Theory of hard LHC collisions for pedestrians

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European School on High Energy Physics, Portugal, September 2017

Outline

1) Introduction

2) Drell-Yan process at leading and next-to-leading orders in perturbative QCD

3) Transverse momentum distribution of a lepton pair

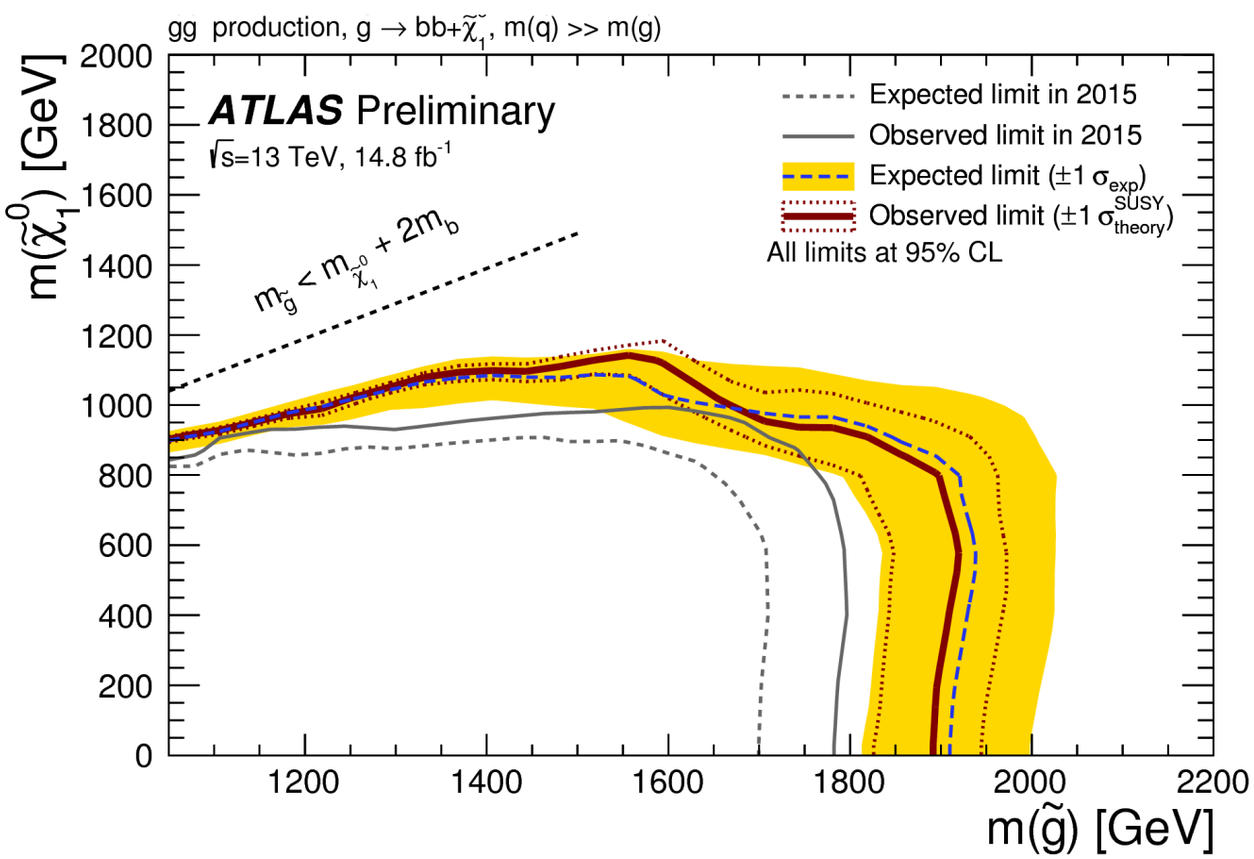
4) Physics of parton distribution functions

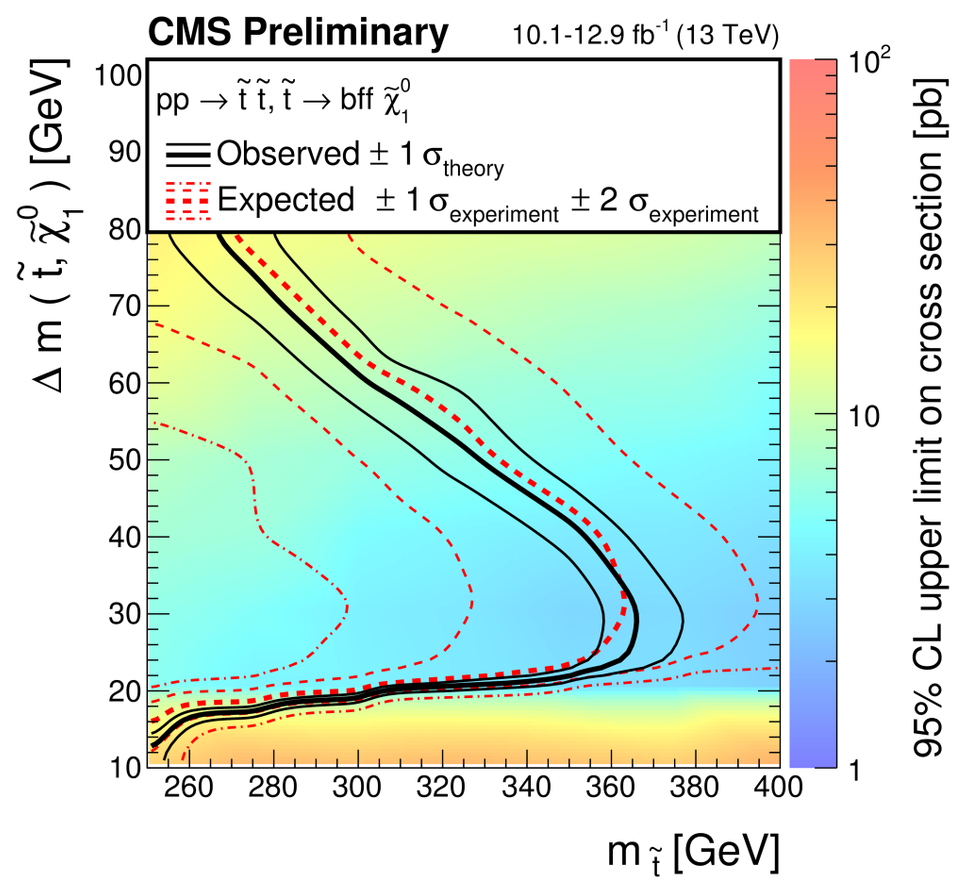
5) Basics of parton showers

6) Basics of jets

A challenge

The LHC experiments struggle to get past the ``Standard Model barrier" and discover physics beyond it. As many exclusion limits improve (increase), we face the prospect of having to understand how to find New Physics either in complex final states (if it is light) or in tails of distributions (if it is heavy). This forces us to consider if precision studies at the LHC -- a hadron collider (!) -- are possible.





Exclusion limits for stops and gluinos after ICHEP2016

Precision physics at the LHC?

Systematic precision studies at hadron colliders, aimed at discovering new effects, have never been attempted before. Indeed, hadrons are composite particles kept together by a poorly understood strong force. If we can't understand (compute) properties of one proton, can we confidently describe what happens when two collide?

The most important prerequisite for the success of precision physics program at the LHC is the understanding of hadron collisions from first principles. This requirement is stronger than the ability of our tools (parton showers, fixed order calculations, resummations etc.) to describe data; this can happen by accident or because multiple tunes are available.

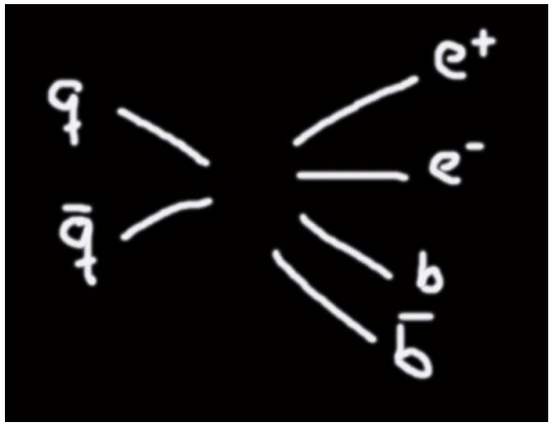
We need to understand, parametrically, the approximations that are made on the way from the Standard Model Lagrangian to a particular measurement and why these particular approximations are justified in each case.

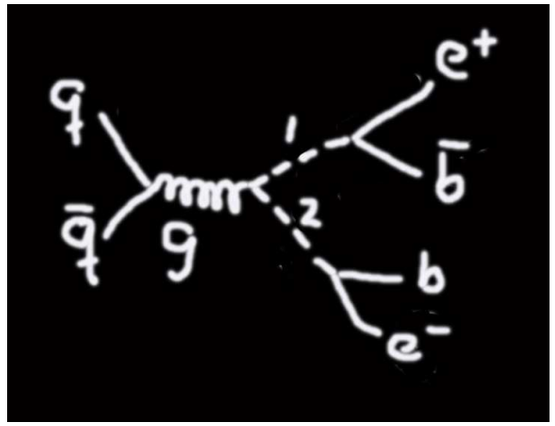
We need to be sure that the framework that we use is solid and that it is improvable (and if not, then what is its ultimate limit).

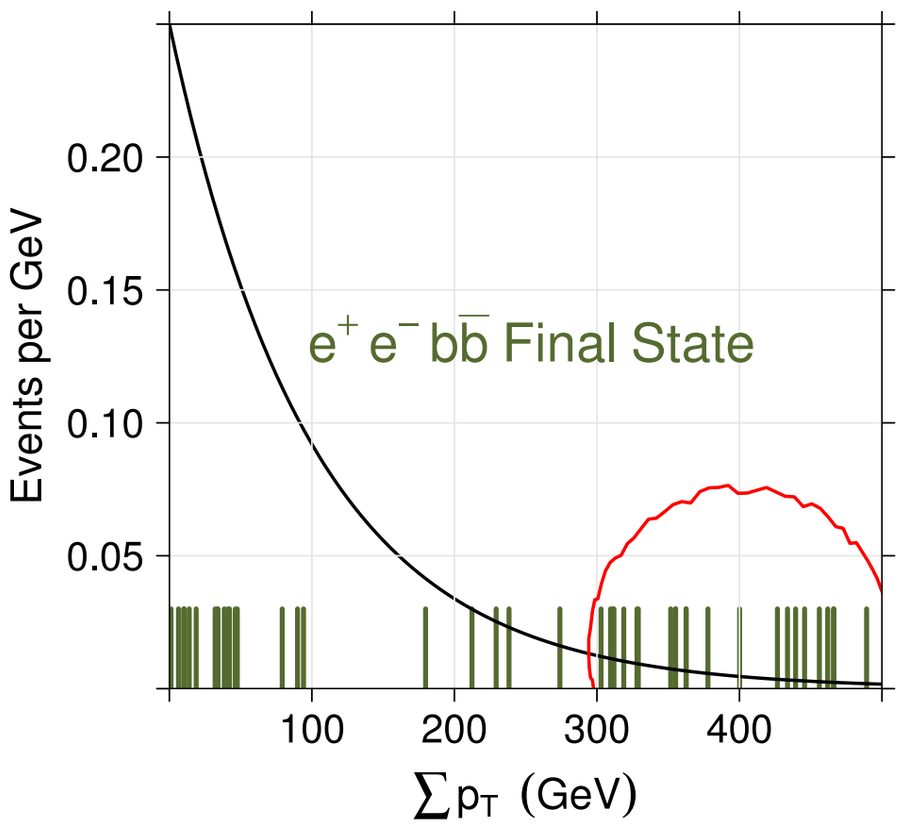
In short, we need to start asking questions about the foundations of what we do to describe hard hadron collisions and keep in mind that a significant fraction of the current lore and ideas date back to times when even an order-of-magnitude understanding of hadron collider physics was considered a successes; currently, we strive for more.

The new paradigm of discovery

BARD: Interpreting New Frontier Energy Collider Physics







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In contemporary high energy physics experiments, it is not uncommon to observe discrepancies between data and Standard Model predictions. Most of these discrepancies have been explained away over time. To convincingly demonstrate that an observed effect is evidence of physics beyond the Standard Model, it is necessary to prove it is (1) not a likely statistical fluctuation, (2) not introduced by an imperfect understanding of the experimental apparatus, (3) not due to an inadequacy of the implementation of the Standard Model prediction, and (4) interpretable in terms of a sensible underlying theory. Those who object to (4) as being necessary fail to appreciate that most hypothesis development in science occurs before, rather than after, publication. This last criterion is essential, and will likely point the way to other discrepancies that must exist if the interpretation is correct.

Main idea behind this paper was to search systematically for a correlated set of deviations from the SM predictions and a possibility to explain them with a single NP hypothesis. With null search results from the LHC, this idea becomes extremely timely...

The original wishlist

Knuteson then came up with the ``NLO wishlist '', i.e. a list of processes whose reliable description he thought was instrumental for making his idea a reality. Incidentally, this wishlist started a concerted effort by theorists to improve ways and means to perform NLO computations -- the beginning of the NLO revolution.

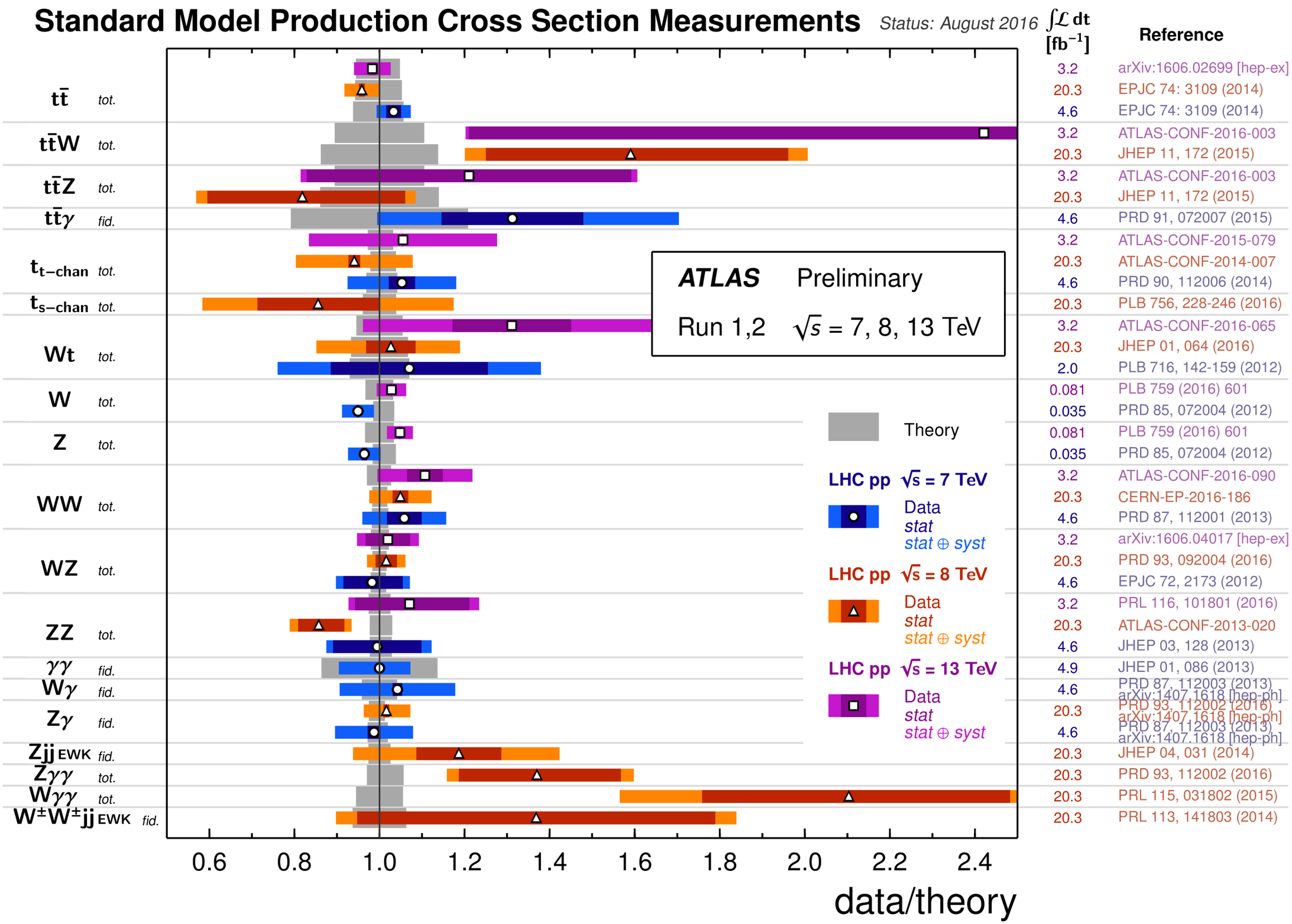
Note that we have ticked off one cross section from the first list

An experimenter's wishlist

Run II Monte Carlo Workshop

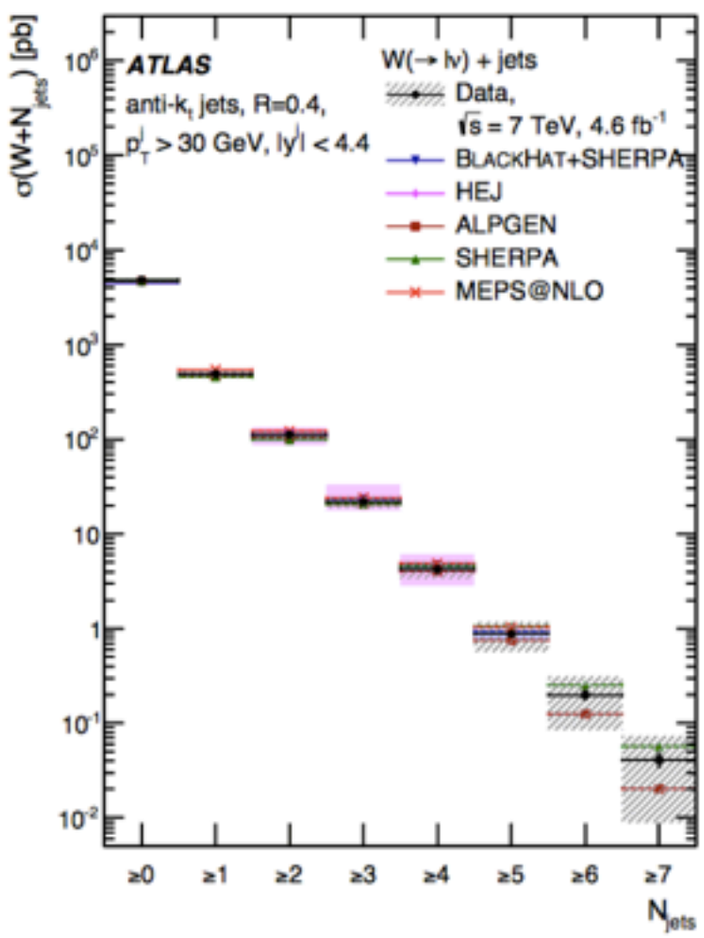
Single Boson	Diboson	Triboson	Heavy Flavour
$W+ \leq 5j$	$WW+ \leq 5j$	$WWW+ \leq 3j$	$t\bar{t}+ \leq 3j$
$W + b\bar{b} \leq 3j$	$W + b\bar{b}+ \leq 3j$	$WWW + b\bar{b}+ \leq 3j$	$t\bar{t} + \gamma+ \leq 2j$
$W + c\bar{c} \leq 3j$	$W + c\bar{c}+ \leq 3j$	$WWW + \gamma\gamma+ \leq 3j$	$t\bar{t} + W+ \leq 2j$
$Z+ \leq 5j$	$ZZ+ \leq 5j$	$Z\gamma\gamma+ \leq 3j$	$t\bar{t} + Z+ \leq 2j$
$Z + b\bar{b}+ \leq 3j$	$Z + b\bar{b}+ \leq 3j$	$ZZZ+ \leq 3j$	$t\bar{t} + H+ \leq 2j$
$Z + c\bar{c}+ \leq 3j$	$ZZ + c\bar{c}+ \leq 3j$	$WZZ+ \leq 3j$	$t\bar{b} \leq 2j$
$\gamma+ \leq 5j$	$\gamma\gamma+ \leq 5j$	$ZZZ+ \leq 3j$	$b\bar{b}+ \leq 3j$
$\gamma + b\bar{b} \leq 3j$	$\gamma\gamma + b\bar{b} \leq 3j$		
$\gamma + c\bar{c} \leq 3j$	$\gamma\gamma + c\bar{c} \leq 3j$		
	$WZ+ \leq 5j$		
	$WZ + b\bar{b} \leq 3j$		
	$WZ + c\bar{c} \leq 3j$		
	$W\gamma+ \leq 3j$		
	$Z\gamma+ \leq 3j$		

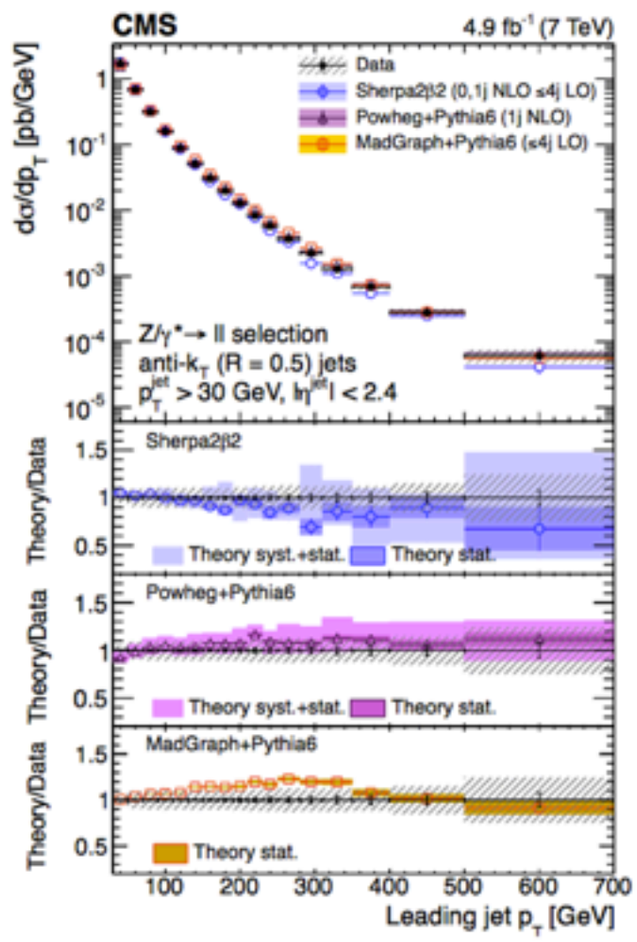
What has been studied and how well?

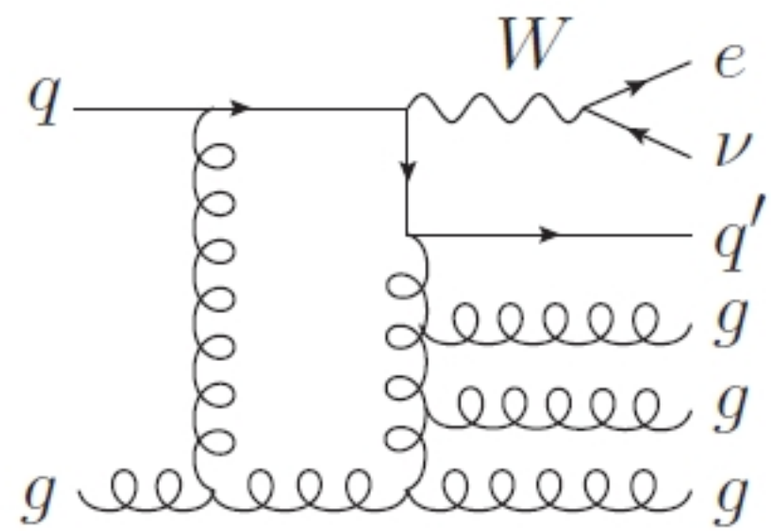


What has been studied and how well?

Vector boson production in association with jets (up to seven !) . Need to describe kinematic properties of jets; difficult but sufficient statistics to be sensitive to O(1-10) percent accuracy. Physics: determination of PDFs; backgrounds to BSM searches.





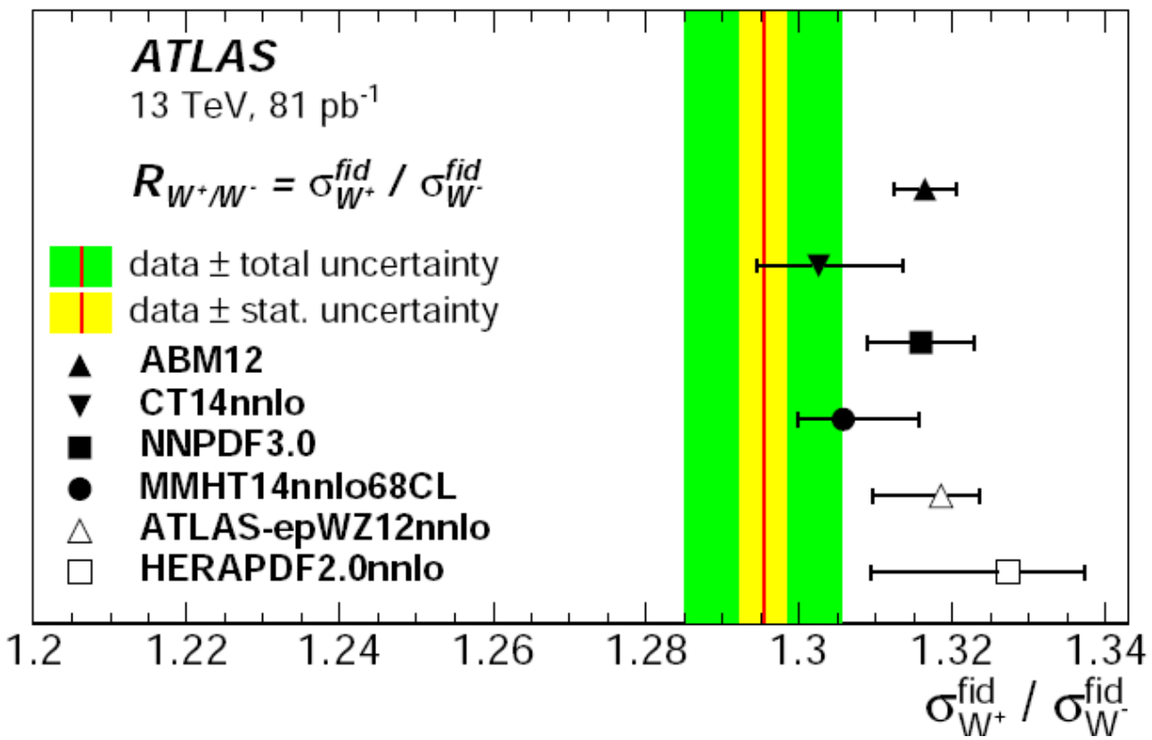


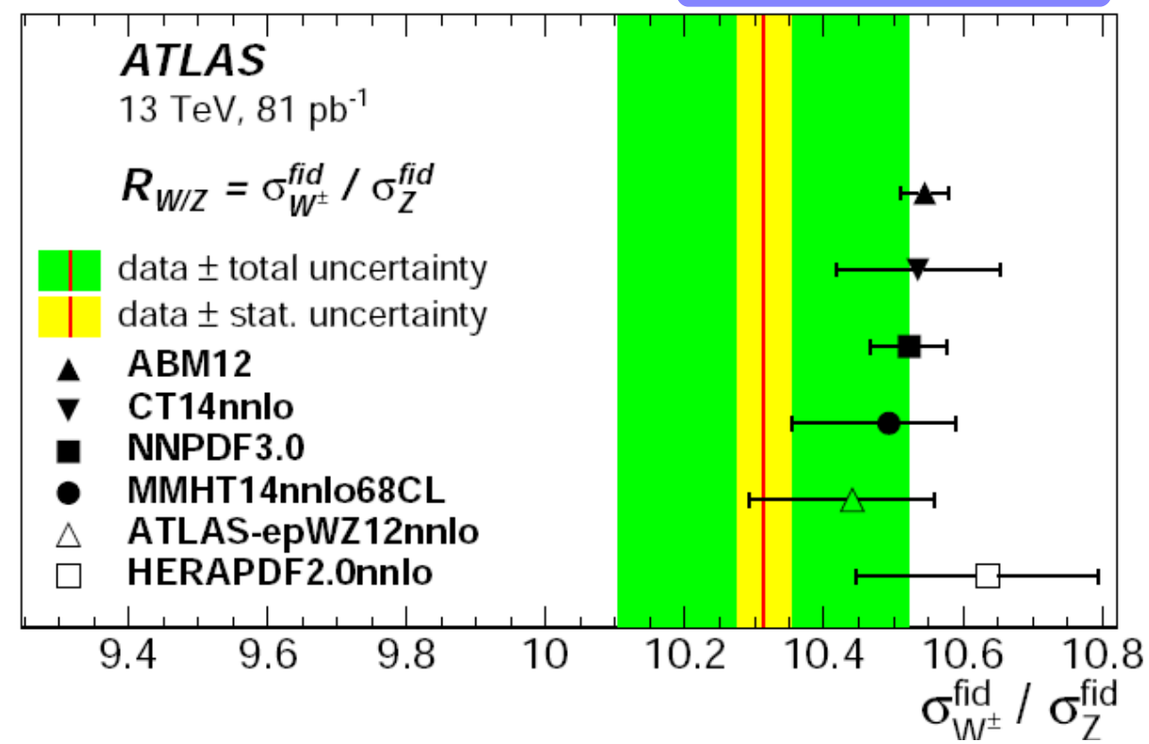
Gigantic number of tree-level and one-loop diagrams needed to compute cross-sections and kinematic distributions. Very hard if traditional methods have to be used.

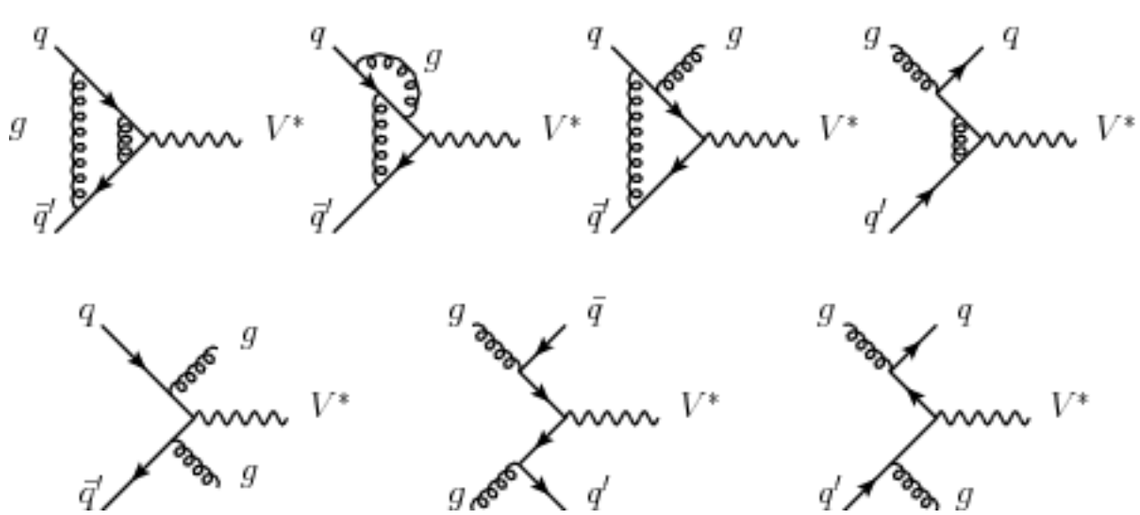
What has been studied and how well?

Drell-Yan process (inclusive production of Z,W,virtual photons). Few percent measurements; need accurate predictions. Requires at least second order in pQCD; Physics: backgrounds, PDFs, EW parameters (mass of the W, Weinberg angle, etc.).

arXiv:1603.09222







Two- (and more) loop diagrams; control over final state kinematics. Joint action of real emission contributions and virtual loops.

What has been studied and how well?

Higgs production and decay channels are known with high precision, largely in line with the forthcoming phenomenology requirements.

1) all major Higgs decay channels are known to NNLO QCD (or higher) and NLO EW

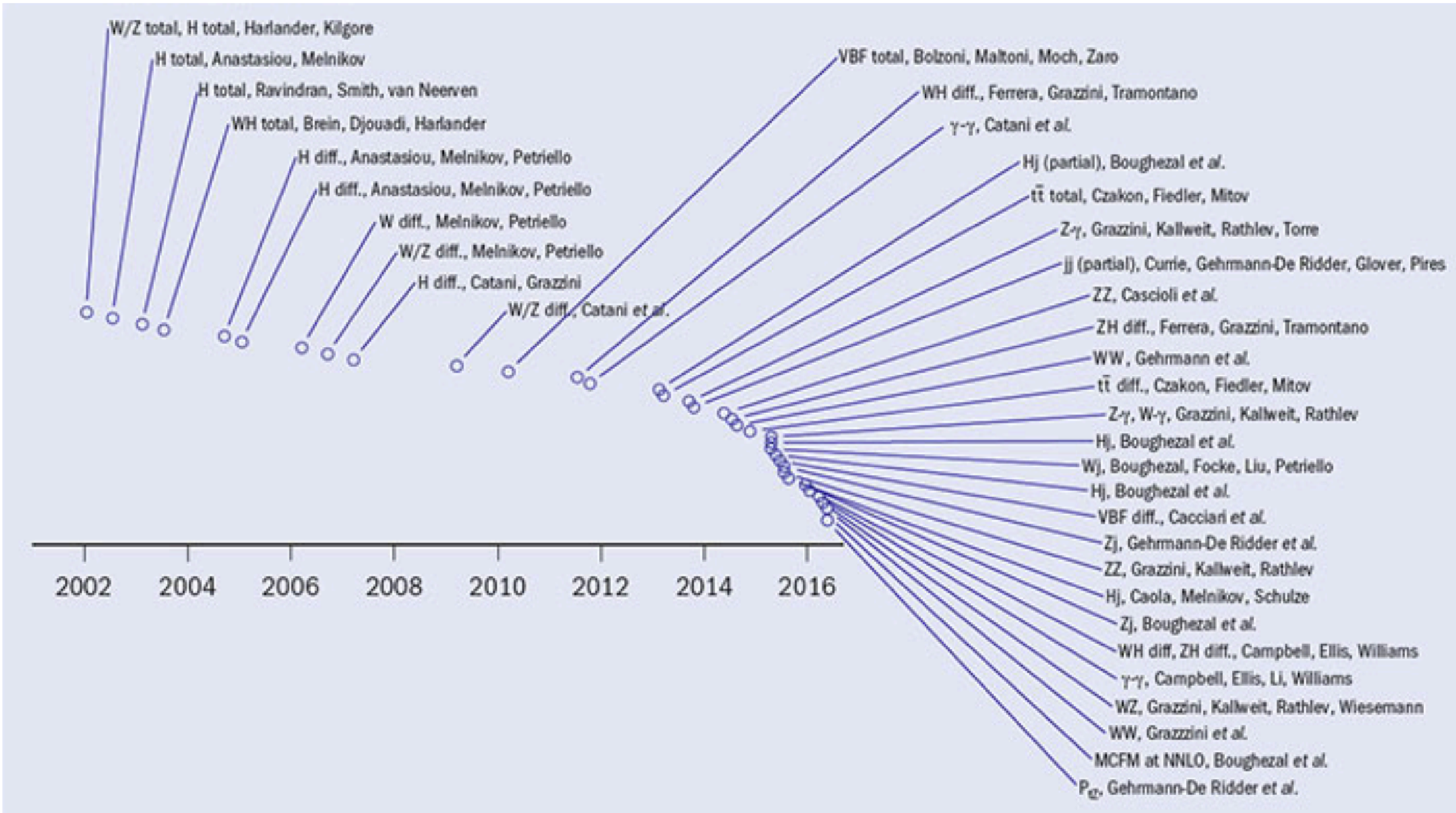
1) all major Higgs production channels are currently known through (at least) NLO QCD (many through NNLO, one through N3LO), and through NLO electroweak.

2) Many associated Higgs production processes with high jet multiplicity are also known at least through NLO QCD (H+3j)

3) Matching/merging of NLO QCD (and NNLO QCD for simple cases) results with parton showers is available thanks to MC@NLO, Powheg, Sherpa etc..

H+0 jet	N^3LO	O(3-5 %)	10 pb	fully inclusive
H+1 jet	N^2LO	O(7%)	7 pb	fully exclusive; Higgs decays, infinite mass
H+2 jet	NLO	O(20%)	1.5 pb	matched/merged
H+3 jet	NLO	O(20%)	0.4 pb	matched/merged/almost
WBF	N^2LO	O(1%)	1.5 pb	exclusive, no VBF cuts
WBF	N^2LO	O(5%)	0.2 pb	exclusive, VBF cuts
ZH, WH	N^2LO	O(2-3%)	O(1) pb	decays to bottom
ttH	NLO	O(5%)	0.2pb	decays, off-shell effects

The second (NNLO) revolution

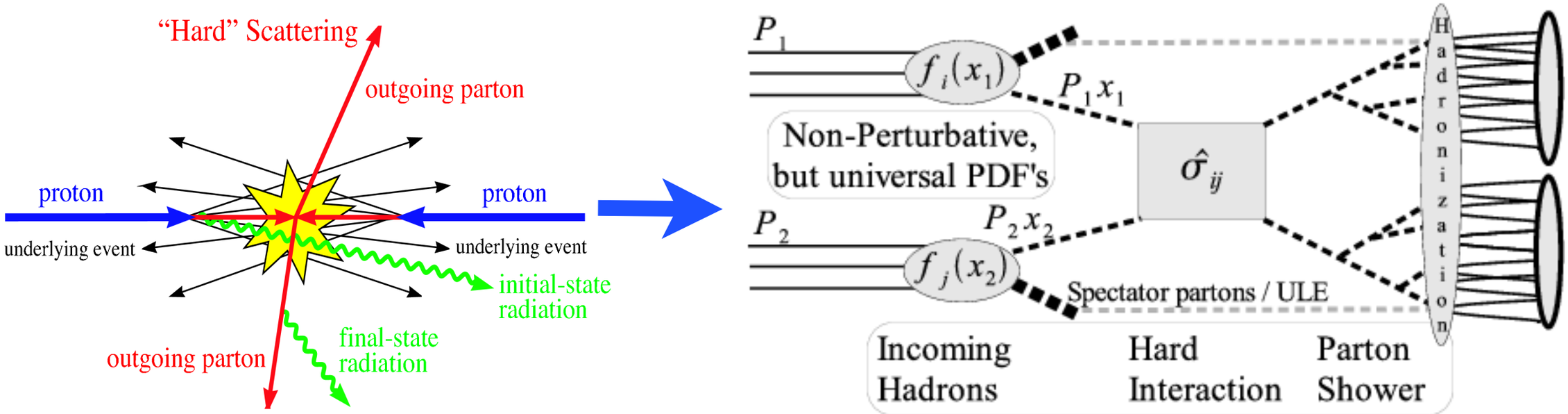


Where do all these results come from?

Sufficiently inclusive hard hadron processes can be described by the collinear factorization formula. In this framework, proton can be thought of as a beam of partons (quarks and gluons), each carrying a fraction of proton energy (Bjorken x). The probabilities to find partons with definite energy fractions are called parton distribution functions (PDFs). These objects are determined from fits to experimental data.

The goal therefore is to compute partonic cross sections for particular observables and then integrate these cross sections together with PDFs over Bjorken x's.

$$\mathrm{d}\sigma = \int \mathrm{d}x_1 \mathrm{d}x_2 f_i(x_1) f_j(x_2) \mathrm{d}\sigma_{ij}(x_1, x_2) F_J \left(1 + \mathcal{O}(\Lambda_{\mathrm{QCD}}/Q)\right)$$



LHC: the world of quarks and gluons

Hard scattering processes at the LHC can be understood in terms of quarks and gluons; only limited knowledge about protons is needed. Physics of quarks and gluons is governed by a field theory known as QCD. QCD is a non-abelian SU(3)-gauge theory, so it is quite complicated. Here is what we need to know about it.

1) The Lagrangian $$\mathcal{L}_{\rm QCD} = \sum \bar{q}_j \left(i\hat{D} - m_j \right) q_j - \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu}$$

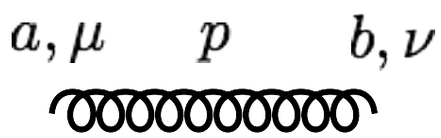
$$D_\mu = \partial_\mu - i g_s T^a A^a_\mu, \quad G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i g_s f^{abc} A^b_\mu A^c_\nu. \qquad [T^a, T^b] = i f_{abc} T^c$$

2) Degrees of freedom: quarks (up, down, strange, charm, bottom, top) and gluons. We will focus on the first five quarks that can be considered massless for the physics of hard processes that we want to describe.

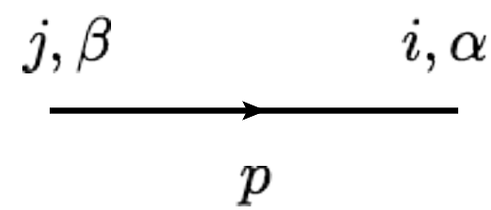
3) Color charges; interactions between quarks and gluons are determined by the color charges. Color charges are described by the so-called Casimir invariants of the SU(3) group: C_F for the quark and C_A for the gluon. Quarks can appear in three and gluons in eight color-charged states.

$$C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}, \qquad C_A = N_c = 3$$

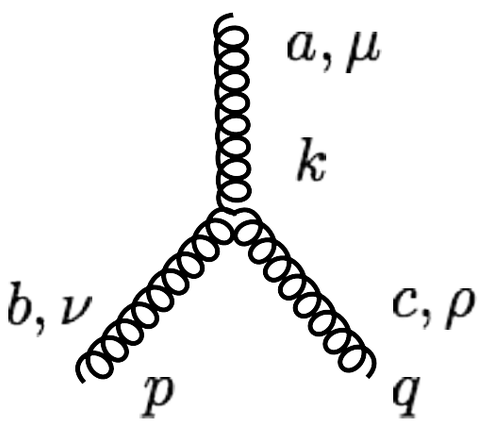
QCD Feynman rules



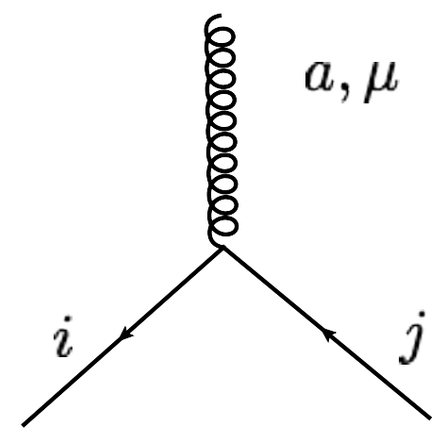
$$= \frac{-i\delta^{ab}}{p^2}\left(-g_{\mu\nu} + \xi\frac{p_\mu p_\nu}{p^2}\right)$$



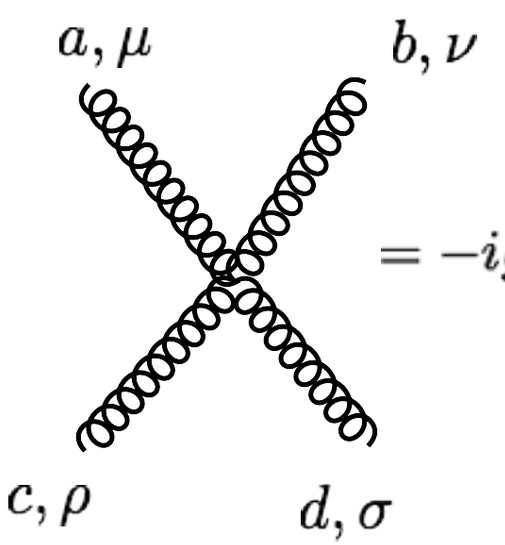
$$= \left(\frac{i}{\hat{p}}\right)_{\alpha\beta}\delta^{ij}$$



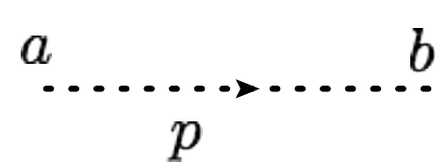
$$= g_s f^{abc}\big(\; g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu \big)$$



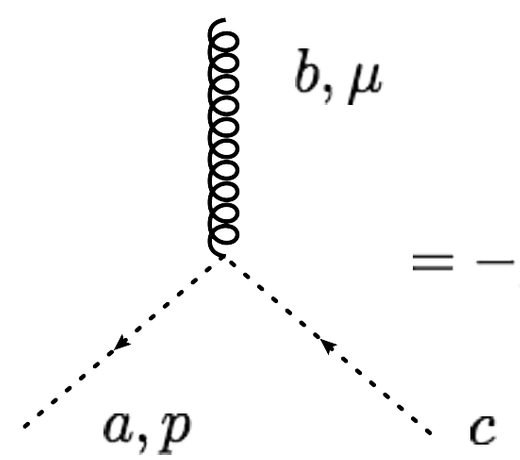
$$= ig_s\gamma^\mu T^a_{ij}$$



$$= -ig_s^2\Big[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bed}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\Big]$$



$$= \frac{i\delta^{ab}}{p^2}$$



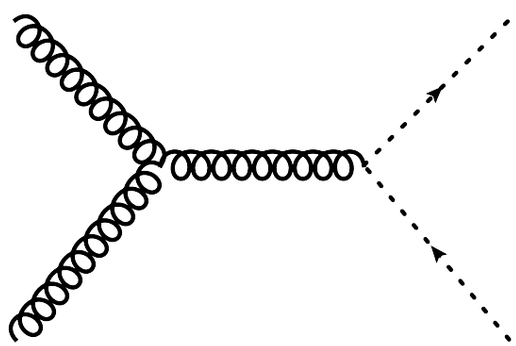
$$= -g_s f^{abc}p^\mu$$

Amplitudes, external states etc.

Starting from the QCD Feynman rules -- and given initial and final states for which we would like to know the scattering amplitudes -- we can put together Feynman diagrams and calculate them.

This sounds simple but there are are many things one has to worry about: large number of diagrams, complex computations, loop corrections, renormalization and so on.

One has to be careful to only use physical degrees of freedom to describe initial and final states; if this is not done, we will have to explicitly account for contributions of the ghosts particles !



$$gg \to Gh\,\bar{G}h$$

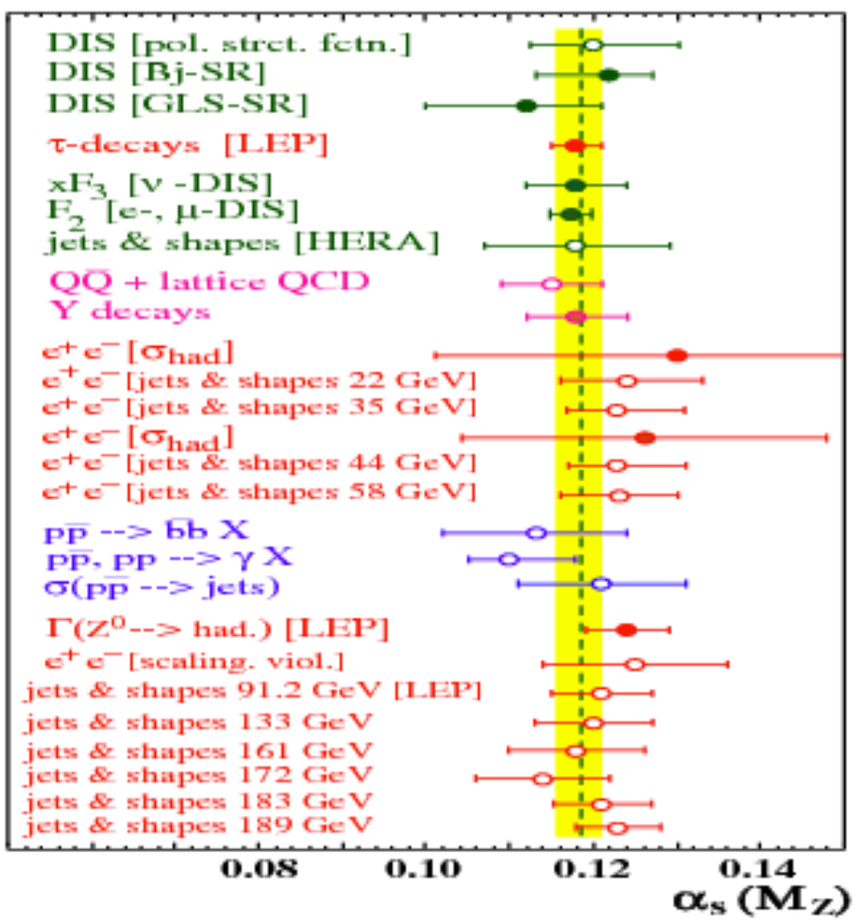
The external states are described by spinors (quarks) that satisfy the (massless) Dirac equation and two polarization vectors for gluons.

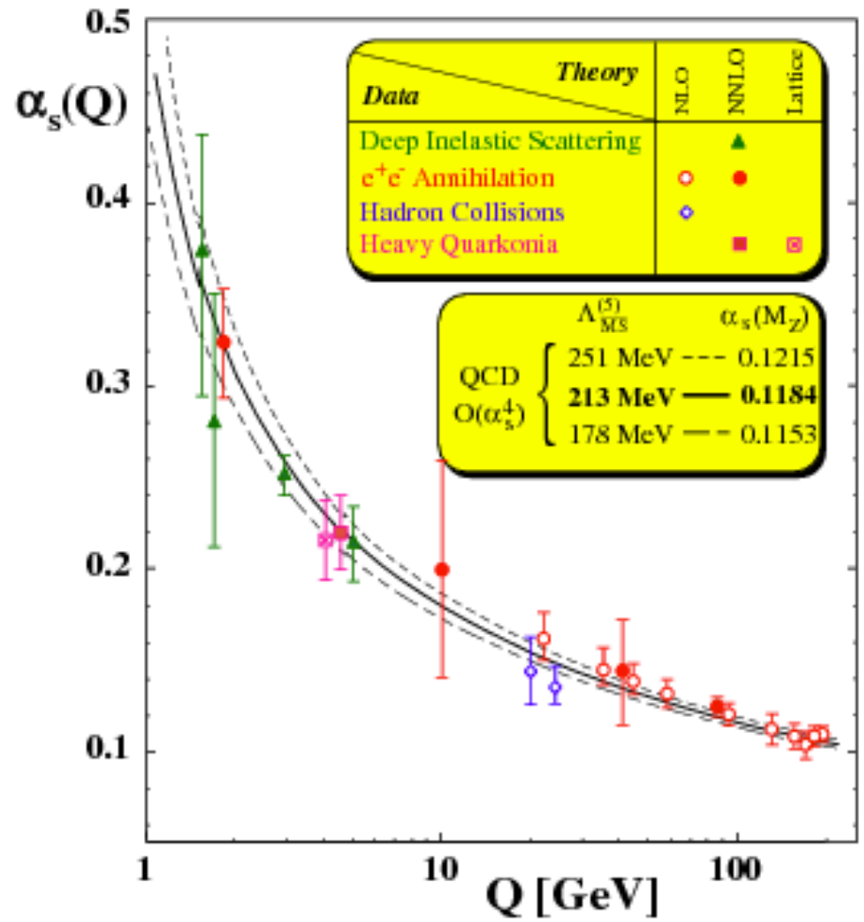
$$\hat{p}\,u_p = 0, \qquad k \cdot \epsilon = 0$$

$$k = (k_0, 0, 0, k_0) \Rightarrow \epsilon_+ = \frac{1}{\sqrt{2}}(0, 1, i, 0), \quad \epsilon_- = \frac{1}{\sqrt{2}}(0, 1, -i, 0).$$

The running coupling constant

It is possible to absorb large quantum corrections to the so-called running coupling constant. The coupling constant in QCD runs in such a way that it decreases at large momenta transfers or short distances. This phenomenon, known as the asymptotic freedom, enables us to describe perturbatively hard scattering processes at the LHC.





$$\alpha_s(\mu) = \frac{1}{\beta_0 \log \frac{\mu^2}{\Lambda^2}},$$

$$\beta_0 = \frac{33 - 2n_f}{12\pi} \sim 0.5|_{n_f=5}$$

$$\Lambda \approx 300 \text{ MeV}$$

Choice of scale in QCD coupling should be correlated with kinematics; for collider physics purposes, the transverse momentum of a jet relative to an emitter is often a good choice. We will see an example where understanding of this fact plays a crucial role in getting the physics right.

The importance of infrared dynamics in (p)QCD

Another aspect of QCD that is very essential for a variety of issues that we need to address when we talk about physics at the LHC is the behavior of QCD scattering amplitudes in soft and collinear (unresolved) limits. The reason this is a very important point is as follows:

1) Theoretically, we can only make solid statements about infra-red safe observables (i.e. that are independent of long-distance (potentially non-perturbative) dynamics);

2) Soft and collinear limits of amplitudes lead to not-integrable singularities in cross section computations. Obtaining finite fixed order predictions in high orders of pQCD requires us to understand how soft and collinear singularities cancel in the inclusive quantities (a general statement is provided by so-called Kinoshita-Lee-Nauenberg theorem).

3) Soft and collinear limits often determine enhanced contributions to scattering amplitudes and cross sections. Important for the re-summations, understanding of PDFs and parton showers.

4) Soft and collinear emissions dominate high-multiplicity final states. Crucial for parton showers, evolution to large distances and eventual non-perturbative transition from partons to hadrons.

The importance of infrared dynamics

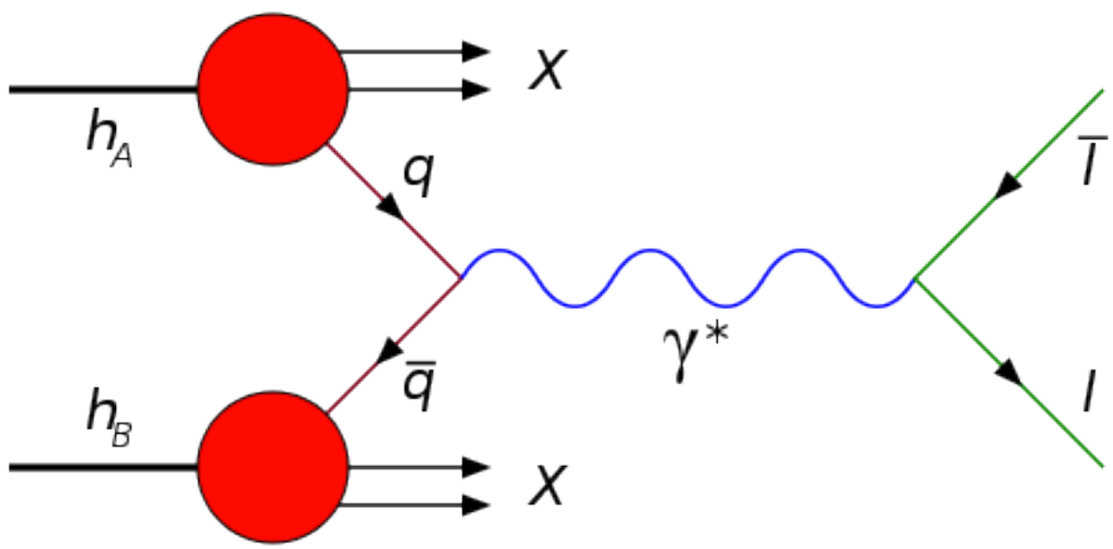
In what follows, we will look at different ways to describe hard hadron collider processes, emphasizing the role of infra-red limits for these descriptions.

We will mostly talk about the Drell-Yan (and similar) processes and will, occasionally, discuss QED instead of QCD, for simplicity.

We will discuss

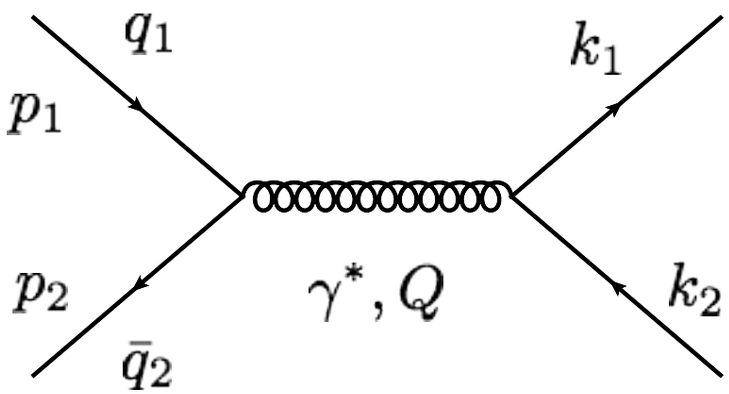
- Drell-Yan process at leading and next-to-leading orders in perturbative QCD;
- transverse momentum distribution of a lepton pair;
- physics of parton distribution functions;
- basics of parton showers and jet physics

The Drell-Yan process at leading and next-to-leading order



Drell-Yan process at LO

At leading order in pQCD, production of a lepton pair is described by a single Feynman diagram. We square this diagram and sum over polarizations and colors of the external particles. We use the resulting amplitude squared to compute the cross section, employing the standard formulas.



$$\mathcal{M}_{q_1\bar{q}_2} = \frac{i\delta_{q_1\bar{q}_2}\delta_{ij}{Q_q}^2 e^2}{Q^2}\,[\bar{u}(k_1)\gamma_\mu v(k_2)]\;[\bar{v}(p_2)\gamma^\mu u(p_1)]$$

$$Q = p_1 + p_2 = k_1 + k_2$$

$$\sum_{\text{pol, col}} |\mathcal{M}|^2 = \frac{32(e^2 Q_q)^2 N_c}{Q^4}\,[(k_1p_2)(k_2p_1) + (k_1p_1)(k_2p_2)]$$

$$\frac{\mathrm{d}^3k}{(2\pi)^3 2k^0} = [\mathrm{d}k]$$

$$\mathrm{d}\sigma_{q\bar{q}} = \frac{1}{4(p_1p_2)} \times \frac{1}{4N_c^2}\sum_{\text{pol, col}} |\mathcal{M}|^2 \times [\mathrm{d}k_1][\mathrm{d}k_2](2\pi)^4\delta(p_1 + p_2 - k_1 - k_2)$$

The phase space

We can not do much with the expression that we just wrote down unless we remove the energy-momentum conserving delta-function. Although the manipulations below is an overkill for the simple problem we are discussing, in general they show how to split the phase-space into the production and the decay parts.

$$
\begin{aligned}
&[\mathrm{d}k_1][\mathrm{d}k_2](2\pi)^4\delta(p_1+p_2-k_1-k_2)\\
&\quad= \mathrm{d}^4Q\;\delta(Q-k_1-k_2)[\mathrm{d}k_1][\mathrm{d}k_2](2\pi)^4\delta(p_1+p_2-Q)\\
&\quad= \mathrm{d}M^2\delta(Q^2-M^2)\;\mathrm{d}^4Q\;\delta(p_1+p_2-Q)\times[\mathrm{d}k_1][\mathrm{d}k_2](2\pi)^4\delta(Q-k_1-k_2)\\
&\quad= \mathrm{d}M^2\delta(M^2-s)\times[\mathrm{d}k_1][\mathrm{d}k_2](2\pi)^4\delta(Q-k_1-k_2)|_{Q^2=M^2}
\end{aligned}
$$

$$
\boxed{\mathrm{d}\sigma_{q\bar{q}} = \frac{4\pi^2\alpha^2Q_q^2}{3s}\frac{[(k_1p_2)(k_2p_1)+(k_1p_1)(k_2p_2)]}{Q^4}\times\mathrm{d}M^2\delta(s-M^2)\;\mathrm{dLips}(Q;M^2;k_1,k_2)}
$$

$$
\boxed{\mathrm{dLips}(Q;M^2;k_1,k_2) = [\mathrm{d}k_1][\mathrm{d}k_2](2\pi)^4\delta(Q-k_1-k_2)|_{Q^2=M^2} = \frac{1}{8\pi}\frac{\mathrm{d}\varphi\mathrm{d}\cos\theta}{4\pi}}
$$

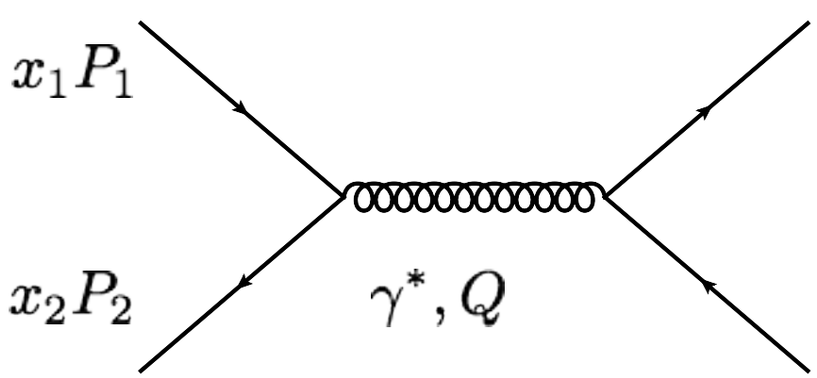
The last equation is valid in the rest frame of the dilepton pair.

The differential cross section

We can now use the above formulas to compute the DY cross section at leading order. We use rapidity and invariant mass of a lepton pair to fix Bjorken x's, compute vector Q, go to its rest frame, generate lepton momenta and boost them back to the lab frame.

$$\mathrm{d}\sigma = \sum_q \int \mathrm{d}x_1 \mathrm{d}x_2 f_q(x_1) f_{\bar{q}}(x_2) \mathrm{d}\sigma_{q\bar{q}}(x_1 P_1, x_2 P_2) + (q \leftrightarrow \bar{q})$$

$$M^2 = Q^2 = (x_1 P_1 + x_2 P_2)^2 = x_1 x_2 S, \quad Y = \frac{1}{2} \ln \frac{Q_0 + Q_z}{Q_0 - Q_z}$$



$$x_1 = \sqrt{\frac{M^2}{S}} e^Y, \quad x_2 = \sqrt{\frac{M^2}{S}} e^{-Y},$$

$$\mathrm{d}x_1 \mathrm{d}x_2 = \frac{\mathrm{d}M^2 \mathrm{d}Y}{S},$$

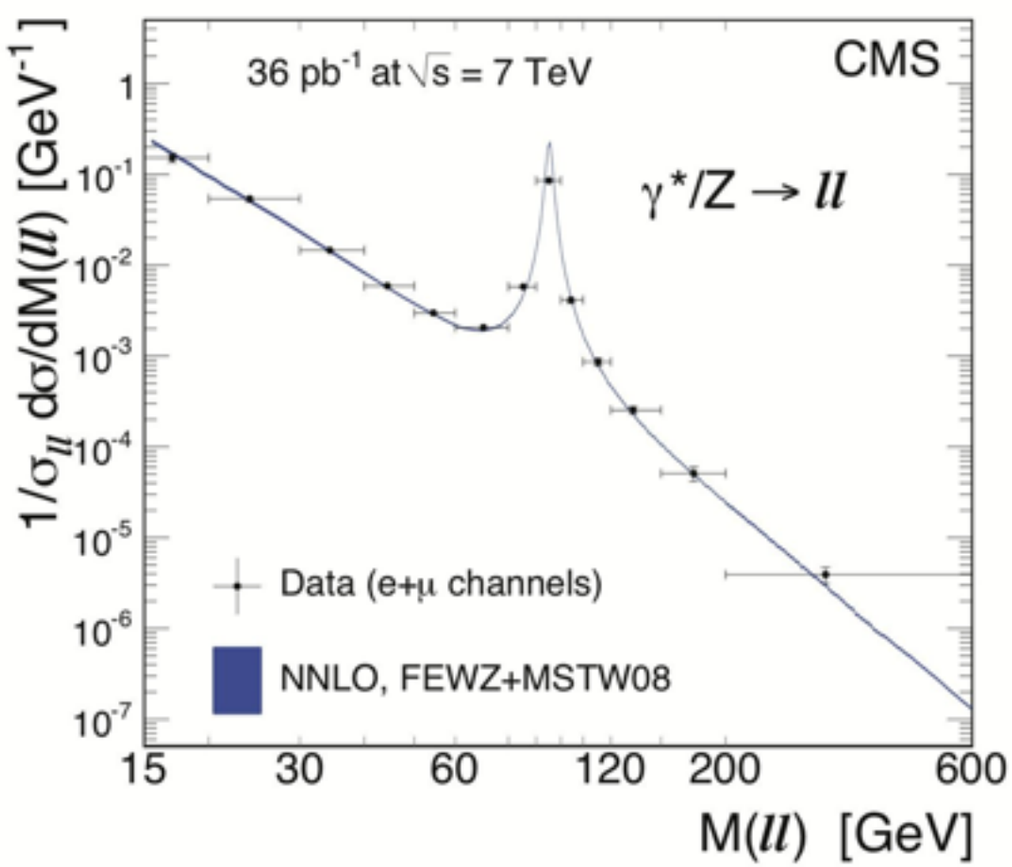
$$0 < M^2 < S, \quad -\frac{1}{2} \ln \frac{S}{M^2} < Y < \frac{1}{2} \ln \frac{S}{M^2}$$

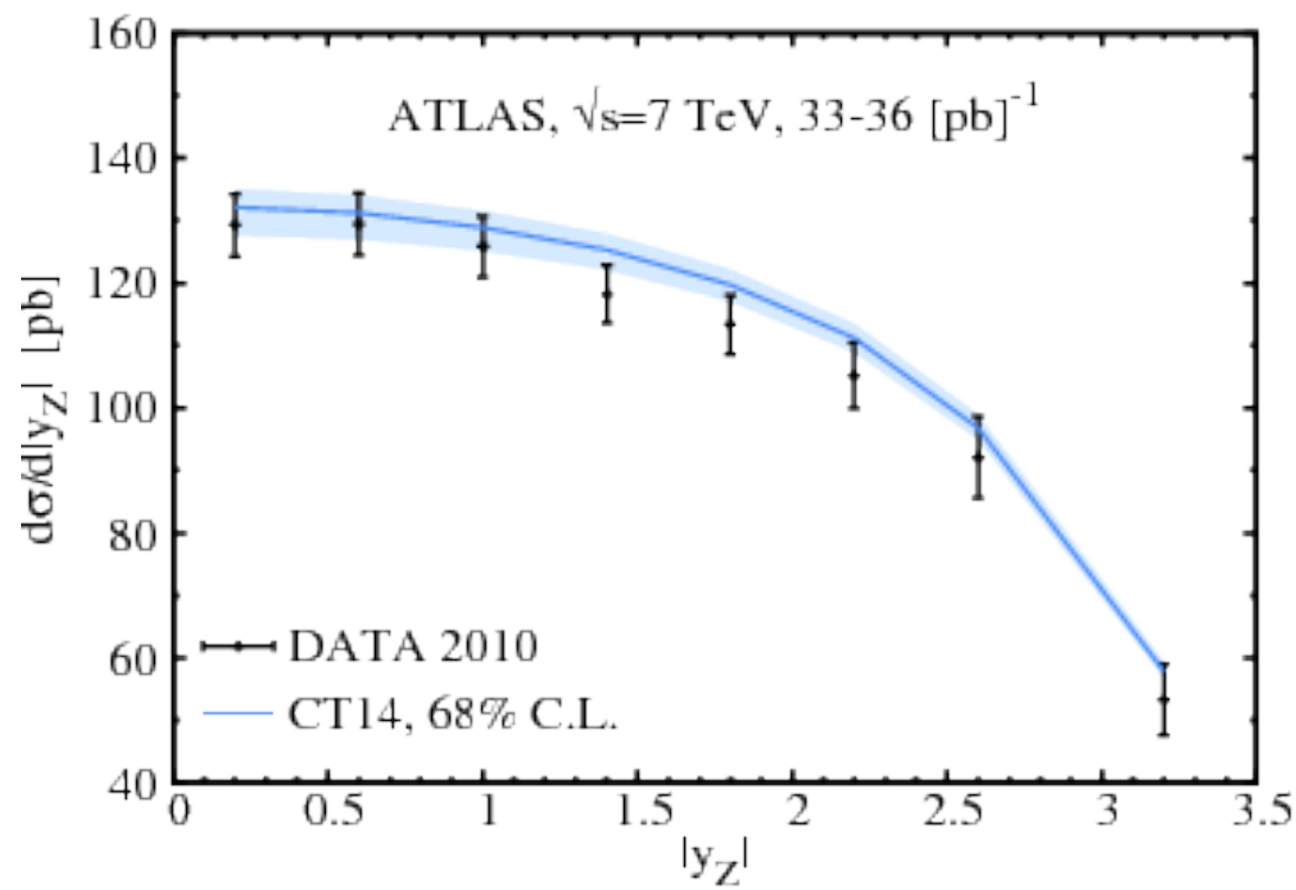
$$p_1 = x_1 P_1, \quad p_2 = x_2 P_2, \quad Q = p_1 + p_2,$$

$$k_{1,2}^{\mathrm{restQ}} = \sqrt{\frac{Q^2}{2}} (1, \pm \sin\theta \cos\phi, \pm \sin\theta \sin\phi, \pm \cos\theta)$$

The differential cross section

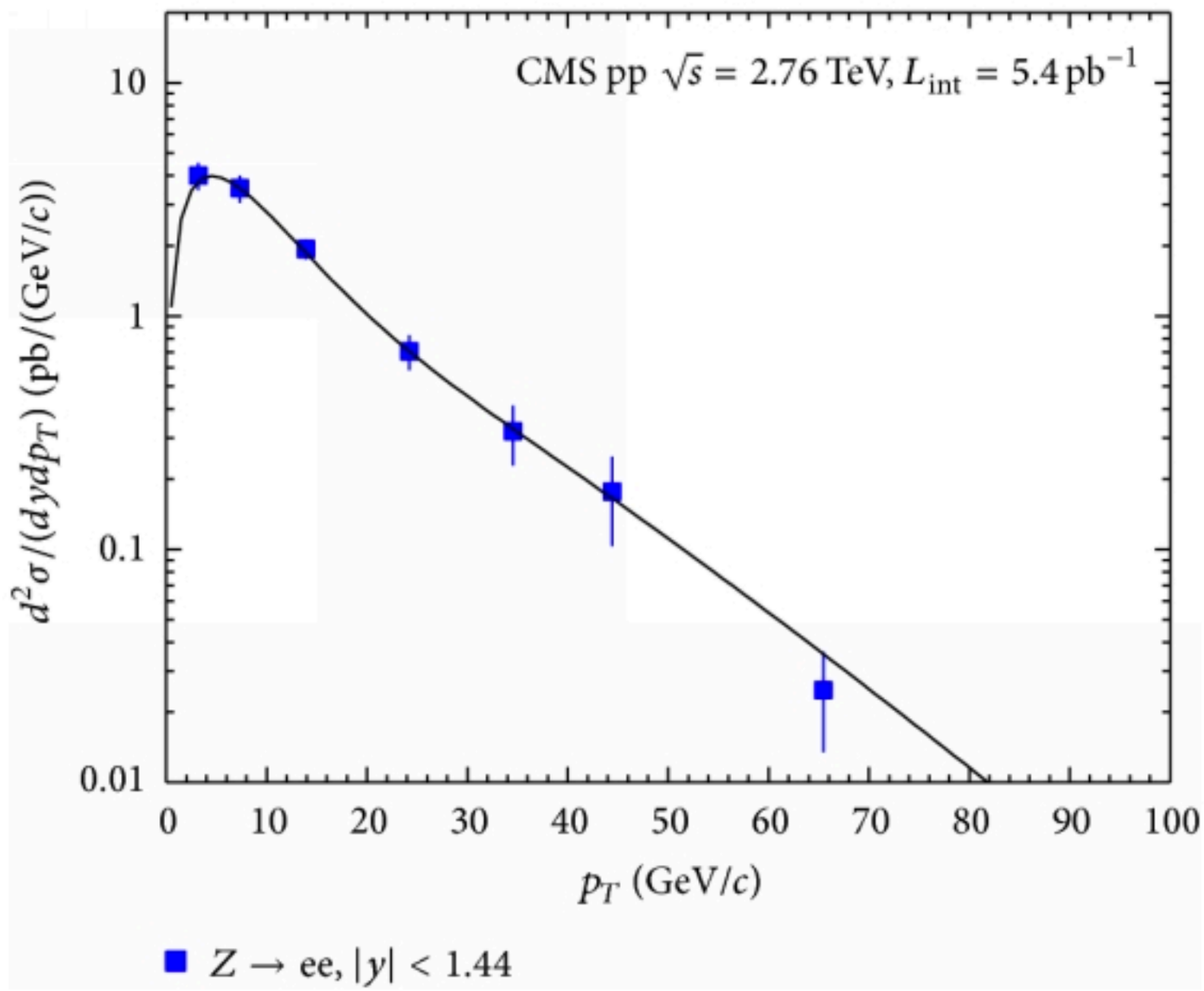
The differential cross section that we just computed provides non-trivial information about invariant mass and rapidity distribution of a lepton pair. However, the lepton pair has vanishing transverse momentum; this is unfortunate since in reality the transverse momentum distribution of a lepton pair is non-trivial and extends from low to high p_t.





The differential cross section

The transverse momentum distribution of a lepton pair is generated by emission(s) of gluons. These emissions can be soft (and multiple), producing a vector boson with low transverse momentum) or hard (and then one does not need many gluons to produce significant recoil). We will start with the discussion of the latter case (one emission, big recoil).

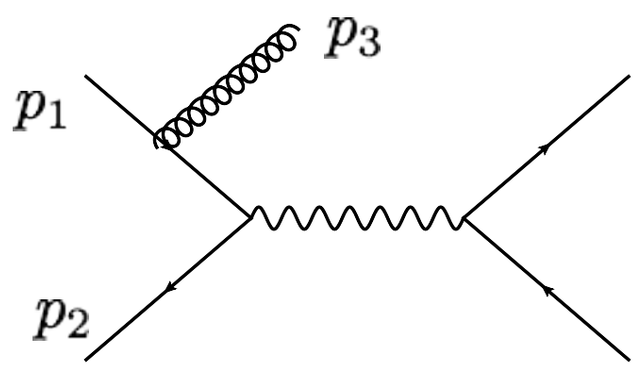


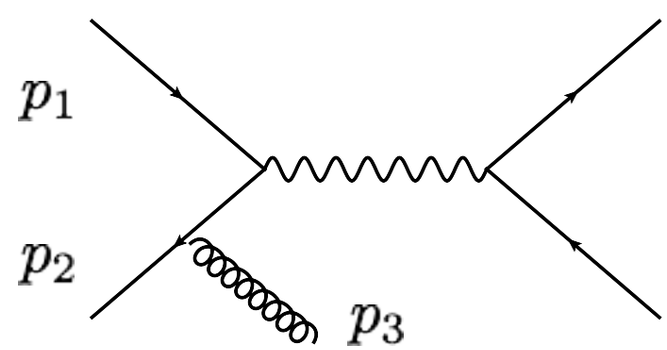
The differential cross section at NLO

A gluon emission generates non-trivial transverse momentum distribution of a lepton pair and, if we only count the number of lepton pairs produced in proton collisions, contributes also to the production rate.

However, if we attempt to compute its contribution to the total cross section, we find that it is infinitely large! We will see this on the next slide. To make our analysis simple, we need to introduce some notation.

It is convenient to separate an integration over the gluon momentum and consider the rest, i.e. a phase-space for the lepton pair, the matrix element and the observable (that is supposed to be infra-red and collinear safe) as a generic function that satisfies certain conditions if the energy of the gluon becomes soft or if the momentum of the gluon becomes collinear to colliding quark or anti-quark.





$$\mathrm{d}\sigma_R = \frac{1}{2s}\int [\mathrm{d}g_3] F_{\mathrm{LM}}(1,2,g_3), \qquad [\mathrm{d}g_3] = \frac{\mathrm{d}^{d-1}p_3}{(2\pi)^3 2E_3}\theta(E_{\max}-E_3), \quad d = 4-2\epsilon$$

$$F_{\mathrm{LM}}(1,2,3) = N_A \mathrm{dLips}(p_1+p_2-p_3,k_1,k_2)\sum |\mathcal{M}(1,2,3)|^2 F_j(1,2,3)$$

A digression: dimensional regularization

To regularize infra-red and collinear divergences, I will be using dimensional regularization. For our purposes, use of dimensional regularization amounts to working in a space-time whose dimensionality is smaller than four.

Many integrals that diverge in four dimensions in the infra-red and collinear limits become convergent; they exhibit a particular dependence on (d-4) that reminds us that the d->4 limit does not exist.

$$d = 4 - 2\epsilon \qquad \mathrm{d}^{d-1}k = E^{d-2}\mathrm{d}E\,\mathrm{d}\Omega^{d-1}$$

$$\Omega^{(d)} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \qquad \mathrm{d}\Omega^{(d-1)} = \mathrm{d}\cos\theta(1-\cos^2\theta)^{(d-4)/2}\mathrm{d}\Omega^{(d-2)}$$

$$\int_0^{E_{\max}} \frac{\mathrm{d}k_0}{k_0^{1+2\epsilon}} = -\frac{1}{2\epsilon}E_{\max}^{-2\epsilon} \qquad \Gamma(1+x\epsilon) \approx 1, \quad z\Gamma(z) = \frac{\Gamma(1+z)}{z}$$

$$\int_{-1}^{1} \mathrm{d}\cos\theta\,(1-\cos^2\theta)^{-\epsilon-1} = [\cos\theta \to 1-2x] = 2^{-1-2\epsilon}\int_0^1 (x(1-x))^{-\epsilon-1} = -2^{-2\epsilon}\frac{\Gamma(1-\epsilon)^2}{\epsilon\,\Gamma(1-2\epsilon)}$$

$$\int \frac{\mathrm{d}^{d-1}k}{2k_0}\frac{2p_1p_2}{(kp_1)(kp_2)}\theta(E_{\max}-k_0) = \Omega^{(d-2)}\int_0^{E_{\max}}\frac{\mathrm{d}k_0}{k_0^{1+2\epsilon}}\int_{-1}^{1}\mathrm{d}\cos\theta(1-\cos^2\theta)^{-\epsilon-1} \sim \frac{1}{\epsilon^2}$$

Soft and collinear limits

We will now study why the integration over the gluon momentum can not be performed. We begin with the soft limit (i.e. the energy of the gluon vanishes). There is an infra-red divergence in the integral for the real emission cross section which implies that this cross section can not be computed without an infra-red regulator (as already mentioned, we will use dimensional regularization). However, the divergence is logarithmic; this implies that it is determined by the leading power singularity of the matrix element squared.

$$\mathcal{M} = g_s T^a_{ij}\ \bar{v}(p_2)\left[\frac{\gamma^\mu(\hat{p}_1-\hat{p}_3)\hat{\epsilon}}{(p_1-p_3)^2} - \frac{\hat{\epsilon}(\hat{p}_2-\hat{p}_3)\gamma^\mu}{(p_2-p_3)^2}\right]u(p_1)\times\frac{ie^2Q_q^2}{(p_1+p_2-p_3)^2}\bar{u}(k_1)\gamma^\mu v(k_2)$$

$$\lim_{p_3\to 0}\mathcal{M}(1,2,3) \approx g_s T^a_{ij}\ \left[\frac{p_1\epsilon}{p_1p_3} - \frac{p_2\epsilon}{p_2p_3}\right]M^{\rm nc}_{\rm Born}(1,2) \qquad \Rightarrow \qquad |\mathcal{M}(1,2,3)|^2 \approx E_3^{-2}|\mathcal{M}_{\rm Born}(1,2)|^2$$

$$[{\rm d}g_3] \sim E_3{\rm d}E_3 \Rightarrow {\rm d}\sigma_R \sim \int_0 E_3{\rm d}E_3|\mathcal{M}(1,2,3)|^2 \sim \int_0 \frac{{\rm d}E_3}{E_3} \to \infty$$

$$[{\rm d}g_3] \sim E_3^{1-2\epsilon}{\rm d}E_3 \Rightarrow {\rm d}\sigma_R \sim \int_0 E_3^{1-2\epsilon}{\rm d}E_3|\mathcal{M}(1,2,3)|^2 \sim \int_0 \frac{{\rm d}E_3}{E_3^{1+2\epsilon}} \to \frac{1}{\epsilon}$$

infra-red (soft) divergencies, (un)regulated

Soft and collinear limits

It is convenient to describe the soft limit of the entire function $F_{LM}(1,2,4)$ since it is this function that enters the cross section computation. We define the operator that extracts the soft divergence at the leading power and obtain

$$S_3 F_{\rm LM}(1,2,3) = \lim_{p_3 \to 0} F_{\rm LM}(1,2,3) = g_s^2 C_F \, \frac{2p_1p_2}{(p_1p_3)(p_2p_3)} F_{\rm LM}(1,2)$$

Note that the $F_{LM}(1,2,3)$ contains the phase space of a lepton pair; the above formula implies that in the soft limit, it has to be evaluated for Born (no emission) kinematics.

Note also that the soft limits are universal; they are determined by the color charges of hard partons that are present in a particular process but otherwise are insensitive to the details of the hard scattering process itself.

Indeed, the above formula applies to Drell-Yan process, Z-production, WW-production, ZZ-production). It also applies to the Higgs production in gluon fusion provided that we replace the quark color charge (C_F) with the gluon color charge C_A.

Soft and collinear limits

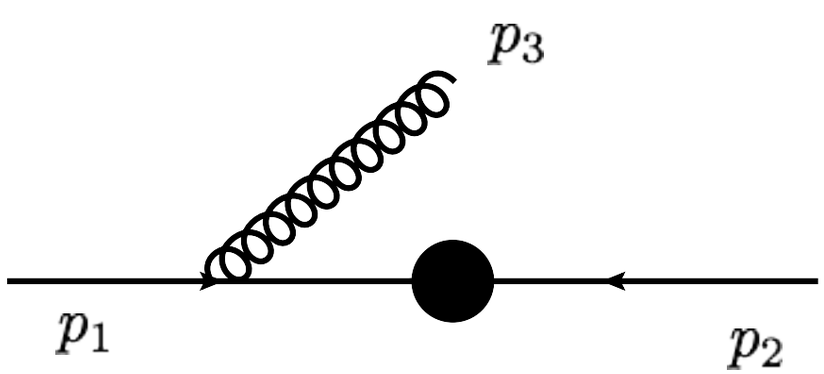
The other potentially singular kinematic region is called ``collinear". The collinear limit corresponds to gluon being emitted along the momentum of a quark or an anti-quark. We will consider the first situation, for definiteness.

To understand collinear singularities, it is useful to parametrize the momenta in a particular way, known as the Sudakov decomposition.

$$p_3 = xp_1 + \beta p_2 + p_\perp, \quad p_1 p_\perp = p_2 p_\perp = 0.$$

$$(p_1 - p_3)^2 = -2E_1 E_3 (1 - \cos\theta_{13}) \approx E_1 E_3 \theta_{13}^2 = -s\beta.$$

$$p_3^2 = 0 \Rightarrow x\beta s - p_{3\perp}^2 = 0; \quad x \sim O(1), \quad \beta = \frac{p_{3\perp}^2}{xs}$$



$$\boxed{x \sim 1, \quad \beta \sim \theta_{13}^2, \quad p_{3\perp} \sim s\theta_{13}}$$

$$[\mathrm{d}g_3] \sim \mathrm{d}\cos\theta_{13} \sim \theta_{13}\, \mathrm{d}\theta_{13}$$

Soft and collinear limits

To understand how collinear singularities appear in scattering amplitudes, we note that propagators that are present in two diagrams that contribute to the production of a lepton pair have different 1||3 collinear limits. However, naively, we can not discard non-singular diagrams since their interference with the singular ones appears to be non-integrable. We will now show that this is not the case (in physical gauges).

$$(p_1 - p_3)^2 \sim \theta_{13}^2, \qquad (p_2 - p_3)^2 \sim 1$$

$$\mathcal{M} \sim \frac{A}{(p_1 - p_3)^2} + B \Rightarrow |\mathcal{M}|^2 \sim \left[\frac{A}{(p_1 - p_3)^2}\right]^2 + \frac{2AB}{(p_1 - p_3)^2}$$

$$\mathcal{M}(1,2,3)_{1||3,\text{sing}} \approx g_s T^a_{ij} \bar{v}(p_2) \frac{\gamma^\mu \left[(1-x)\hat{p}_1 - \beta \hat{p}_2 - \hat{p}_\perp\right] \hat{\epsilon} u(p_1)}{(p_1 - p_3)^2} \times \frac{e^2 Q_q}{(p_1 + p_2 - p_3)^2} \bar{u}(k_1)\gamma_\mu v(k_2)$$

$$\approx g_s T^a_{ij} \bar{v}(p_2) \frac{\gamma^\mu \left[(1-x)2(p_1\epsilon) - \hat{p}_\perp \hat{\epsilon}\right] u(p_1)}{(p_1 - p_3)^2} \times \frac{e^2 Q_q}{(p_1 + p_2 - p_3)^2} \bar{u}(k_1)\gamma_\mu v(k_2)$$

$$(p_3\epsilon) = 0 \Rightarrow \quad x(p_1\epsilon) + \beta(p_2\epsilon) + (p_\perp\epsilon) = 0. \qquad p_1\epsilon \approx -\frac{p_\perp \epsilon}{x} \sim \mathcal{O}(\theta_{13}).$$

$$\lim_{1||3} \mathcal{M} \approx \frac{A}{\theta_{13}} + B \Rightarrow \lim_{1||3} \mathcal{M}^2 \sim \frac{A^2}{\theta_{13}^2}$$

These considerations imply that no information about non-singular diagram (or diagrams, in general) is needed for understanding the collinear limit (in physical gauges).

Soft and collinear limits

A straightforward computation then gives the result for the amplitude in the collinear limit. Similar to the soft case, it is convenient to introduce an operator that extracts the relevant singular behavior from the amplitude. The collinear limit is determined by the corresponding splitting function -- a universal function that is independent of the underlying hard process. In contrast to the soft limit, the hard matrix elements does not fully decouple -- it depends on the longitudinal fraction of momenta carried away by the emitted gluon.

$$C_{31} F_{\mathrm{LM}}(1,2,3) \approx \frac{2g_s^2}{(p_1-p_3)^2} P_{qq}\left(\frac{E_1}{E_1-E_3}\right) F_{LM}(1-3,2)$$

$$P_{qq}(z) = C_F\left[\frac{1+z^2}{1-z} - \epsilon(1-z)\right] \qquad F_{\mathrm{LM}}(1-3,2) = F_{\mathrm{LM}}\left(1\cdot\frac{E_1-E_3}{E_1},2\right)$$

$$(p_1-p_3)^2 \sim \theta_{13}^2, \quad [\mathrm{d}g_3] \sim \theta_{13}\mathrm{d}\theta_{13} \Rightarrow \mathrm{d}\sigma_R \sim \int_0 \frac{\mathrm{d}\theta_{13}}{\theta_{13}} \to \infty$$

Similar to the soft limit, the collinear limit of the matrix element squared can not be integrated over the emission angle in four dimensions. The divergence is fully determined by the leading singularity of the amplitude. In dimensional regularization, collinear divergences turn into 1/(d-4) poles.

Soft and collinear limits

Here is the summary of the situation with the gluon emission:

1) for a non-soft, non-collinear emission (resolved), the matrix element is finite and integrable; no infra-red regulator is needed.

2) for a soft or collinear emission, the matrix element is not integrable without an infra-red or collinear regulator. The non-integrability (divergence) is entirely determined by the leading power singularity of the amplitude squared.

3) in both soft and collinear limits, the matrix element with additional gluon factorizes, at leading power, into a universal functions (eikonal, splitting) and the Born matrix element that is independent of the momentum of the emitted gluon (almost independent, c.f. the collinear limit).

$$S_3 F_{\rm LM}(1,2,3) = \lim_{p_3 \to 0} F_{\rm LM}(1,2,3) = g_s^2 C_F \, \frac{2p_1p_2}{(p_1p_3)(p_2p_3)} F_{\rm LM}(1,2)$$

$$C_{31} F_{\rm LM}(1,2,3) \approx \frac{2g_s^2}{(p_1-p_3)^2} P_{qq}\left(\frac{E_1}{E_1-E_3}\right) F_{LM}(1-3,2)$$

The question is then how to separate resolved emissions from the unresolved ones. Two ways are known and used: slicing and subtraction. We will describe the second one.

Nested subtraction

It is convenient to denote the integration over the relevant phase-space by angle brackets. We extract soft and collinear singularities by an iterated application of soft and collinear operators that we introduced earlier.

$$\mathrm{d}\sigma_R = \frac{1}{2s}\int [\mathrm{d}g_3] F_{\mathrm{LM}}(1,2,g_3), \qquad [\mathrm{d}g_3] = \frac{\mathrm{d}^{d-1}p_3}{(2\pi)^3 2E_3}\theta(E_{\max}-E_3), \quad d = 4-2\epsilon$$

$$F_{\mathrm{LM}}(1,2,3) = N_A \mathrm{dLips}(p_1+p_2-p_3,k_1,k_2)\sum |\mathcal{M}(1,2,3)|^2$$

$$\mathrm{d}\sigma_R = \langle F_{\mathrm{LM}}(1,2,3)\rangle$$

$$\begin{aligned}\langle F_{\mathrm{LM}}(1,2,3)\rangle &= \langle (I-S_3)F_{\mathrm{LM}}(1,2,3)\rangle + \langle S_3 F_{\mathrm{LM}}(1,2,3)\rangle \\ &= \langle ((I-C_{31}-C_{32})\,(I-S_3)F_{\mathrm{LM}}(1,2,3)\rangle + \langle (C_{31}+C_{32})\,(I-S_3)F_{\mathrm{LM}}(1,2,3)\rangle \\ &+ \langle S_3 F_{\mathrm{LM}}(1,2,3)\rangle\end{aligned}$$

The first term on the right hand side is finite and can be integrated without an infrared regulator. The second and the third terms are potentially divergent and require further analysis.

Nested subtraction

Consider the second and the third term. They can be significantly simplified.

$$\begin{aligned}\langle F_{\rm LM}(1,2,3)\rangle &= \langle (I-S_3)F_{\rm LM}(1,2,3)\rangle + \langle S_3 F_{\rm LM}(1,2,3)\rangle \\ &= \langle ((I-C_{31}-C_{32})\,(I-S_3)F_{\rm LM}(1,2,3)\rangle + \langle (C_{31}+C_{32})\,(I-S_3)F_{\rm LM}(1,2,3)\rangle \\ &+ \langle S_3 F_{\rm LM}(1,2,3)\rangle\end{aligned}$$

$$\boxed{\begin{aligned}&\langle (C_{31}+C_{32})\,(I-S_3)F_{\rm LM}(1,2,3)\rangle + \langle S_3 F_{\rm LM}(1,2,3)\rangle \\ &= \langle (C_{31}+C_{32})\,F_{\rm LM}(1,2,3)\rangle + \langle [(C_{31}+C_{32})\,S_3 - S_3]F_{\rm LM}(1,2,3)\rangle\end{aligned}}$$

The last term vanishes for the back-to-back kinematics:

$$S_3 F_{\rm LM}(1,2,3) = g_s^2 C_F \frac{2p_1p_2}{(p_1p_3)(p_2p_3)} F_{\rm LM}(1,2)$$

$$C_{31}S_3 F_{\rm LM}(1,2,3) = g_s^2 C_F\ \frac{2E_1}{E_3}\frac{1}{p_1p_3} F_{\rm LM}(1,2);$$

$$C_{32}S_3 F_{\rm LM}(1,2,3) = g_s^2 C_F\ \frac{2E_2}{E_3}\frac{1}{p_2p_3} F_{\rm LM}(1,2).$$

$$\left[\frac{2E_1}{E_3}p_2p_3 + \frac{2E_2}{E_3}p_1p_3\right] = \frac{2p_1p_2}{(p_1p_3)(p_2p_3)} \Rightarrow \langle [(C_{31}+C_{32})\,S_3 - S_3]F_{\rm LM}(1,2,3)\rangle = 0$$

Nested subtraction

We obtain the final result for the real emission contribution. The O_{NLO} term is finite and can be computed in four-dimensions. The term with two collinear operators is divergent. However, this term has a simplified matrix element because of the collinear approximation so that partial integration over the gluon phase-space become possible.

$$\langle F_{\rm LM}(1,2,3)\rangle = \langle \mathcal{O}_{\rm NLO} F_{\rm LM}(1,2,3)\rangle + \langle (C_{31}+C_{32})\, F_{\rm LM}(1,2,3)\rangle,$$

$$\mathcal{O}_{\rm NLO} F_{\rm LM}(1,2,3) = ((I - C_{31} - C_{32})\,(I - S_3) F_{\rm LM}(1,2,3)$$

$$\langle C_{31} F_{LM}(1,2,3)\rangle = \int [{\rm d}g_3]\, \frac{g_s^2}{(-2)E_1 E_3 \rho_{13}} P_{qq}\left(\frac{E_1}{E_1 - E_3}\right)(2s)^{-1} F_{\rm LM}(1-3,2)$$

$$\rho_{13} = 1 - \cos\theta_{13} \qquad \int_{-1}^{1} \frac{{\rm d}\cos\theta_{13}(1-\cos^2\theta_{13})^{-\epsilon}}{\rho_{13}} = \frac{2^{-2\epsilon}\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

$$E_3 = E_1(1-z), \quad C_{31} F_{\rm LM}(1,2,3) = \frac{g_s^2}{E_3^2 \rho_{13}}\,[(1-z)P_{qq}(z)]\,\frac{F_{\rm LM}(z\cdot 1,2)}{z}$$

$$\langle C_{31} F_{\rm LM}(1,2,3)\rangle = -\frac{[\alpha_s]}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}(2E_1)^{-2\epsilon}\int_{z_{\rm min}}^{1} \frac{{\rm d}z}{(1-z)^{2\epsilon}} P_{qq}(z) \times \langle \frac{F_{\rm LM}(z\cdot 1,2)}{z}\rangle$$

$$[\alpha_s] = \frac{\alpha_s \mu^{2\epsilon} e^{\epsilon\gamma_E}}{2\pi\Gamma(1-\epsilon)} \qquad z_{\rm min} = 1 - \frac{E_{\rm max}}{E_1}$$

We need to extract divergencies from the remaining integration over z

Nested subtraction

To extract divergencies related to the integration over z, we introduce the plus-prescription which allows us to extract the divergences and regularize the remainder.

$$\langle C_{31} F_{\rm LM}(1,2,3)\rangle = -\frac{[\alpha_s]}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}(2E_1)^{-2\epsilon}\int\limits_{z_{\rm min}}^{1}\frac{{\rm d}z}{(1-z)^{2\epsilon}}P_{qq}(z)\times\langle\frac{F_{\rm LM}(z\cdot 1,2)}{z}\rangle.$$

$$P_{qq}(z)=\frac{2C_F}{1-z}+P_{qq}^{\rm reg}(z),\qquad P_{qq}^{\rm reg}(z)=-(1+z)-\epsilon(1-z)\qquad G(z)=\langle\frac{F_{\rm LM}(z\cdot 1,2)}{z}\rangle$$

$$\int\limits_{z_{\rm min}}^{1}\frac{{\rm d}z}{(1-z)^{2\epsilon}}P_{qq}(z)\,G(z)=\int\limits_{0}^{1}{\rm d}z\left[\frac{2C_F}{(1-z)^{1+2\epsilon}}+(1-z)^{-2\epsilon}P_{qq}^{\rm reg}(z)\right]G(z)$$

$$\int\limits_{0}^{1}{\rm d}z\frac{2C_F}{(1-z)^{1+2\epsilon}}G(z)=-\frac{C_F G(1)}{\epsilon}+\int\limits_{0}^{1}{\rm d}z\frac{2C_F}{(1-z)^{1+2\epsilon}}(G(z)-G(1))=$$

$$-\frac{C_F G(1)}{\epsilon}+\sum_{n=0}^{\infty}\frac{(-1)^n(2\epsilon)^n}{n!}\int\limits_{0}^{1}\mathcal{D}_n(z)G(z)\qquad\qquad \mathcal{D}_n(z)=\left[\frac{\ln^n(1-z)}{1-z}\right]_+$$

$$\int\limits_{0}^{1}{\rm d}z\,\mathcal{D}_n(z)G(z)=\int\limits_{0}^{1}{\rm d}z\frac{\ln^n(1-z)}{1-z}\,[G(z)-G(1)]$$

Nested subtraction

It is convenient to rewrite the integral over z through the leading order Altarelli-Parisi splitting function; this function appears in the so-called DGLAP evolution equation that we will discuss later.

We then find divergences proportional to tree-level and boosted tree-level matrix elements; in the latter case, the boost is described by the Altarelli-Parisi splitting function (and additional finite contributions).

$$\langle C_{31} F_{\rm LM}(1,2,3)\rangle = -\frac{[\alpha_s]}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}(2E_1)^{-2\epsilon}\int\limits_{z_{\rm min}}^{1}\frac{{\rm d}z}{(1-z)^{2\epsilon}}P_{qq}(z)\times\langle\frac{F_{\rm LM}(z\cdot 1,2)}{z}\rangle.$$

$$P_{qq}^{(0)}(z) = C_F\,\left(2\mathcal{D}_0(z) - (1+z) + \frac{3}{2}\,\delta(1-z)\right) \qquad \mathcal{D}_n(z) = \left[\frac{\ln^n(1-z)}{1-z}\right]_+$$

$$\langle C_{31} F_{\rm LM}(1,2,3)\rangle = -\frac{[\alpha_s]}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}s^{-\epsilon}\times$$

$$\left[\left(-\frac{C_F}{\epsilon}+\frac{3C_F}{2}\right)\langle F_{\rm LM}(1,2)\rangle + \int\limits_0^1 {\rm d}z\; P_{qq,R}(z)\langle\frac{F_{\rm LM}(z\cdot 1,2)}{z}\right]$$

$$P_{qq,R}(z) = P_{qq}^{(0)}(z) + \epsilon P_{qq,R}^{(\epsilon)}(z) + \mathcal{O}(\epsilon^2),$$

$$P_{qq,R}^{(\epsilon)}(z) = C_F\left[2(1+z)\ln(1-z) - (1-z) - 4\mathcal{D}_1(z)\right]$$

Nested subtraction

We add the collinear limit of the emission off the anti-quark leg and obtain the final result for the real-emission cross section. Note that all divergences have been extracted without integration over resolved kinematic configurations. The divergences are proportional to Born cross sections, including the boosted one. The boost is described by the Altarelli-Parisi splitting function.

$$
\begin{aligned}
2s\mathrm{d}\sigma^R =& 2[\alpha_s]s^{-\epsilon}\left(\frac{C_F}{\epsilon^2}+\frac{3C_F}{2\epsilon}\right)\times\frac{\Gamma^2(1-2\epsilon)}{\Gamma(1-2\epsilon)}\langle F_{\mathrm{LM}}(1,2)\rangle \\
&-\frac{[\alpha_s]s^{-\epsilon}}{\epsilon}\frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\int_0^1 \mathrm{d}z\, P_{qq}^{(0)}(z)\langle\frac{F_{\mathrm{LM}}(z1,2)}{z}+\frac{F_{\mathrm{LM}}(1,z2)}{z}\rangle \\
&-[\alpha_s]s^{-\epsilon}\frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\int_0^1 \mathrm{d}z\, P_{qq,R}(z)^{(\epsilon)}\langle\frac{F_{\mathrm{LM}}(z1,2)}{z}+\frac{F_{\mathrm{LM}}(1,z2)}{z}\rangle+\langle\mathcal{O}_{\mathrm{NLO}}F_{\mathrm{LM}}(1,2,4)\rangle
\end{aligned}
$$

To cancel these divergences, we have to identify other contributions to the NLO cross section that have matching kinematics. There are two candidates: a) virtual corrections since they have Born kinematics; b) PDFs since their modifications boost the final state along the collision axis without giving it a transverse momentum.

Virtual corrections

We begin with the virtual corrections. We can compute them explicitly. We can also make use of the observation by S. Catani who pointed out that the structure of infra-red divergences of one- (and two-) loop (UV renormalized) amplitudes is known in general. In the one-loop case, the divergences are proportional to the Born matrix element, and depend on the color charges of colliding partons and the kinematic invariants.

$$\mathcal{M}_{\text{full}} = \mathcal{M}_0 + \frac{\alpha_s}{2\pi}\mathcal{M}_{1-\text{loop}} + \mathcal{O}(\alpha_s^2),$$
$$\mathcal{M}_{1-\text{loop}} = \hat{I}_1(\epsilon)\, M_0 + \mathcal{M}^{\text{fin}}_{1-\text{loop}}$$

$$\hat{I}_1(\epsilon) = -\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left[\frac{C_F}{\epsilon^2} + \frac{3C_F}{2\epsilon}\right]\left(\frac{\mu^2}{-s}\right)^{-\epsilon}$$

Squaring the amplitude and accounting for the interference, we obtain the contribution of the virtual diagrams to the cross section. Note that the divergent part is fully determined by the leading order cross section; on the contrary, the finite part is process-dependent and needs to be explicitly computed on a case-by-case basis.

$$2s\mathrm{d}\sigma_V = -2[\alpha_s]\cos(\epsilon\pi)\left(\frac{C_F}{\epsilon^2} + \frac{3C_F}{2\epsilon}\right) \times s^{-\epsilon}\langle F_{\text{LM}}(1,2)\rangle + \langle F^{\text{fin}}_{\text{LV}}(1,2)\rangle$$

Real and virtual corrections, combined

Combining real and virtual contributions, we obtain

$$
\begin{aligned}
2s\mathrm{d}\sigma^R =& 2[\alpha_s]s^{-\epsilon}\left(\frac{C_F}{\epsilon^2}+\frac{3C_F}{2\epsilon}\right)\times\frac{\Gamma^2(1-2\epsilon)}{\Gamma(1-2\epsilon)}\langle F_{\mathrm{LM}}(1,2)\rangle \\
&-\frac{[\alpha_s]s^{-\epsilon}}{\epsilon}\frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\int_0^1 \mathrm{d}z\, P_{qq}^{(0)}(z)\langle\frac{F_{\mathrm{LM}}(z1,2)}{z}+\frac{F_{\mathrm{LM}}(1,z2)}{z}\rangle \\
&-[\alpha_s]s^{-\epsilon}\frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\int_0^1 \mathrm{d}z\, P_{qq,R}(z)^{(\epsilon)}\langle\frac{F_{\mathrm{LM}}(z1,2)}{z}+\frac{F_{\mathrm{LM}}(1,z2)}{z}\rangle+\langle\mathcal{O}_{\mathrm{NLO}}F_{\mathrm{LM}}(1,2,4)\rangle
\end{aligned}
$$

$$
2s\mathrm{d}\sigma_V = -2[\alpha_s]\cos(\epsilon\pi)\left(\frac{C_F}{\epsilon^2}+\frac{3C_F}{2\epsilon}\right)\times s^{-\epsilon}\langle F_{\mathrm{LM}}(1,2)\rangle+\langle F_{\mathrm{LV}}^{\mathrm{fin}}(1,2)\rangle
$$

$$
\boxed{
\begin{aligned}
2s\mathrm{d}\sigma^{R+V} =& -\frac{[\alpha_s]s^{-\epsilon}}{\epsilon}\frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\int_0^1 \mathrm{d}z\, P_{qq}^{(0)}(z)\langle\frac{F_{\mathrm{LM}}(z1,2)}{z}+\frac{F_{\mathrm{LM}}(1,z2)}{z}\rangle \\
&+\frac{2\pi^2 C_F}{3}\frac{\alpha_s}{2\pi}\langle F_{\mathrm{LM}}(1,2)\rangle-\frac{\alpha_s}{2\pi}\int_0^1 \mathrm{d}z\, P_{qq,R}^{(\epsilon)}(z)\langle\frac{F_{\mathrm{LM}}(z1,2)}{z}+\frac{F_{\mathrm{LM}}(1,z2)}{z}\rangle+\langle\mathcal{O}_{\mathrm{NLO}}F_{\mathrm{LM}}(1,2,4)\rangle
\end{aligned}
}
$$

Collinear renormalization and the final result

The remaining divergence is associated with the boosted Born kinematics. It can be removed after re-definition of the parton distribution functions. To see that this is possible, one should recognize that the boosted kinematics that appears in the divergent contribution is produced by a change in the PDF (similar to the coupling constants, PDFs must be determined from the experimental measurements).

$$\mathrm{d}\sigma = \sum_{ij} \int \mathrm{d}x_1 \mathrm{d}x_2 f_i(x_1) f_j(x_2) \mathrm{d}\sigma_{ij}(x_1, x_2)$$

$$f_i^{\mathrm{bare}} \to \left[1 + \frac{\alpha_s(\mu)}{2\pi\epsilon} P_{ij}^{(0)} + \ldots\right] \otimes f_j(\mu) \quad [f_1 \otimes f_2](z) = \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 f_1(x_1) f_2(x_2) \delta(z - x_1 x_2)$$

Putting everything together and canceling divergences, we obtain the final formula for the NLO corrections to the fully differential cross section. This formula contains only finite quantities that are split according to the underlying Born configurations. Note that we did not require explicit form of the matrix elements to achieve the cancellation of the infra-red and collinear divergences. However, we assumed that soft and collinear limits of the matrix elements are consistent with QCD. Also, it is crucial that observables that are considered are infra-red and collinear safe.

$$2s\mathrm{d}\sigma^{\mathrm{NLO}} = \langle F_{\mathrm{LV}}^{\mathrm{fin}}(1,2) + \frac{\alpha_s}{2\pi}\frac{2\pi^2}{3} C_F F_{\mathrm{LM}}(1,2)\rangle + \langle \mathcal{O}_{\mathrm{NLO}} F_{\mathrm{LM}}(1,2,4)\rangle$$
$$+ \frac{\alpha_s}{2\pi}\int_0^1 \mathrm{d}z \left[P_{qq}^{(0)}(z)\ln\frac{s}{\mu^2} - P_{qq,R}^{(\epsilon)}(z)\right] \langle \frac{F_{\mathrm{LM}}(z1,2)}{z} + \frac{F_{\mathrm{LM}}(1,z2)}{z}\rangle$$

Summary of the recent developments

1) Computations of cross sections and kinematic distributions at a fully differential level is currently possible at LO, NLO and NNLO. This is an important development of the past twenty years (NNLO -- of the past 2-3 years).

2) NLO subtractions and one-loop computations are automated (i.e. there are public programs that can perform NLO computations for complex (in theory -- arbitrary) final states).

3) It is understood (in practice) how to perform fully differential NNLO calculations. Several methods were proposed that allow us to isolate, regularize and cancel infra-red divergencies without integrating over resolved parts of the phase-space.

4) The extension of the NLO technology discussed in this lecture to NNLO is also possible. Such an extension is based on the observation that divergences that appear as the result of correlated soft and collinear emissions, do not appear in QCD amplitudes (they do appear in individual diagrams) as the consequence of the so-called ``color coherence'' phenomenon (we will talk about it later). This allows to extend the described construction -- almost verbatim -- to the NNLO case.

Summary of recent developments

5) NNLO QCD computations are available for many 2->1 and 2->2 processes (we refer to the multiplicity of a Born process, i.e. pp-> V, pp -> H, pp -> 2jets, pp ->H+j, pp -> Z +j, pp->VV etc.).

6) Efficiencies of the existing NNLO computer codes is an important practical issue; improving on that may require an even better understanding of the underlying structures that appear in the soft and collinear limits.