are normal, many distributions are even unknown and coverage might be a real issue

- One may construct many 68% intervals…. $CI = [\mu_L, \mu_U]$

$$\int_{\mu_L}^{\mu_U} f(x \mid \hat{x})dx = 68\%$$

- Which one has a full coverage?
- How can we guarantee a coverage

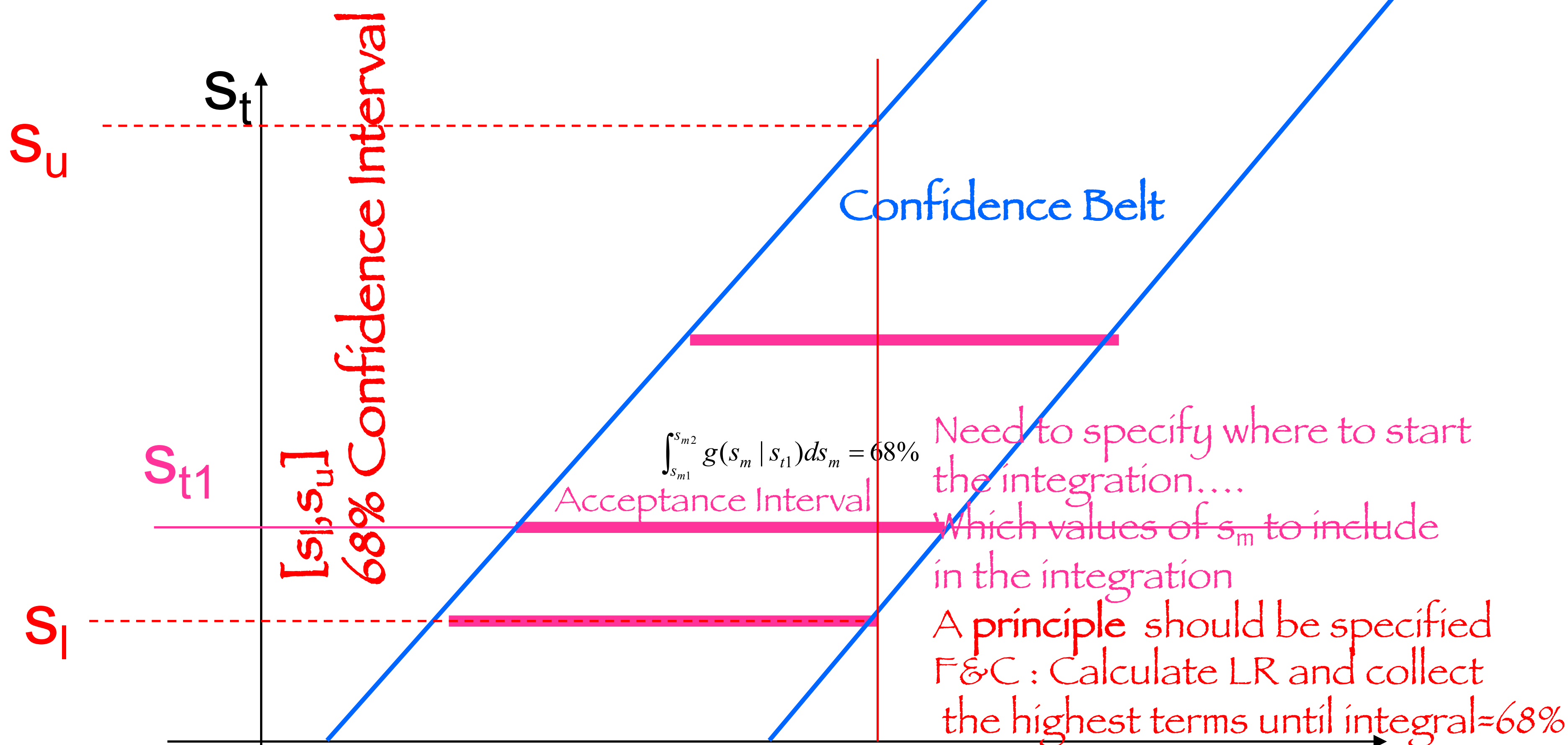
- The QUESTION is NOT how to construct a CI, it is
- HOW TO CONSTRUCT A CI WHICH HAS A COVERAGE @ THE 68% CL

The Frequentist Game a 'la Feldman & Cousins

Or

How to ensure a Coverage

Neyman Construction



$[s_l,s_u]$ 68% Confidence Interval
In **68%** of the experiments the derived **C.I. contains** the **unknown true value of s**

- With Neyman Construction we guarantee a coverage via construction, i.e. for any value of the unknown true s, the Construction Confidence Interval will cover s with the correct rate.

The Flip Flop Way of an Experiment

- The most intuitive way to analyze the results of an experiment would be
  - Construct a test statistics
    e.g. $Q(x) \sim L(x|H_1) / L(x|H_0)$
  - If the significance of the measured $Q(x_{obs})$, is less than 3 sigma, derive an upper limit (just looking at tables), if the result is >5 sigma (and some minimum number of events is observed….), derive a discovery central confidence interval for the measured parameter (cross section, mass….) …..
- **This Flip Flopping policy leads to undercoverage:**
  Is that really a problem for Physicists?
  Some physicists say, for each experiment quote always two results, an upper limit, and a (central?) discovery confidence interval
- Many LHC analyses report both ways.

Frequentist Paradise – F&C Unified with Full Coverage

- Frequentist Paradise is certainly made up of an interpretation by constructing a confidence interval in brute force ensuring a coverage!
- This is the Neyman confidence interval adopted by F&C….

- The motivation:
  - Ensures Coverage
  - Avoid Flip-Flopping – an ordering rule determines the nature of the interval (1-sided or 2-sided depending on your observed data)
  - Ensures Physical Intervals

- Let the test statistics be

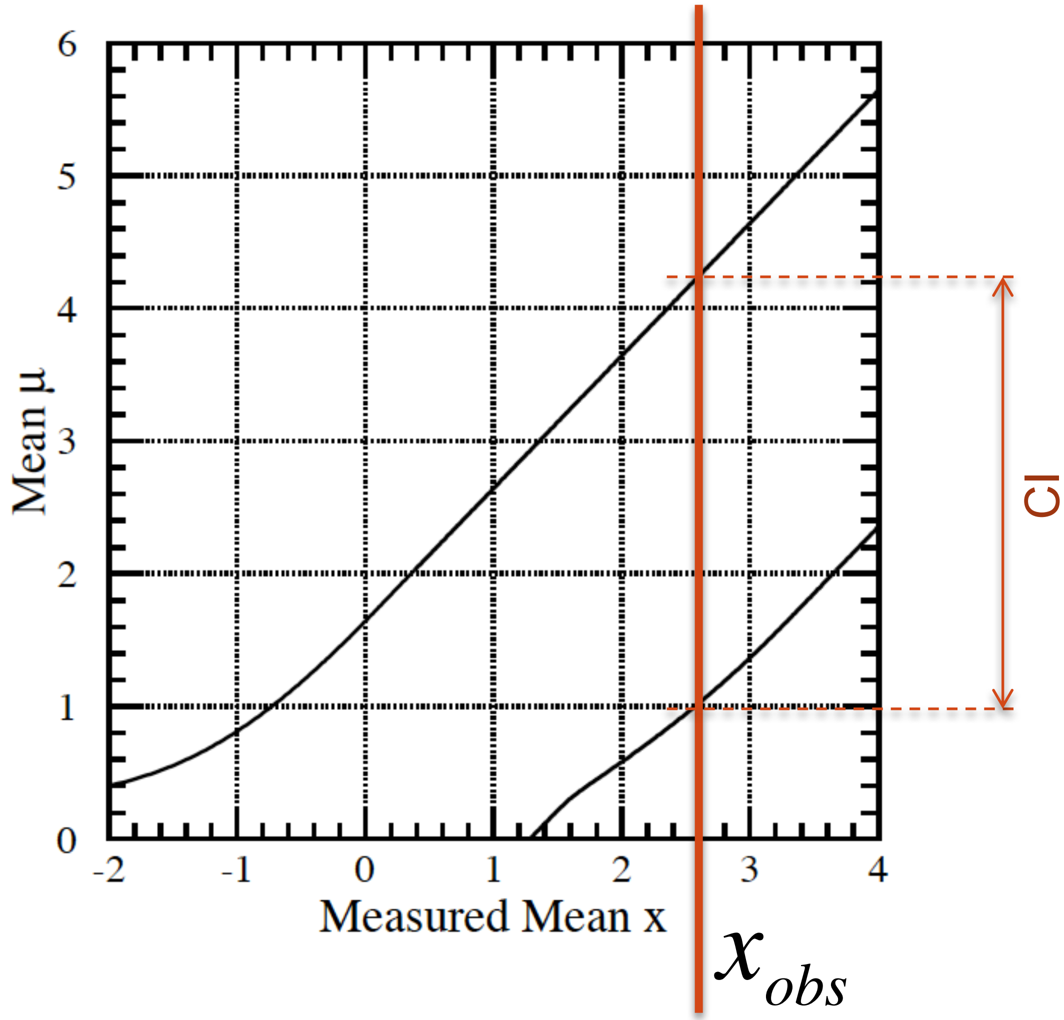
$$Q = \begin{cases} \dfrac{L(s+b)}{L(\hat{s}+b)} & \hat{s} > 0 \\ \dfrac{L(s+b)}{L(b)} & \hat{s} \leq 0 \end{cases}$$

  where $\hat{s}$ is the **physically allowed** mean s that maximizes $L(\hat{s}+b)$
  (If $\hat{s}<0$, denominator is set to $L(b)$.)

- Order by taking the 68% highest Qs

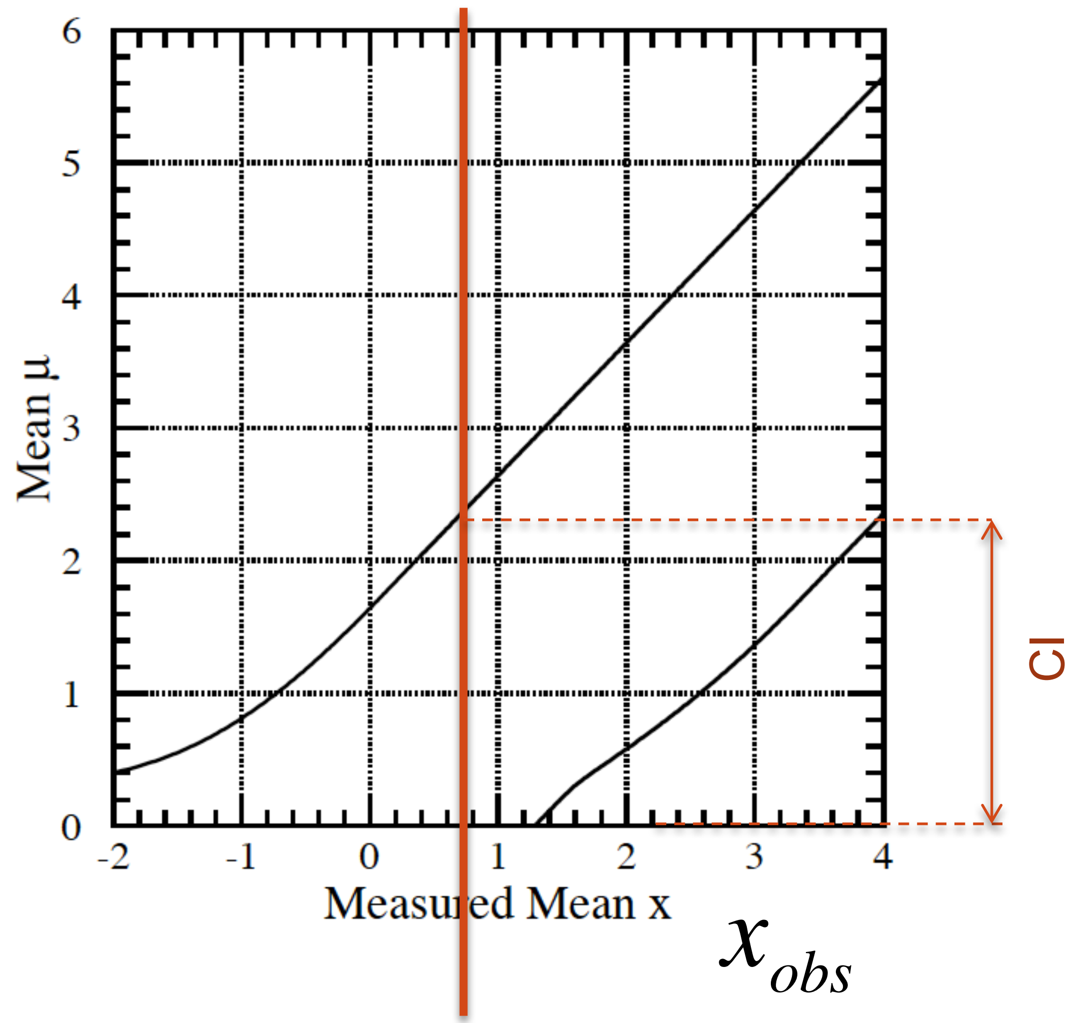
How to tell an Upper limit from a Measurement without Flip Flopping

- A measurement (2 sided)



How to tell an Upper limit from a Measurement without Flip Flopping

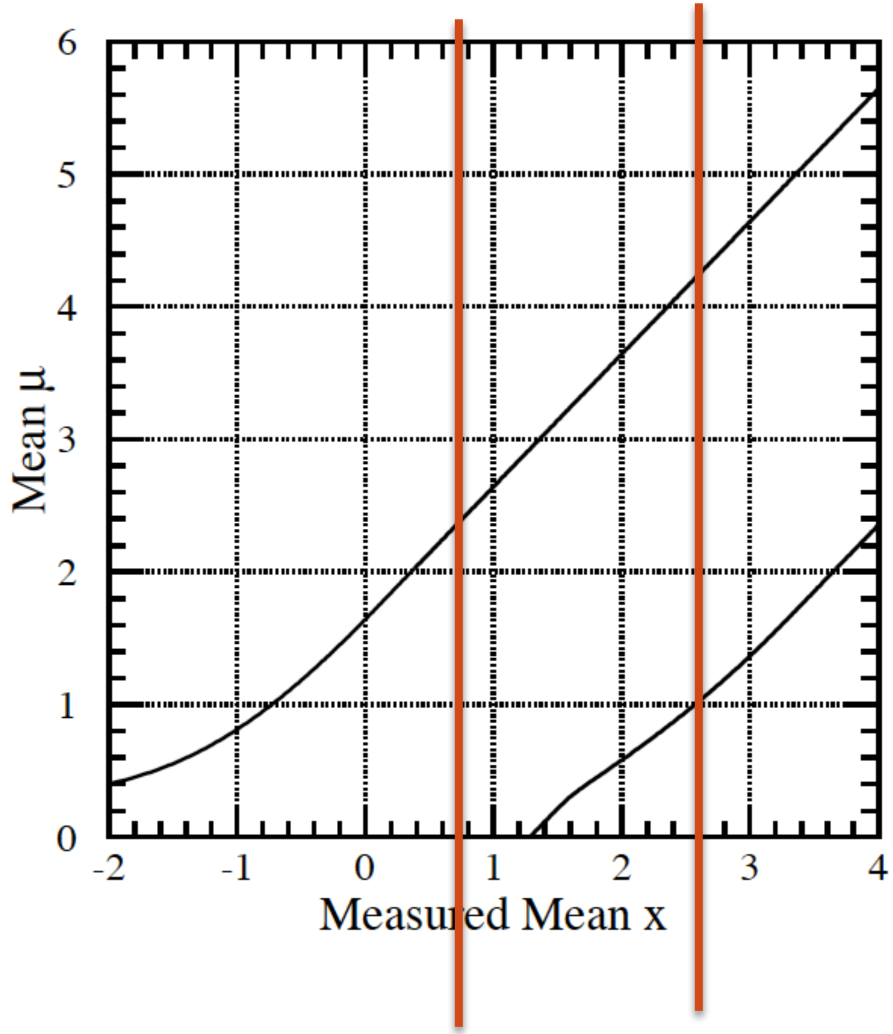
- An upper limit (1 sided)



How to tell an Upper limit from a Measurement without Flip Flopping

- Your observed result will tell you if it's a measurement or an upper limit

- But how to deal with systematics?



Nuisance Parameters

or Systematics

Nuisance Parameters (Systematics)

- There are two kinds of parameters:
  - Parameters of interest (signal strength… cross section… μ)
  - Nuisance parameters (background (b), signal efficiency, resolution, energy scale,…)
- The nuisance parameters carry systematic uncertainties
- There are two related issues:
  - Classifying and estimating the systematic uncertainties
  - Implementing them in the analysis
- The physicist must make the difference between cross checks and identifying the sources of the systematic uncertainty.
  - Shifting cuts around and measure the effect on the observable… Very often the observed variation is dominated by the statistical uncertainty in the measurement.

Implementation of Nuisance Parameters

- Implement by marginalizing (Bayesian) or profiling (Frequentist)
  - One can also use a frequentist test statistics (PL) while treating the NPs via marginalization (Hybrid, Cousins & Highland way)
- Marginalization (Integrating))
  - Integrate the Likelihood, L, over possible values of nuisance parameters (weighted by their prior belief functions -- Gaussian,gamma, others...)
  - Consistent Bayesian interpretation of uncertainty on nuisance parameters

Integrating Out The Nuisance Parameters (Marginalization)

- Our degree of belief in μ is the sum of our degree of belief in μ given θ(nuisance parameter), over "all" possible values of θ
- That's a Bayesian way

$$\boxed{p(\mu\,|\,x) = \int p(x\,|\,\mu,\theta)\pi(\theta)\,d\theta = \int L(\mu,\theta)\pi(\theta)\,d\theta}$$

Credible Interval $CI = [0, \mu_{up}]$

$$\boxed{0.95 = \int_0^{\mu_{up}} p(\mu\,|\,x)\,d\mu}$$

Nuisance Parameters (Systematisc)

- Neyman Pearson (NP) Likelihood Ratio:

$$q^{NP} = -2\ln\frac{L(b(\theta))}{L(s+b(\theta))}$$

- Either Integrate the Nuisance parameters

prior

$$q^{NP}_{Hybrid} = \frac{\int L\big(s+b(\theta)\big)\pi(\theta)\,d\theta}{\int L\big(b(\theta)\big)\pi(\theta)\,d\theta}$$

Cousins & Highland

- Or profile them

$$q^{NP} = -2\ln\frac{L\left(b(\hat{\hat{\theta}}_0)\right)}{L\left(s+b(\hat{\hat{\theta}}_1)\right)}$$

$$\hat{\hat{\theta}}_0 = MLE_{\mu=0}\ of\ L(b(\theta))$$

$$\hat{\hat{\theta}}_1 = MLE_{\mu=1}\ of\ L(s+b(\theta))$$

Nuisance Parameters and Subsidiary Measurements

- Usually the nuisance parameters are auxiliary parameters and their values are constrained by auxiliary measurements
- Example

$$n \sim \mu s(m_H) + b \qquad \langle n \rangle = \mu s + b$$

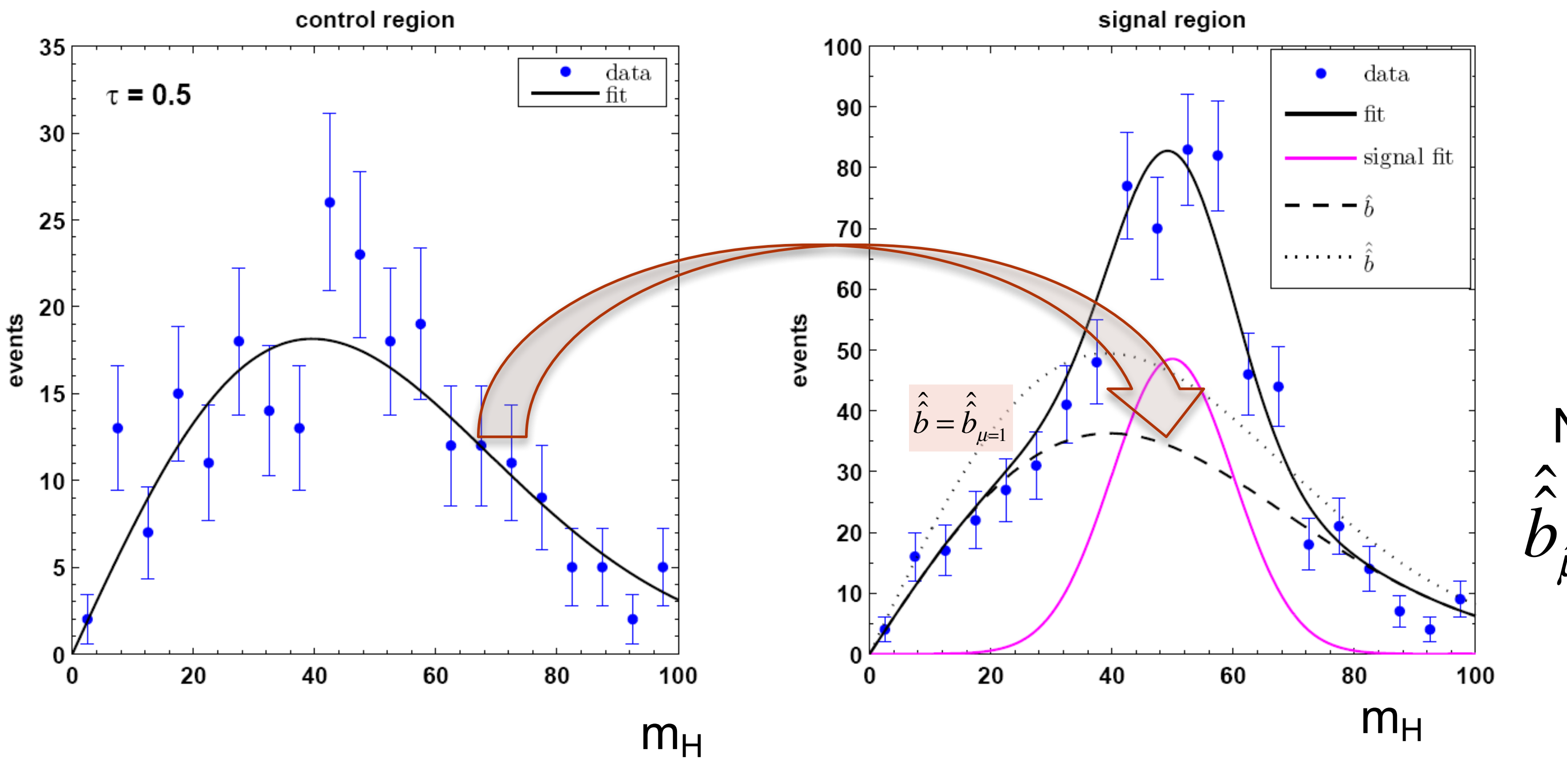
$$m = \tau b$$

$$\boxed{L\big(\mu \cdot s + b(\theta)\big) = Poisson\big(n; \mu \cdot s + b(\theta)\big) \cdot Poisson\big(m; \tau b(\theta)\big)}$$

Mass shape as a discriminator

$$n:\ \mu s(m_H)+b \qquad m \sim \tau b$$

$$L\big(\mu\cdot s+b(\theta)\big)=\prod_{i=1}^{nbins} Poisson\big(n_i;\mu\cdot s_i+b_i(\theta)\big)\cdot Poisson\big(m_i;\tau b_i(\theta)\big)$$



Note

$$\hat{\hat{b}}_{\hat{\mu}}=\hat{b}$$

Profile Likelihood with Nuisance Parameters

$$q_\mu = -2ln\frac{L(\mu s+\hat{\hat{b}}_\mu)}{L(\hat{\mu}s+\hat{b})}$$

$$q_\mu = -2ln\frac{\max_b L(\mu s+b)}{\max_{\mu,b} L(\mu s+b)}$$

$$q_\mu = q_\mu(\hat{\mu}) = -2ln\frac{L(\mu s+\hat{\hat{b}}_\mu)}{L(\hat{\mu}s+\hat{b})}$$

$\hat{\mu}$ MLE of μ

$\hat{b}$ MLE of b

$\hat{\hat{b}}_\mu$ MLE of b fixing μ

$\hat{\hat{\theta}}_\mu$ MLE of θ fixing μ

Wilks theorem in the presence of NPs

- Given n parameters of interest and any number of NPs, then

$$\boxed{\begin{gathered}\lambda(\alpha_i) \equiv \frac{L(\alpha_i, \hat{\hat{\theta}}_j)}{L(\hat{\alpha}_i, \hat{\theta}_j)} \\ q(\alpha_i) \equiv -2\log\lambda(\alpha_i) \sim \chi_n^2\end{gathered}}$$

Classification of Test Statistics

CCGV

Test Statistics	Purpose	Experession	LR
q_0	discovery of positive signal	$q_0 = \begin{cases} -2\ln\lambda(0) & \hat{\mu} \geq 0 \\ 0 & \hat{\mu} < 0 \end{cases}$	$\lambda(0) = \dfrac{L(0,\hat{\hat{\theta}}_0)}{L(\hat{\mu},\hat{\theta})}$
t_μ	2-sided measurement	$t_\mu = -2\ln\lambda(\mu)$	$\lambda(\mu) = \dfrac{L(\mu,\hat{\hat{\theta}}_\mu)}{L(\hat{\mu},\hat{\theta})}$
$\tilde{t}_\mu$	avoid negative signal (FC)	$\tilde{t}_\mu = -2\ln\tilde{\lambda}(\mu)$	$\tilde{\lambda}(\mu) = \begin{cases} \dfrac{L(\mu,\hat{\hat{\theta}}_\mu)}{L(\hat{\mu},\hat{\theta})} & \hat{\mu} \geq 0 \\ \dfrac{L(\mu,\hat{\hat{\theta}}_\mu)}{L(0,\hat{\hat{\theta}}_0)} & \hat{\mu} < 0 \end{cases}$
q_μ	exclusion	$q_\mu = \begin{cases} -2\ln\lambda(\mu) & \hat{\mu} \leq \mu \\ 0 & \hat{\mu} > \mu \end{cases}$	
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Downward fluctuations of the BG do not serve as an evidence against the BG

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Asymptotic Distributions

CCGV

$$q_{null}$$

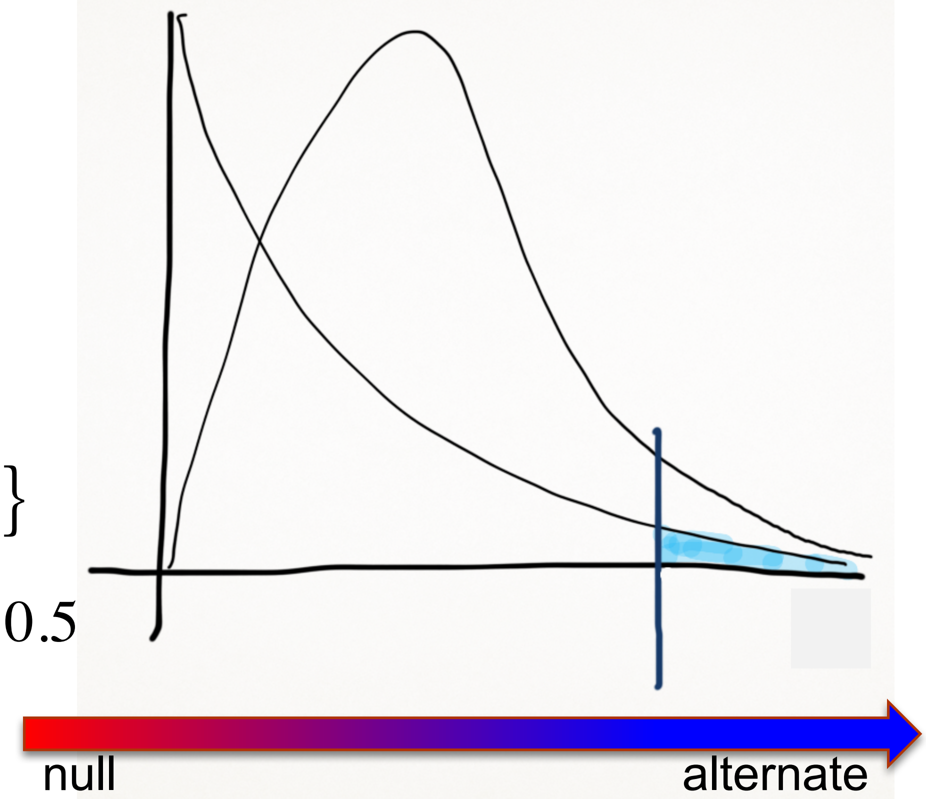
$$f(q_{null} \mid H_{null})$$

$$q_{obs} \equiv q_{null,obs}$$

$$p = \int_{q_{obs}}^{\infty} f(q_{null} \mid H_{null}) dq_{null}$$

$$f(q_{null} \mid H_{alt})$$

$$q_A \equiv q_{null,A} = \begin{array}{l} \{q \mid med\{f(q_{null} \mid H_{alt})\}\} \\ \int_{q_{null,A}}^{\infty} f(q_{null} \mid H_{null}) dq_{null} = 0.5 \end{array}$$



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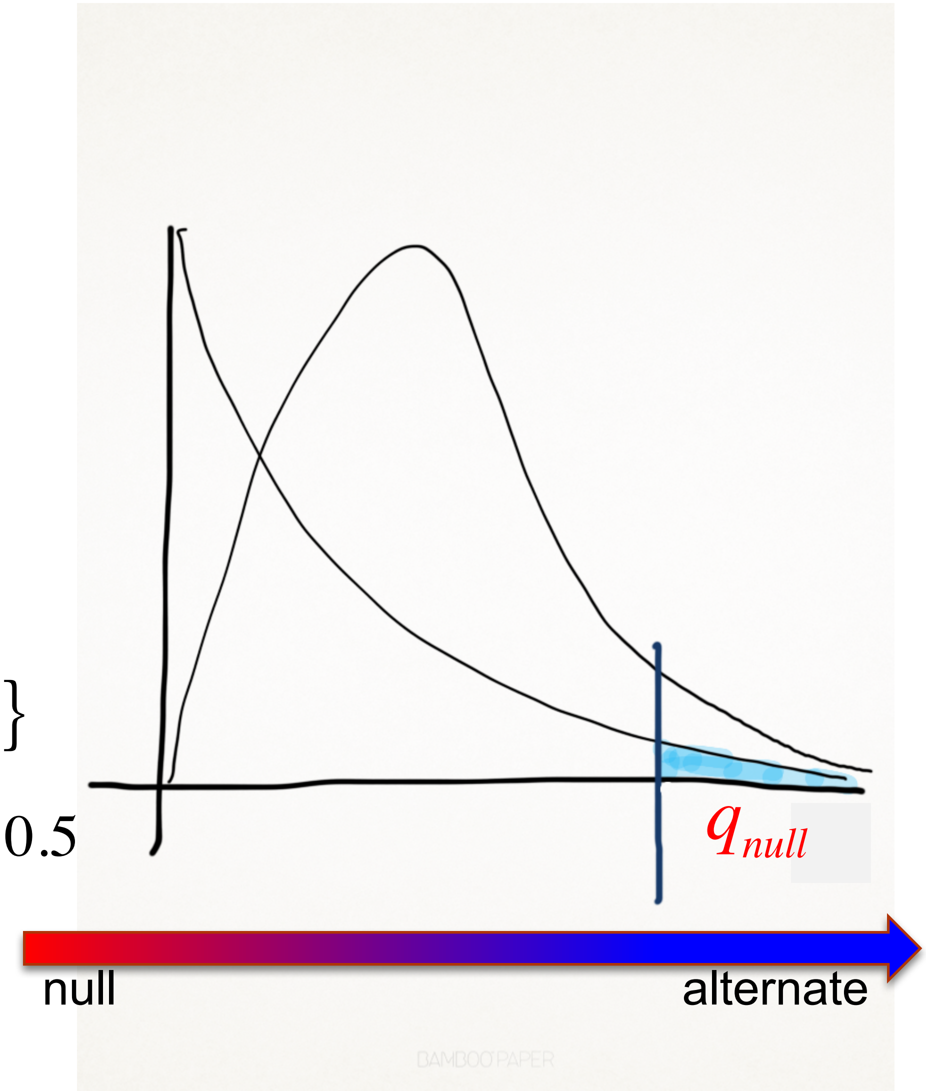

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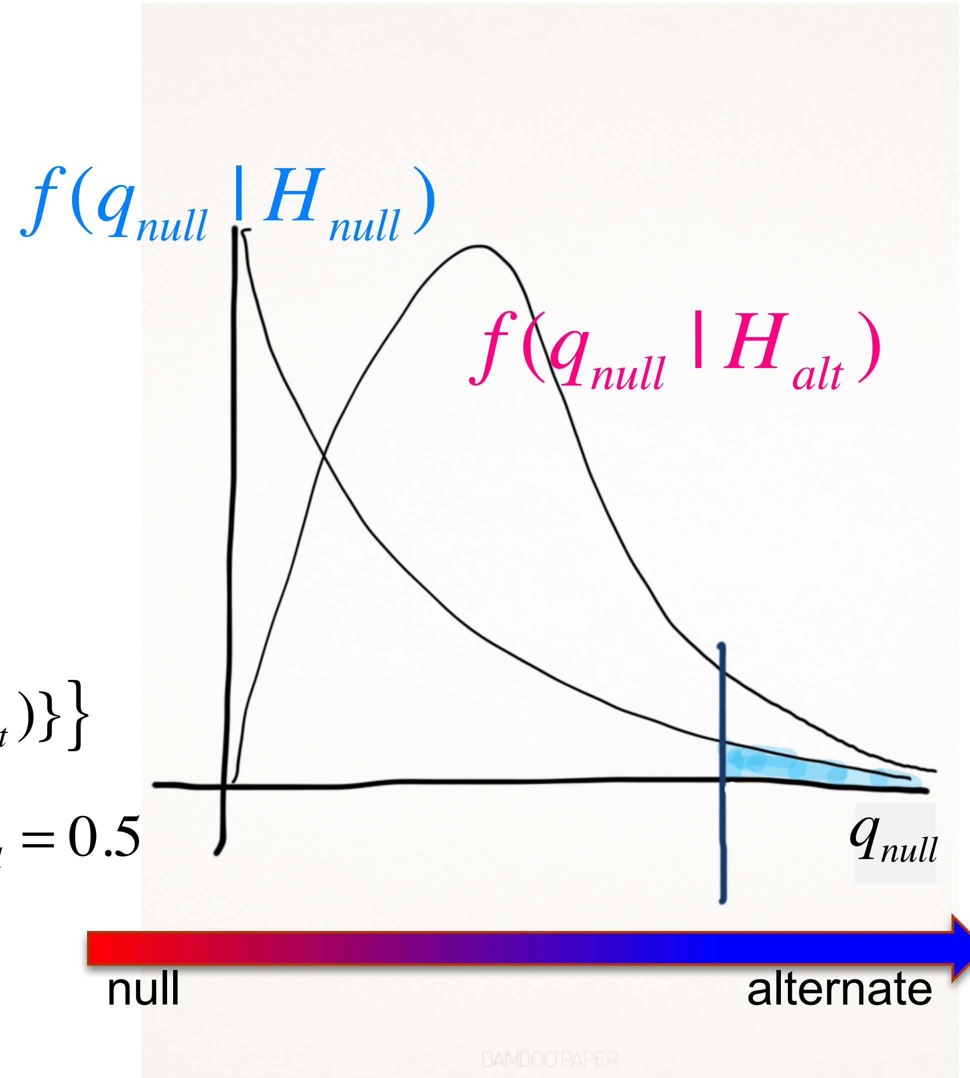

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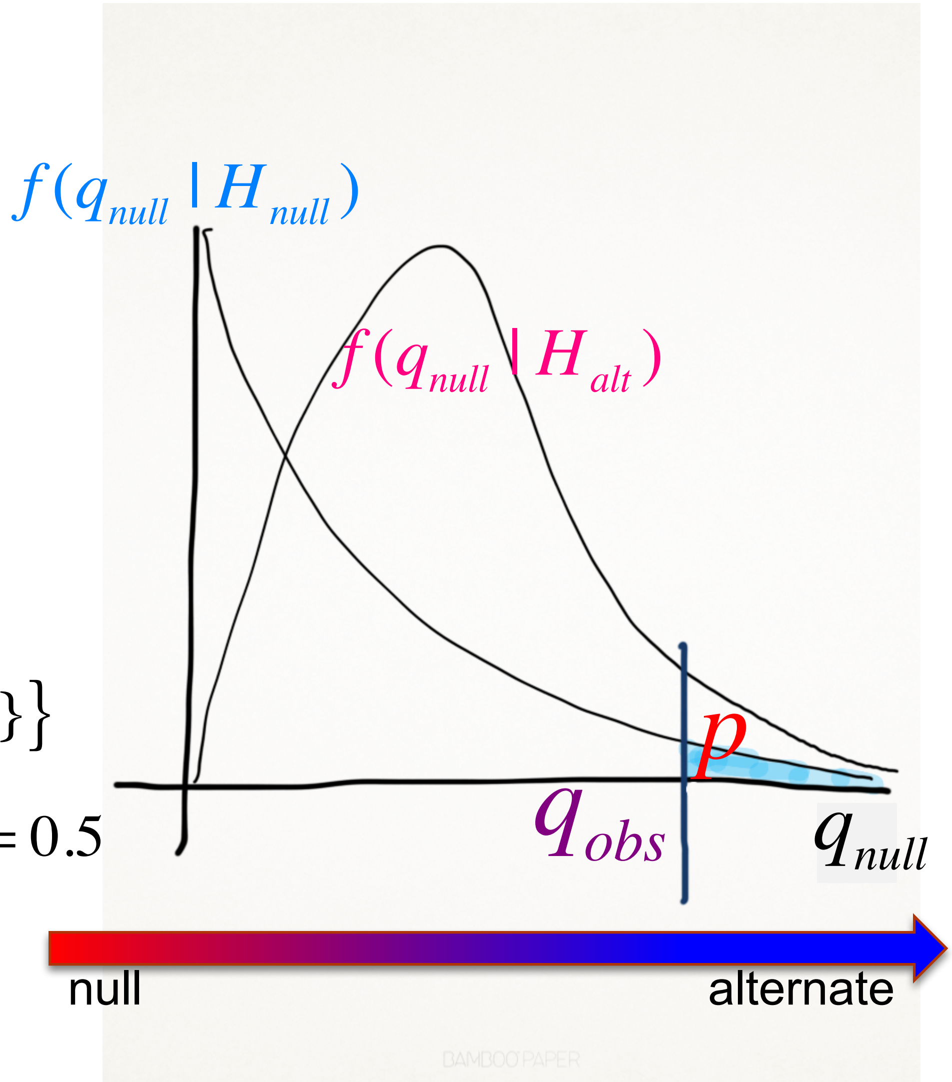

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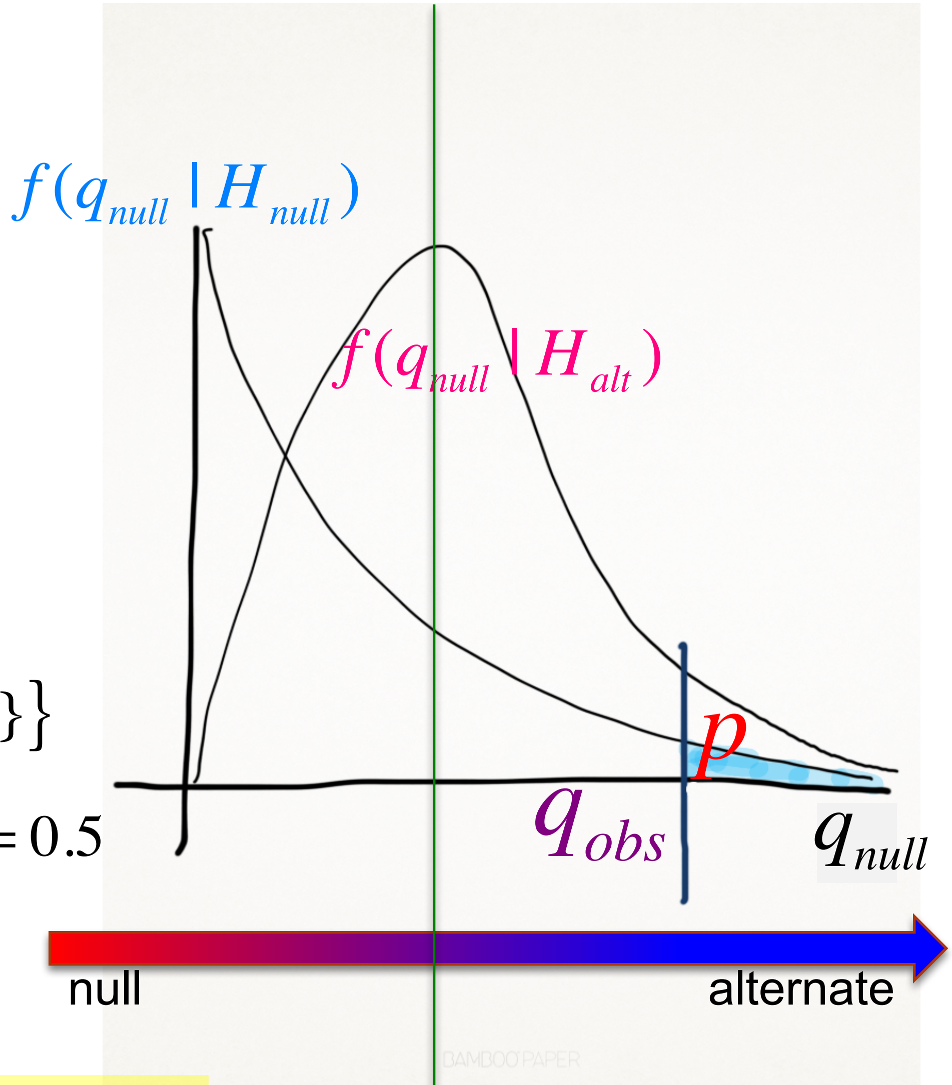


$$Z_{expected} = \sqrt{q_{null,A}}$$

$q_A \equiv q_{null,A}$

Resolving $f(q_{null} | H_{alt})$

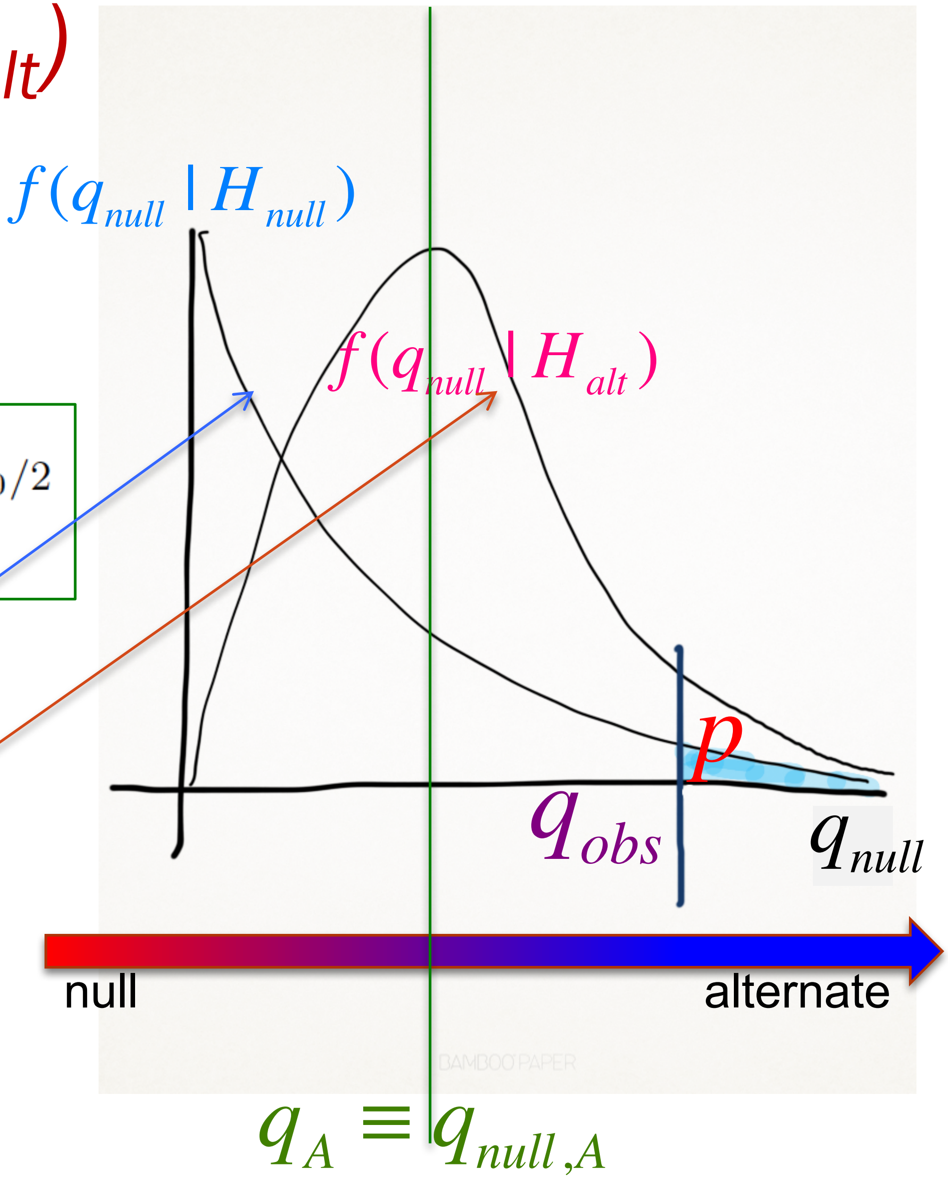
$$n = \mu s + b(\theta)$$

$$q_0 = \begin{cases} -2\ln\lambda(0) & \hat{\mu} \geq 0 \\ 0 & \hat{\mu} < 0 \end{cases}$$

$$f(q_0|0) = \frac{1}{2}\delta(q_0) + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{q_0}}e^{-q_0/2}$$

$$f(q_0 | 0) \sim \frac{1}{2}\chi^2$$

$$f(q_0 | \mu') \sim ?$$



Wald Theorem

- Consider a test of the strength parameter μ, which here can either be zero (for discovery) or nonzero (for an upper limit), and suppose the data are distributed according to a strength parameter μ′

$$\boxed{\langle \hat{\mu} \rangle = \mu'}$$

$$\lambda(\mu) = -2ln \frac{L(\mu s + \hat{\hat{b}}_{\mu})}{L(\hat{\mu} s + \hat{b})} = f(Data) = f(\hat{\mu})$$

- The desired distribution $f(q_{\mu} \mid \mu')$ can be found using a result due to Wald [1946], who showed that for the case of a single parameter of interest,

$$\boxed{-2\ln\lambda(\mu) = \frac{(\mu - \hat{\mu})^2}{\sigma^2} + \mathcal{O}(1/\sqrt{N})}$$

Wald Theorem

- Following the Wald Theorem we find that the 2-sided distributes like a non-central chi squared

$$t_\mu = -2\ln\lambda(\mu)$$

$$\boxed{\lambda(\mu) = \frac{L(\mu,\hat{\hat{\theta}}_\mu)}{L(\hat{\mu},\hat{\theta})}}$$

$$f(t_\mu;\Lambda) = \frac{1}{2\sqrt{t_\mu}}\frac{1}{\sqrt{2\pi}}\left[\exp\left(-\frac{1}{2}\left(\sqrt{t_\mu}+\sqrt{\Lambda}\right)^2\right) + \exp\left(-\frac{1}{2}\left(\sqrt{t_\mu}-\sqrt{\Lambda}\right)^2\right)\right]$$

2 sided CI

$$\Lambda = \frac{(\mu-\mu')^2}{\sigma^2}$$

μ is the tested hypothesis while $\langle\hat{\mu}\rangle = \mu'$

under H_μ, if $\mu' = \mu$

we get Wilks theorem

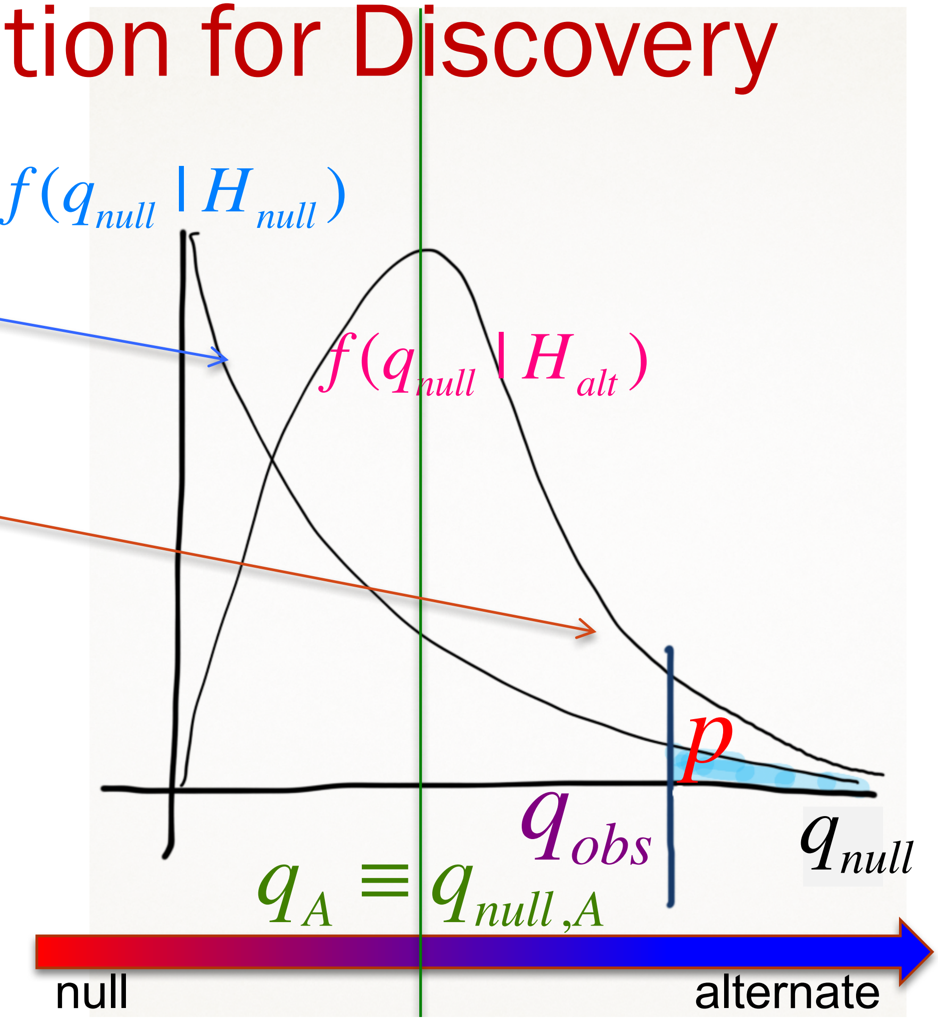
$$f(t_\mu|\mu) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t_\mu}}e^{-t_\mu/2}$$

The rediscovery of the Wald theorem helped to find the asymptotic distributions of all PL test Statistics, including the Neyman Pearson one, calculate the CLs modified p-values the expected sensitivity and save months if not years of computing

Asymptotic Distribution for Discovery

$$\boxed{f(q_0 \mid 0) \sim \frac{1}{2}\chi^2}$$

$$\boxed{f(q_0 \mid \mu') \sim ?}$$



1 sided CI

$$f(q_0|\mu') = \left(1 - \Phi\left(\frac{\mu'}{\sigma}\right)\right)\delta(q_0) + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{q_0}}\exp\left[-\frac{1}{2}\left(\sqrt{q_0} - \frac{\mu'}{\sigma}\right)^2\right]$$

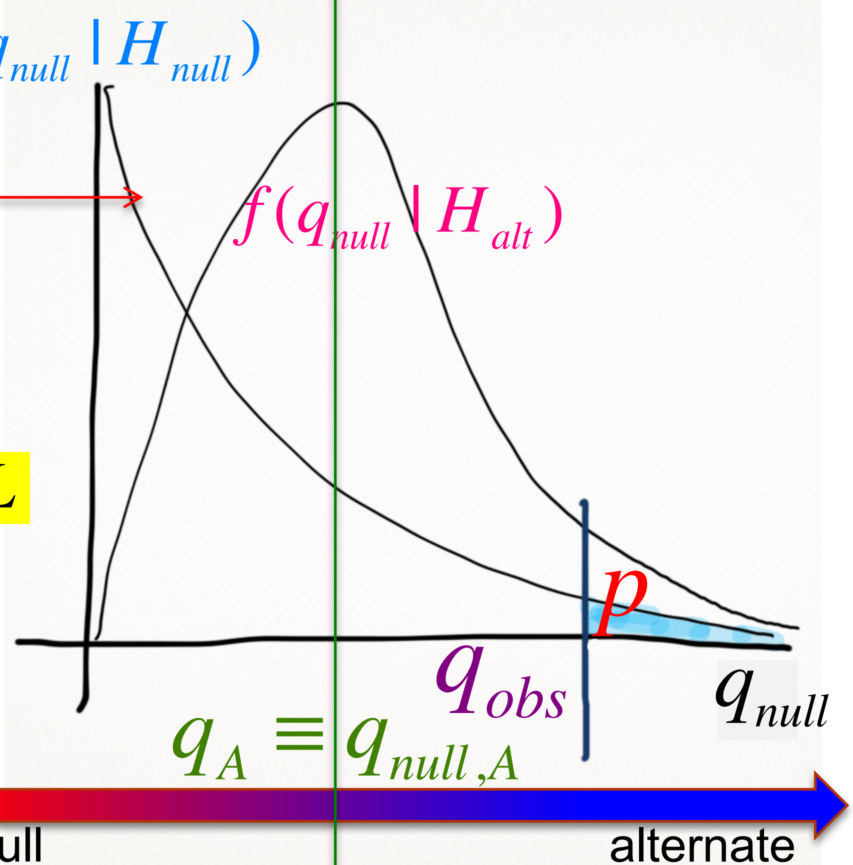
$$\Phi(Z) = 1 - \int_Z^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Asymptotic Distribution for Exclusion

$$f(q_\mu|\mu) = \frac{1}{2}\delta(q_\mu) + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{q_\mu}}e^{-q_\mu/2}$$

$$\mu_{\rm up} = \hat{\mu} + \sigma\Phi^{-1}(1-\alpha)$$

$\alpha = 1 - CL$



1 sided CI

$$f(q_\mu|\mu') = \Phi\left(\frac{\mu'-\mu}{\sigma}\right)\delta(q_\mu) + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{q_\mu}}\exp\left[-\frac{1}{2}\left(\sqrt{q_\mu} - \frac{\mu-\mu'}{\sigma}\right)^2\right]$$

$$\Phi(Z) = 1 - \int_Z^\infty \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$$

Asymptotic Distribution for FC

Depends on the observation
one might get 1-sided or 2-sided CI

3.4 Distribution of $\tilde{t}_\mu$

Assuming the Wald approximation, the statistic $\tilde{t}_\mu$ as defined by Eq. (11) can be written

$$\tilde{t}_\mu = \begin{cases} \frac{\mu^2}{\sigma^2} - \frac{2\mu\hat{\mu}}{\sigma^2} & \hat{\mu} < 0 \,, \\ \frac{(\mu-\hat{\mu})^2}{\sigma^2} & \hat{\mu} \geq 0 \,. \end{cases} \tag{40}$$

From this the pdf $f(\tilde{t}_\mu|\mu')$ is found to be

$$f(\tilde{t}_\mu|\mu') = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\tilde{t}_\mu}}\exp\left[-\frac{1}{2}\left(\sqrt{\tilde{t}_\mu}+\frac{\mu-\mu'}{\sigma}\right)^2\right] \tag{41}$$

$$+ \begin{cases} \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\tilde{t}_\mu}}\exp\left[-\frac{1}{2}\left(\sqrt{\tilde{t}_\mu}-\frac{\mu-\mu'}{\sigma}\right)^2\right] & \tilde{t}_\mu \leq \mu^2/\sigma^2 \,, \\ \frac{1}{\sqrt{2\pi}(2\mu/\sigma)}\exp\left[-\frac{1}{2}\frac{\left(\tilde{t}_\mu-\frac{\mu^2-2\mu\mu'}{\sigma^2}\right)^2}{(2\mu/\sigma)^2}\right] & \tilde{t}_\mu > \mu^2/\sigma^2 \end{cases} \,. \tag{42}$$

The special case $\mu = \mu'$ is therefore

$$f(\tilde{t}_\mu|\mu') = \begin{cases} \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\tilde{t}_\mu}}e^{-\tilde{t}_\mu/2} & \tilde{t}_\mu \leq \mu^2/\sigma^2 \,, \\ \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\tilde{t}_\mu}}e^{-\tilde{t}_\mu/2} + \frac{1}{\sqrt{2\pi}(2\mu/\sigma)}\exp\left[-\frac{1}{2}\frac{(\tilde{t}_\mu+\mu^2/\sigma^2)^2}{(2\mu/\sigma)^2}\right] & \tilde{t}_\mu > \mu^2/\sigma^2 \,. \end{cases} \tag{43}$$

How to determine σ

- To estimate the uncertainty σ there are a few possibilities
  - Given the asymptotic formulae, fit the distribution of

$$f(q_{null} \,|\, H_{alt}) = f(q_{\mu} \,|\, \mu')$$

and extract σ

  - Implement the Wald formula to the Asimov data set and find

$$\sigma_A^2 = \frac{(\mu - \mu')^2}{q_{\mu,A}}$$

where μ is the tested (null) hypothesis and μ' is the alt hypothesis. For discovery, μ=0 while for exclusion μ'=0.

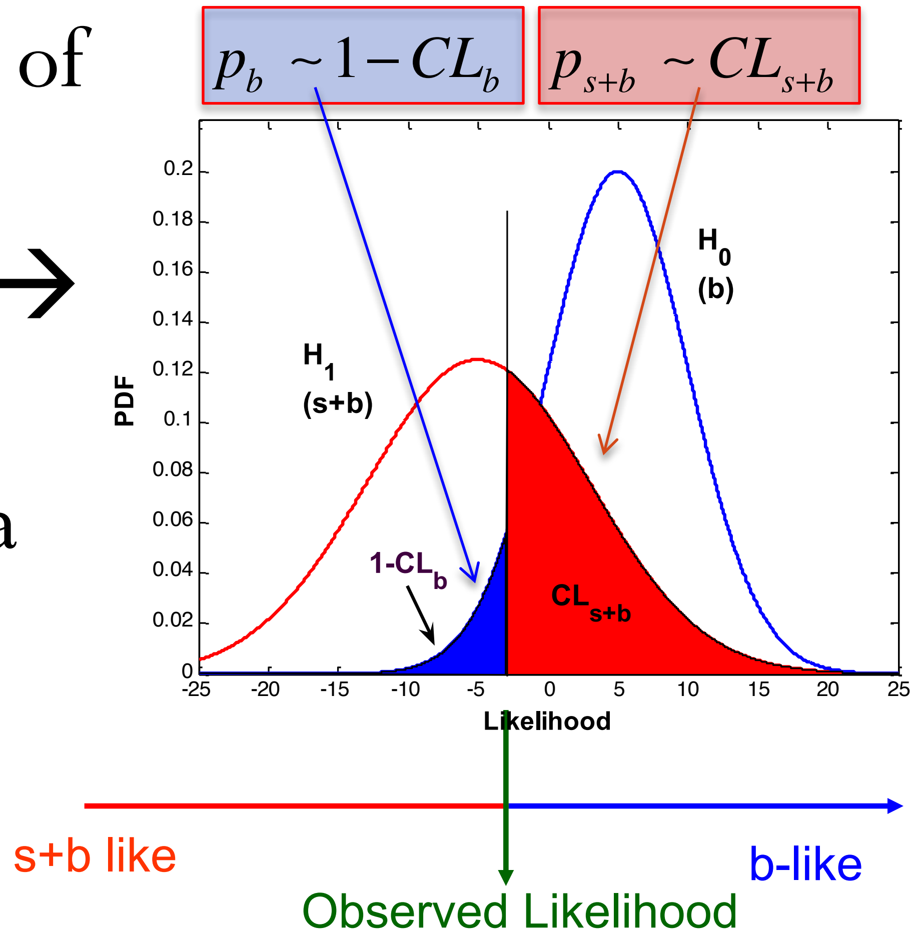
Exclusion and CLs

Case Study:

Exclusion of a Higgs with mass m_H

q_{NP} CL~p equivalence

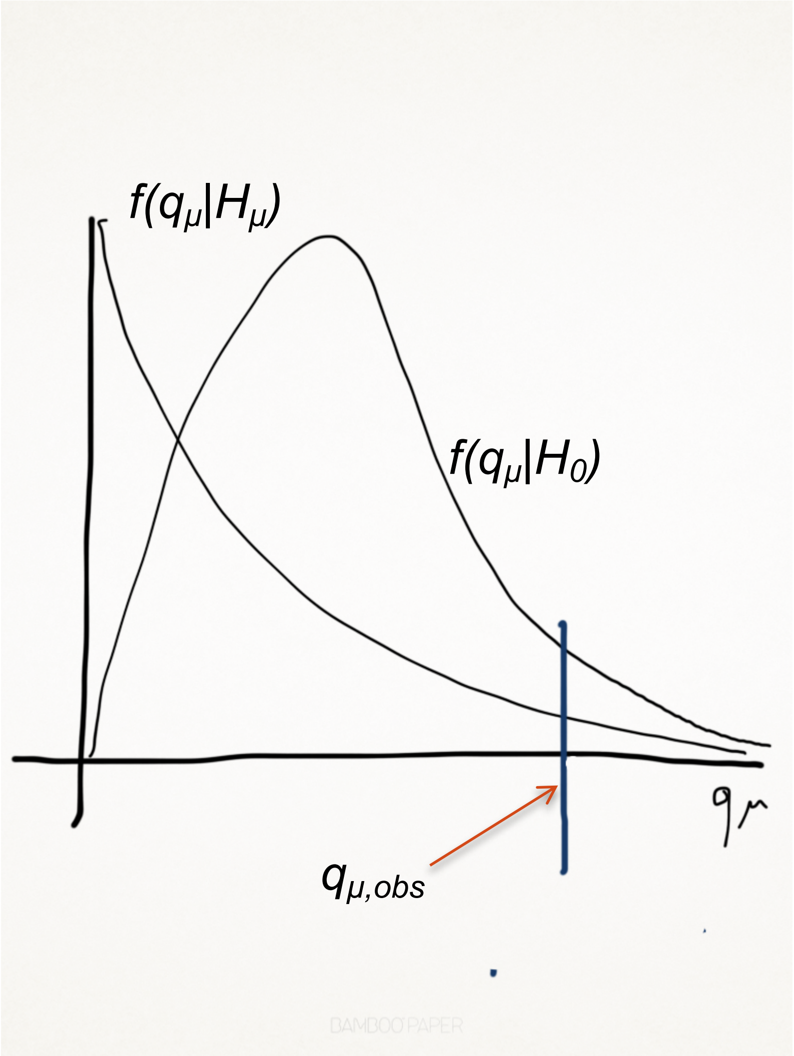
P_b is the incompatibility of the data with the background hypothesis → $CL_b \sim 1 - p_b$ is the compatibility of the data with the background hypothesis



March 2017

PL test statistics

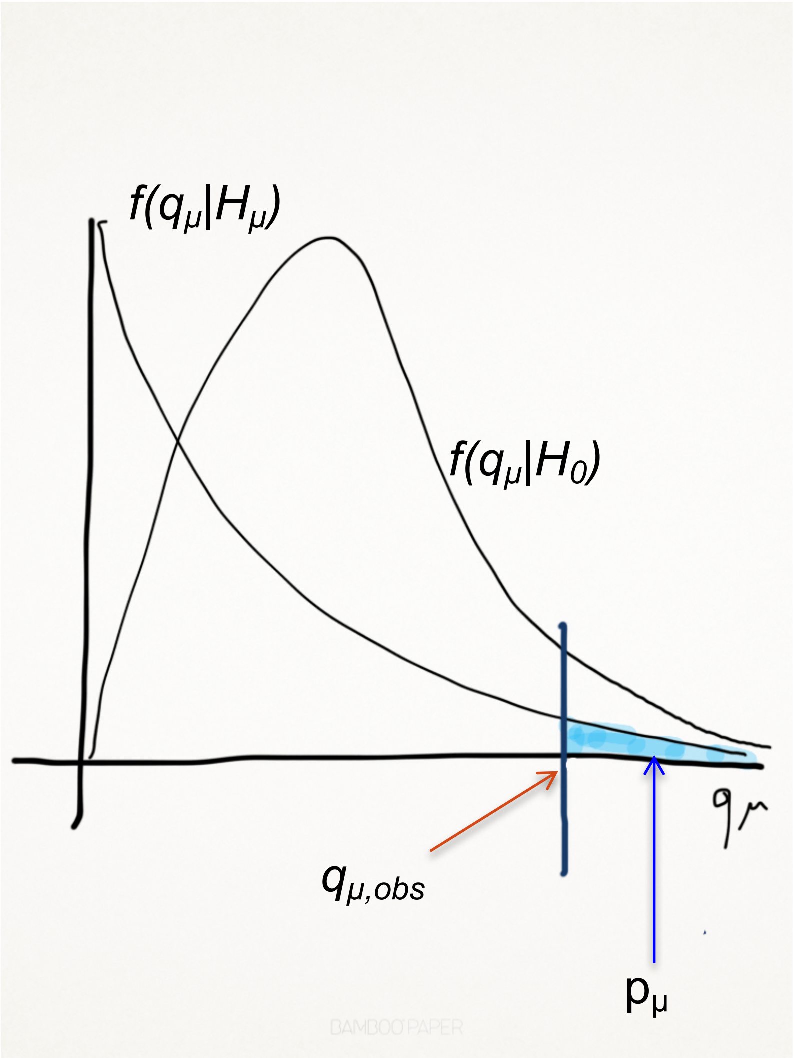
- We test hypothesis H_μ
- We calculate the PL (profile likelihood) ratio with the one observed data
- $q_{\mu,obs}$



- Find the p-value of the signal hypothesis H_μ

$$p_\mu = \int_{q_{\mu,obs}}^{\infty} f(q_\mu \mid H_\mu)\, dq_\mu$$

- In principle if $p_\mu<5\%$, H_μ hypothesis is excluded at the 95% CL
- Note that H_μ is for a given Higgs mass m_H



CLs

- Suppose $<n_b>=100$
- $s(m_{H1})=30$
- Suppose $n_{obs}=102$
- $s+b=130$
- $Prob(n_{obs}\le 102\,|\,130)<5\%$, m_{H1} is excluded at >95% CL

- Now suppose $s(m_{H2})=1$, can we exclude m_{H2}?
- Suppose $n_{obs}=80$, $prob(n_{obs}\le 80\,|\,101)<5\%$, it looks like we can exclude m_{H2}…
  but this is dangerous, because what we exclude is $(s(m_{H2})+b)$ and not s……
- With this logic we could also exclude b (expected b=100)
- To protect we calculate a modified p-value
- We cannot exclude m_{H2}

$$\frac{P(n\le n_o\,|\,s+b)}{P(n\le n_o\,|\,b)}=\frac{\mathrm{Prob}(\mathrm{nobs}\le 80\,|\,101)}{\mathrm{Prob}(\mathrm{nobs}\le 80\,|\,100)}\sim 1$$

CLs – the original derivation

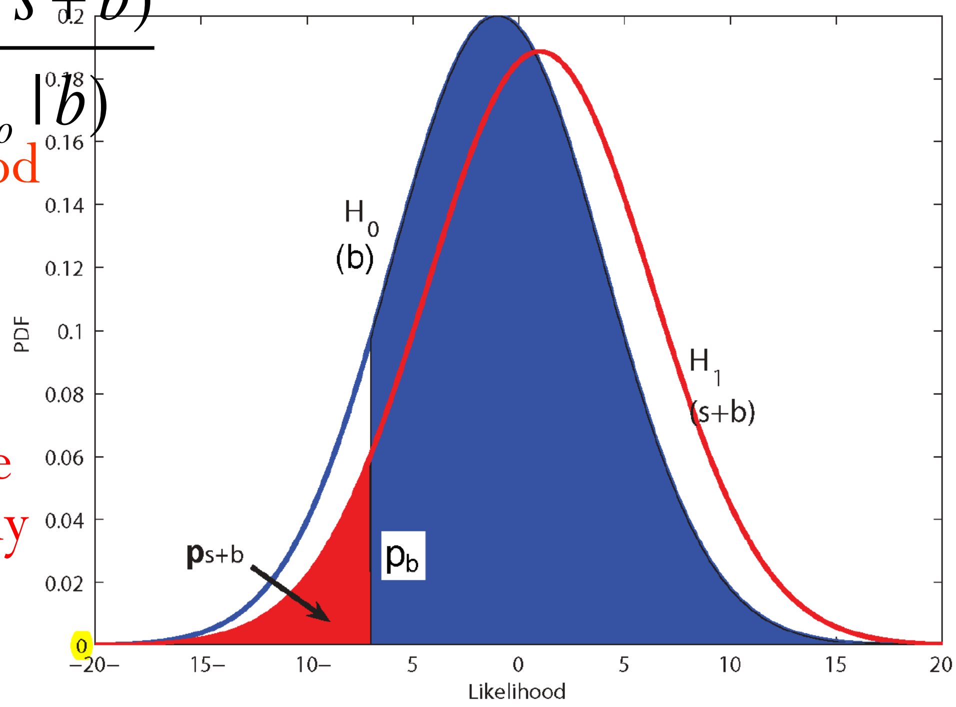
- Inspired by Zech derivation for counting experiments

$$P(n \le n_o | n_b \le n_o, s+b) = \frac{P(n \le n_o \mid s+b)}{P(n \le n_o \mid b)}$$

- A. Read suggested the $\mathrm{CL_s}$ method with

$$\boxed{CL_s = \frac{CL_{s+b}}{CL_b} = \frac{p_{s+b}}{1-p_b}}$$

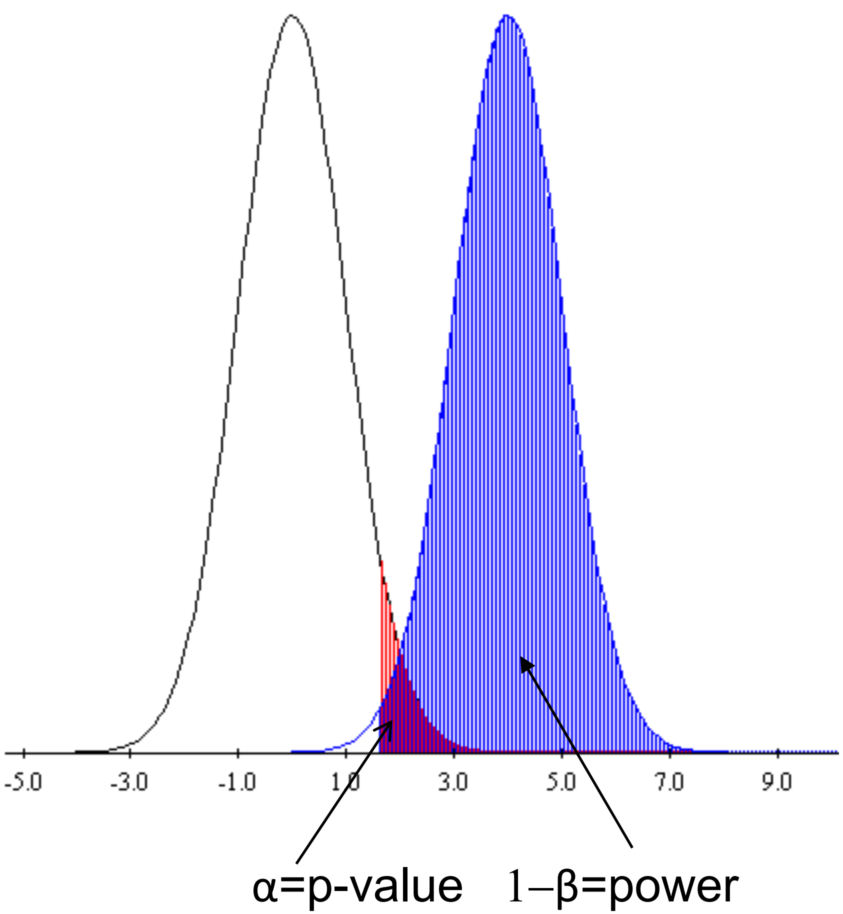
- This means that you will never be able to exclude a signal with a tiny cross section (to which you are not sensitive)



CLs Birenbaum 1962

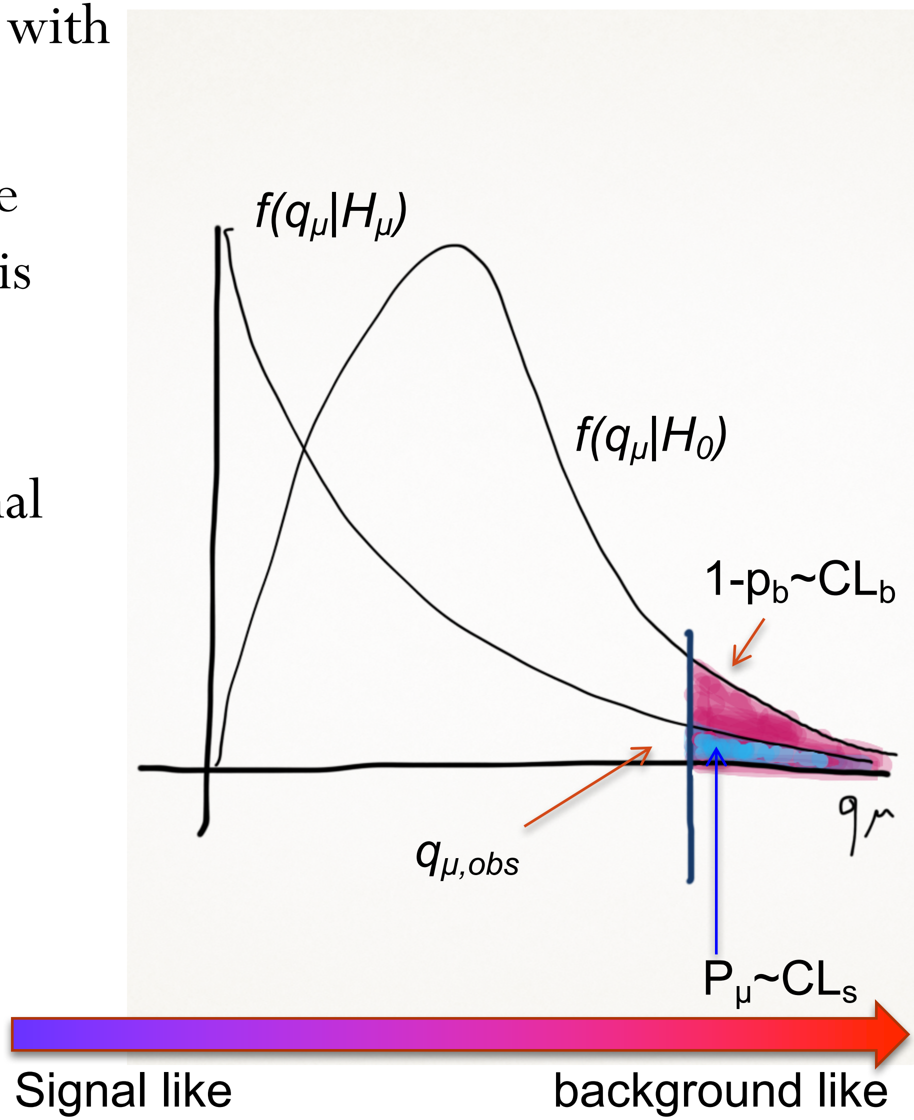
- Birnbaum suggested in 1962 that the the {p-value} / {power} should be used as a measure of the strength of statistical evidence provided by significance tests, rather than the {p-value} alone
- This translates into using a modified {p-value}

$$p'_s = \frac{p_{s+b}}{1 - p_b}$$



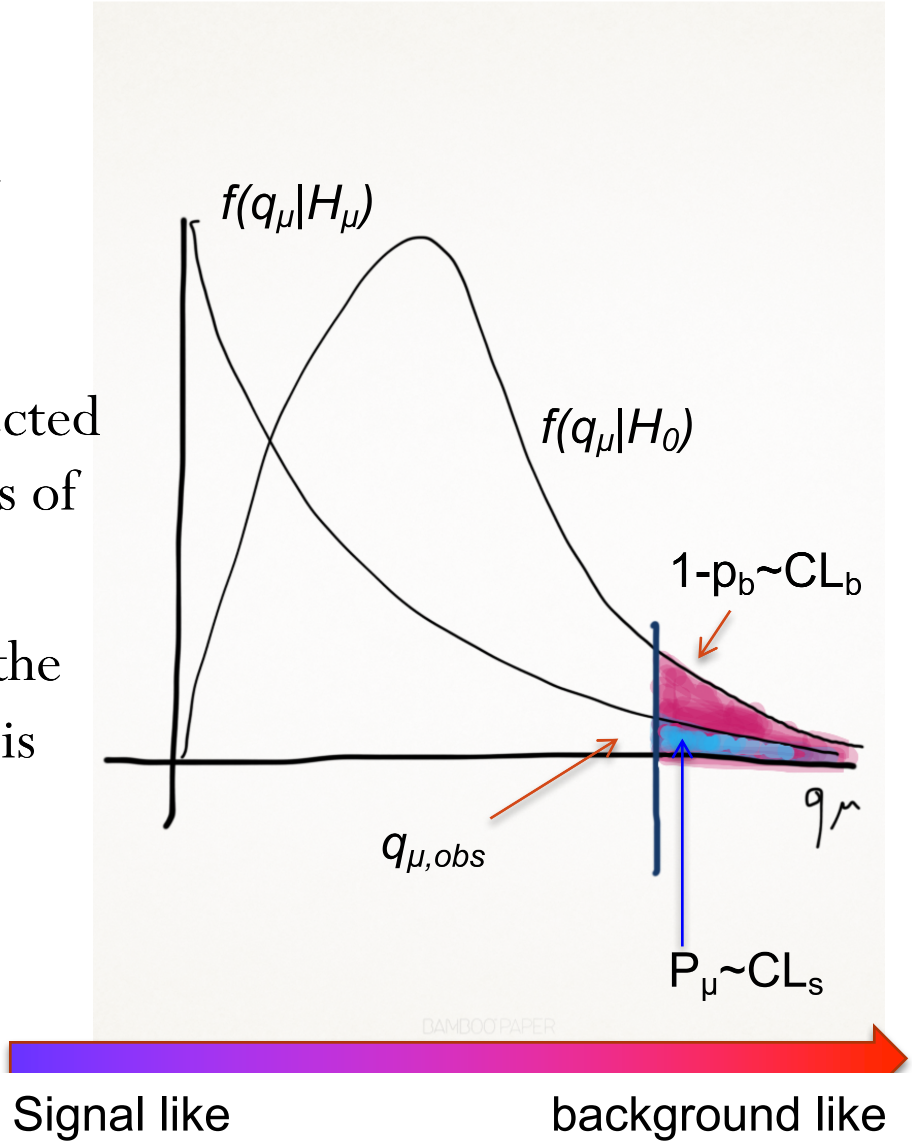
Second Verse- Same as the First (PL) CLb

- P_b is the incompatibility of the data with the background hypothesis →
- $CL_b \sim 1-p_b$ is the compatibility of the data with the background hypothesis and might be very small due to downward fluctuations of the background in the absence of a signal



CLs

- A complication arises when $\mu s+b \sim b$
- When the signal cross section is very small the $s(m_H)+b$ hypothesis can be rejected but at the same time the background hypothesis is almost rejected as well due to downward fluctuations of the background
- These downward fluctuations allow the exclusion of a signal the experiment is not sensitive to



The Modified CLs with the PL test statistic

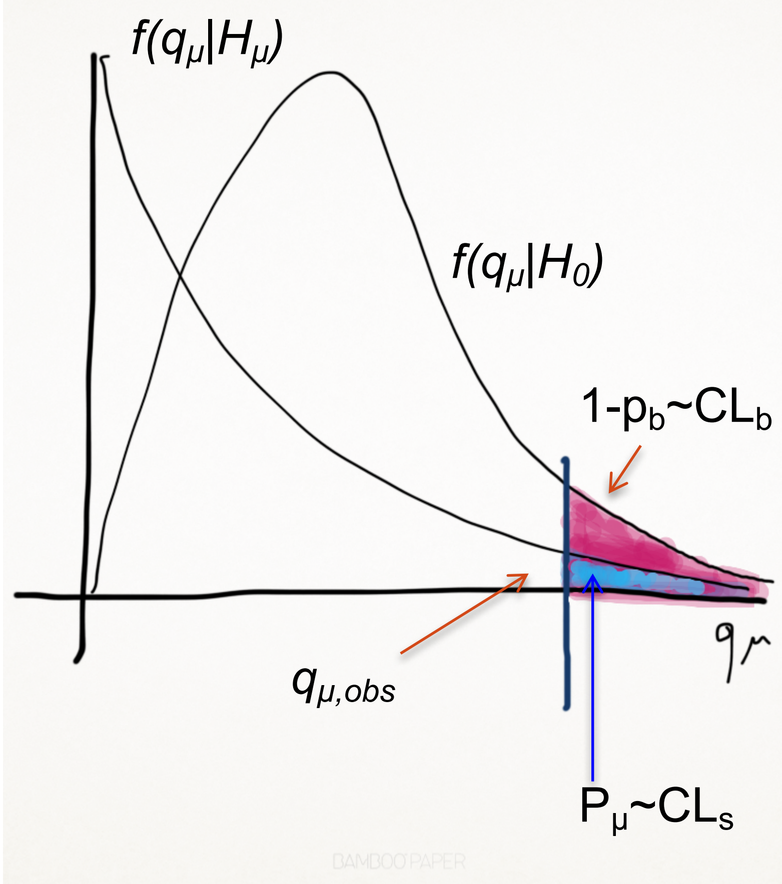
- The CLs method means that the signal hypothesis p-value

p_μ is modified to

$$p_\mu \to p'_\mu = \frac{p_\mu}{1-p_b}$$

$$p_\mu = \int_{\tilde{q}_{\mu,obs}}^{\infty} f(\tilde{q}_\mu|\mu)d\tilde{q}_\mu$$

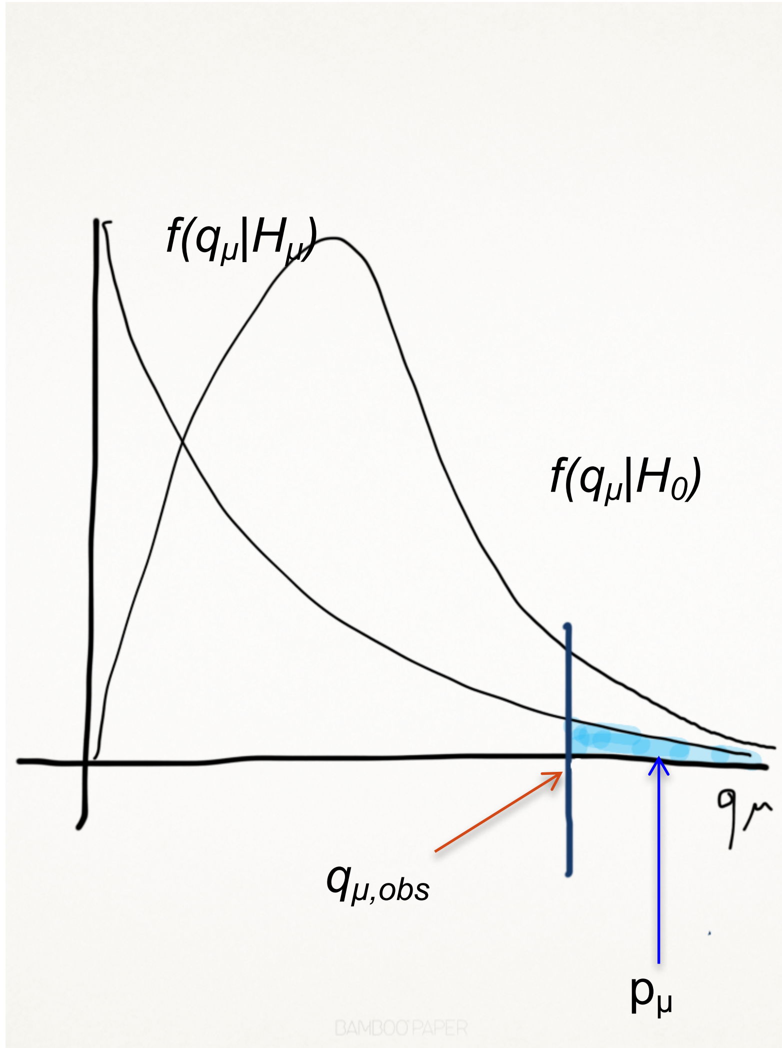
$$p_b = 1 - \int_{\tilde{q}_{\mu,obs}}^{\infty} f(\tilde{q}_\mu|0)d\tilde{q}_\mu$$



- Find the p-value of the signal hypothesis H_μ

$$p_\mu = \int_{q_{\mu,obs}}^{\infty} f(q_\mu \mid H_\mu) dq_\mu$$

- In principle if $p_\mu < 5\%$, H_μ hypothesis is excluded at the 95% CL
- Note that H_μ is for a given Higgs mass m_H



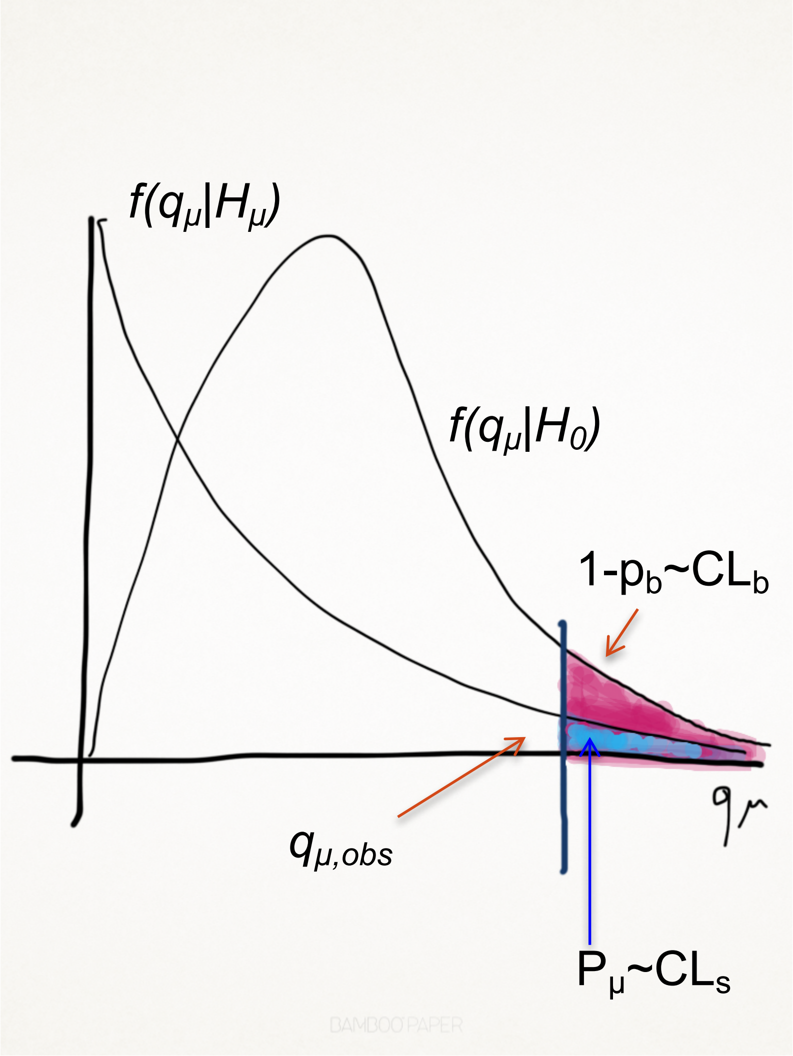
- Find the p-value of the signal hypothesis H_μ

- Find the modified p-value

$$p'_\mu(m_H) = \frac{p_\mu}{1-p_b}$$

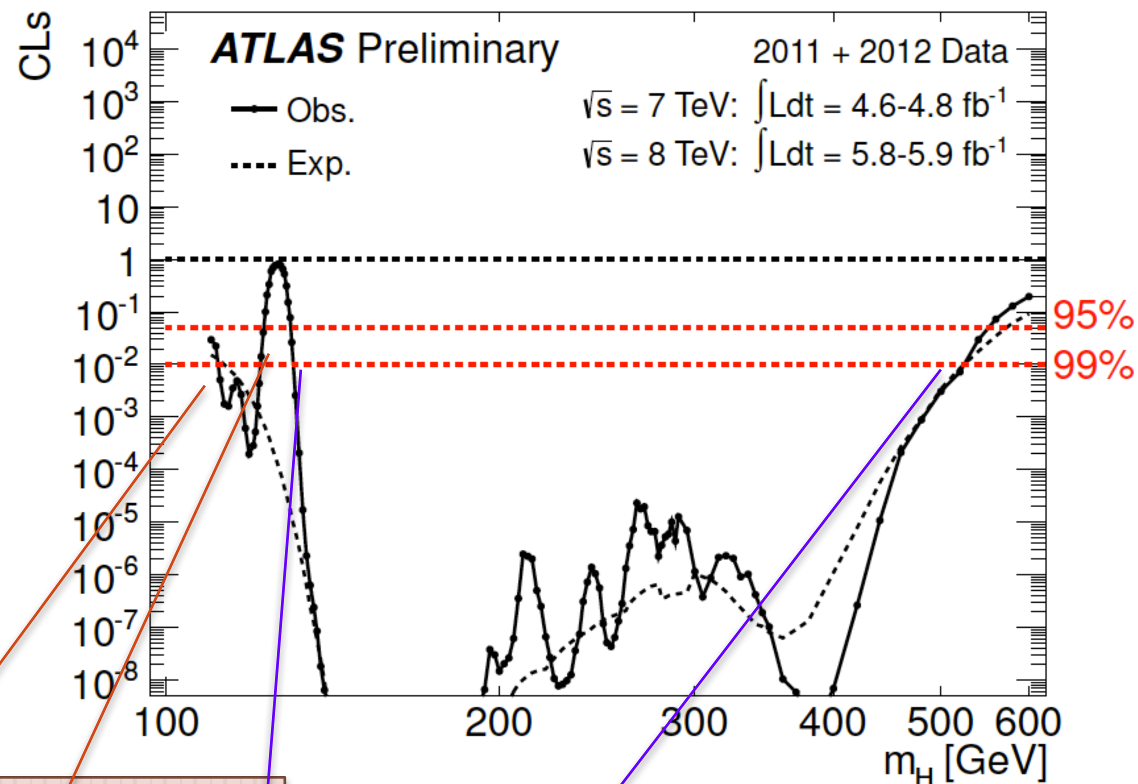
  - To tell if s is excluded, set $\mu=1$ and find

$$p'_1(m_H) = \frac{p_\mu}{1-p_b} \equiv CLs(m_H)$$



Understanding the CLs plot

- Here, for each Higgs mass m_H, one finds the observed p'_s value, i.e. $p'_{\mu},\mu=1$
- This modified p-value, p'_s, is by definition CLs

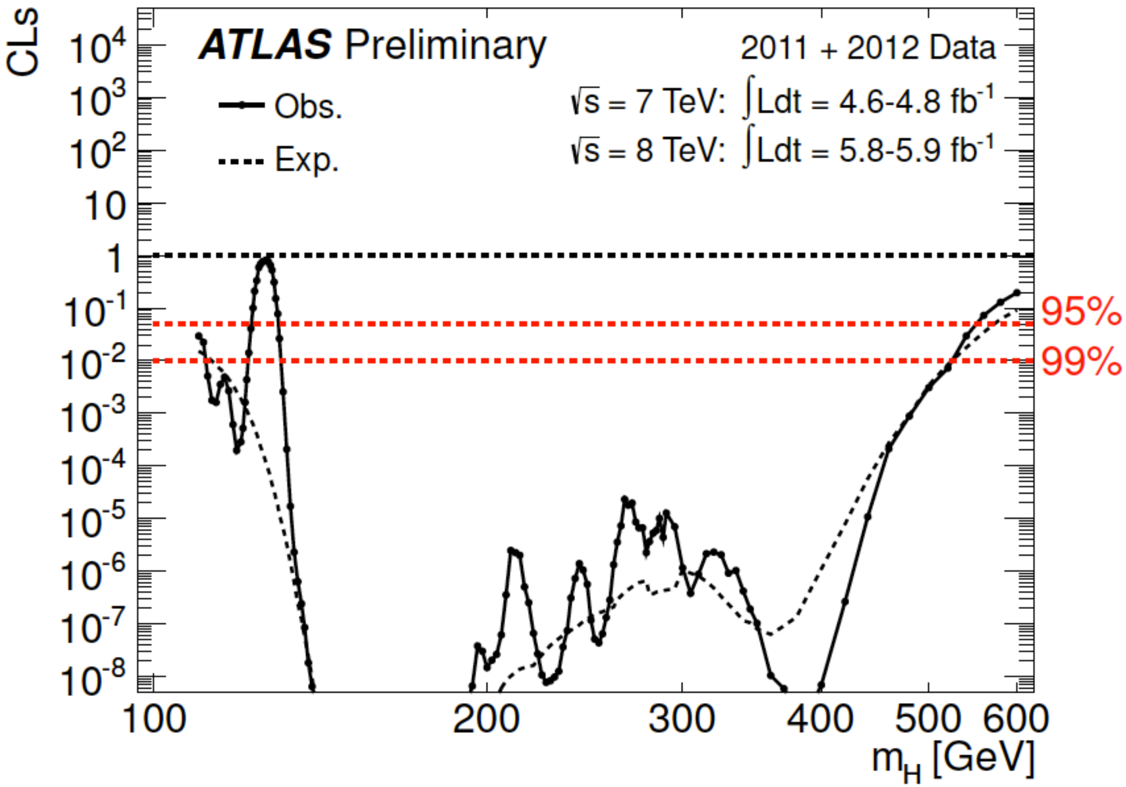


The smaller CLs, the deeper is the exclusion,
Exclusion CL=1-CLs=1-p'_s

to the previous combined search [1]. Figure 2 shows the CL_s values for $\mu = 1$, where it can be seen that the regions between 111.7 GeV to 121.8 GeV and 130.7 GeV to 523 GeV are excluded at the 99% CL.

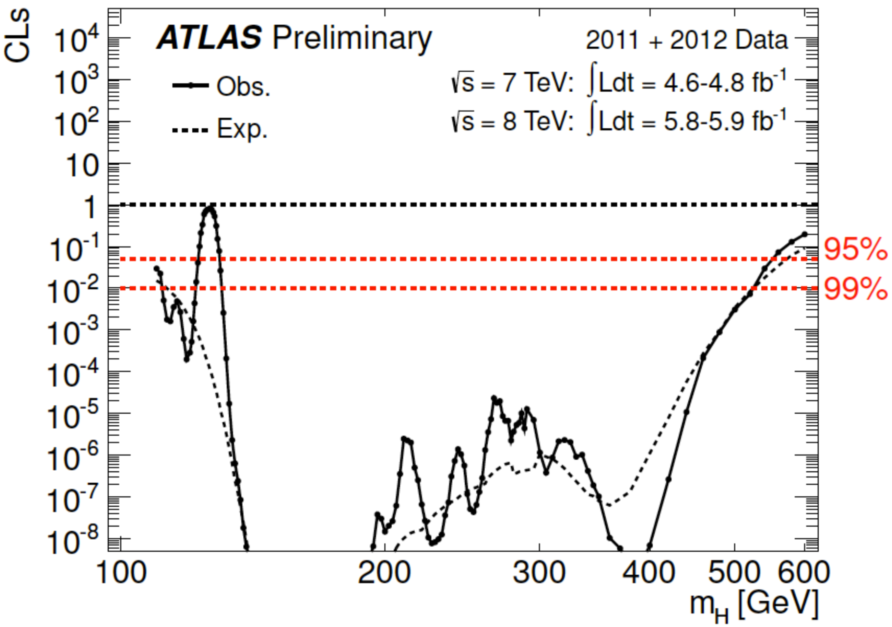
Understanding the CLs plot

- CLs is the compatibility of the data with the signal hypothesis
- The smaller the CLs, the less compatible the data with the prospective signal



- Find the p-value of the signal h... H_μ

$$p_\mu = \int_{q_{\mu,obs}}^{\infty} f(q_\mu \mid H_\mu)\, dq_\mu$$



- Find the modified p-value

$$p'_\mu(m_H) = \frac{p_\mu}{1 - p_b}$$

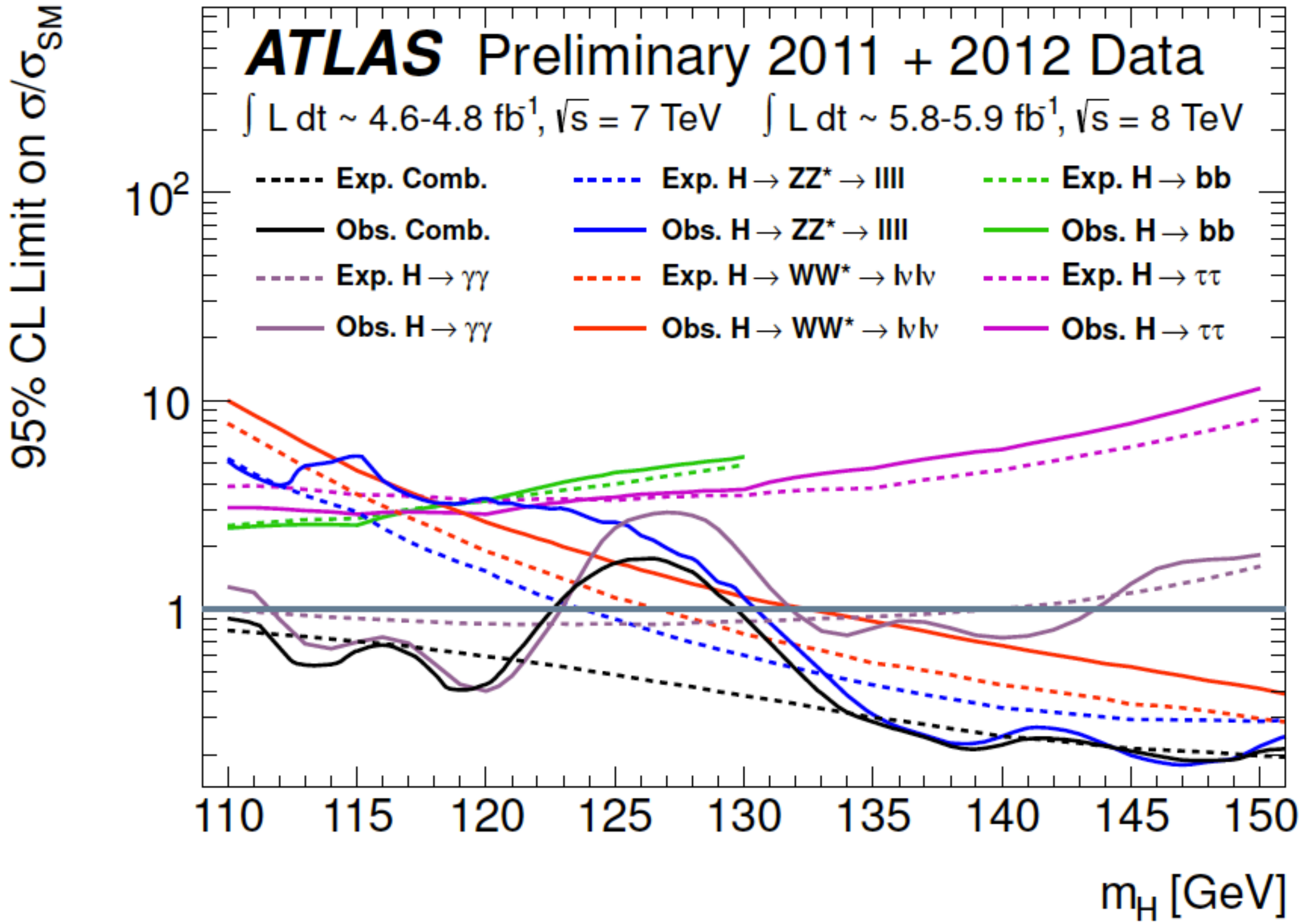
  - Option2: Iterate and find μ for which $p'_\mu(m_H)=5\%$ → μ=μup →
    If μup<1, m_H is excluded at the 95%

For a given data set, in the absence of a signal, the bigger the tested μ is the exclusion is deeper i.e. p'_μ is smaller

Exclusion a Higgs with a mass m_H

- First we fix the hypothesized mass to m_H
- We then test the H_μ $[\mu s(m_H)+b]$ hypothesis
- We find μ_{up}, for which $p'_{\mu up}=5\%$-> the $H_{\mu up}$ hypothesis is rejected at the 95% CL
- This means that the Confidence Interval of μ is $\mu \in [0,\mu_{up}]$
- If $\mu_{up}=\sigma(mH)/\sigma SM(mH)<1$, we claim that a SM Higgs with a mass m_H is excluded at the 95% CL
- A SM Higgs with a mass m_H, such that $\mu(m_H)<1$ is excluded at the 95% CL

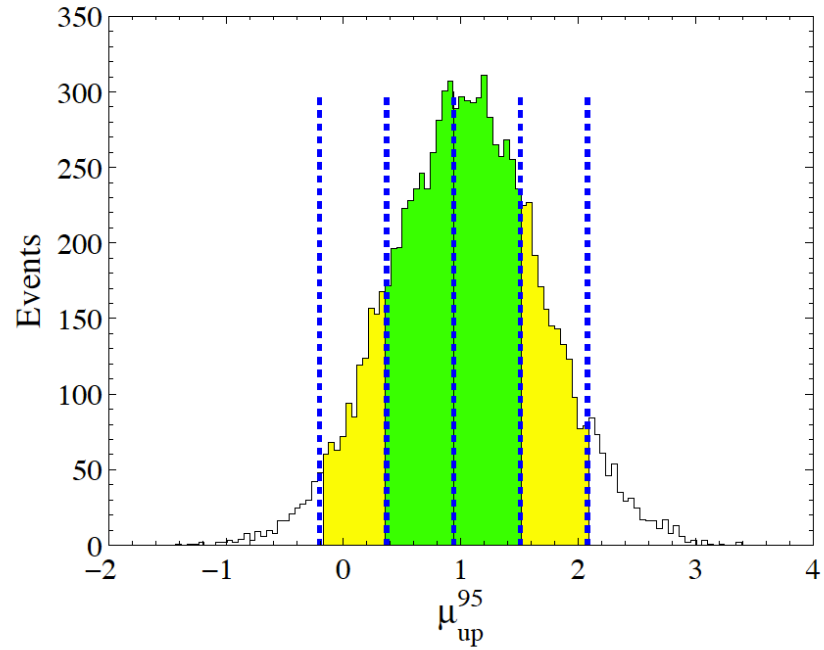
Upper Limit – $\mu_{up}(m_H)$



Sensitivity

- The sensitivity of an experiment to exclude a Higgs with a mass m_H is the median upper limit
- $$\mu_{up}^{med} = med\{\mu_{up} \mid H_0\}$$
- The 68% (green) and 95% (yellow) are the
  1 and 2 σ bands
- The median and the bands can be derived with the Asimov background only dataset n=b

Distribution of the upper limit with background only experiments



The Asimov data set is n=b
-> median upper limit

CCGV Useful Formulae – The Bands

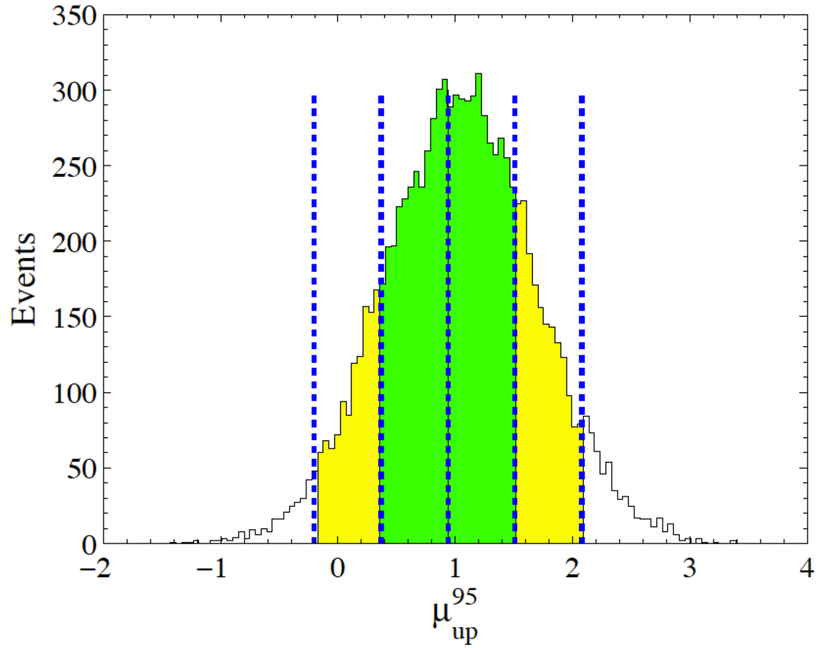
$$\mu_{up}^{med} = \hat{\mu} + \sigma_{\mu_{up}^{med}} \phi^{-1}(1-\alpha)$$

$$\alpha = 0.05 \rightarrow \phi^{-1}(1-\alpha) = \phi^{-1}(0.95) = 1.64$$

$$\mu_{up+N\sigma} = \hat{\mu} + \sigma_{up+N\sigma}\left(\phi^{-1}(1-\alpha) + N\right)$$

$$\sigma^2_{up+N\sigma} = \frac{\mu^2_{up+N\sigma}}{q_{up+N\sigma,A}}$$

Distribution of the upper limit with background only experiments



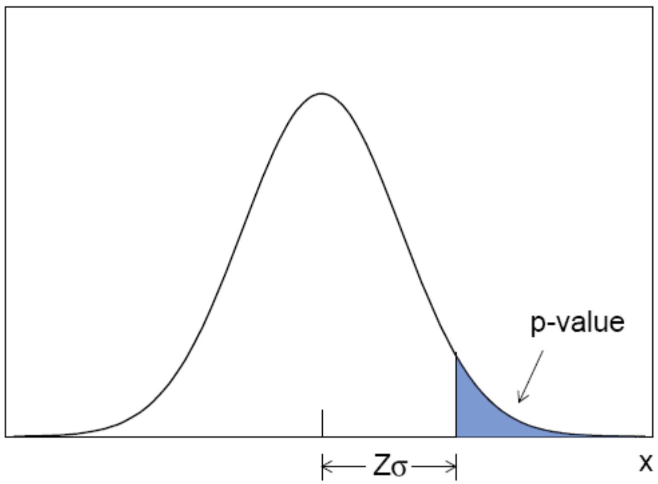
The Asimov data set is n=b
-> median upper limit

Useful Formulae

$$p'_{\mu_{95}} = \frac{1-\Phi(\sqrt{q_{\mu_{95}}})}{\Phi(\sqrt{q_{\mu_{95},A}} - \sqrt{q_{\mu_{95}}})} = 0.05$$

Φ is the cumulative distribution of the standard (zero mean, unit variance) Gaussian.

$q_{\mu_{95},A}$ Is evaluated with the Asimov data set (background only)



$$p = \int_Z^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = 1 - \Phi(Z)$$

$$Z = \Phi^{-1}(1-p)$$

Understanding the Brazil Plot

The expected 95% CL exclusion region covers the m_H range from 110 GeV to 582 GeV. The observed 95% CL exclusion regions are from 110 GeV to 122.6 GeV and 129.7 GeV to 558 GeV. The addition of

- $\mu_{up}=\sigma(m_H)/\sigma_{SM}(m_H)<1 \rightarrow \sigma(m_H)<\sigma_{SM}(m_H) \rightarrow$ SM m_H excluded
- The line $\mu up=1$ corresponds to CLs=5% (p'_s=5%)
- The smaller $\mu up<1$ is, the exclusion of a SM Higgs is deeper$\rightarrow p's<5\%$, $p's=CLs \rightarrow CL=1-p's>95\%$