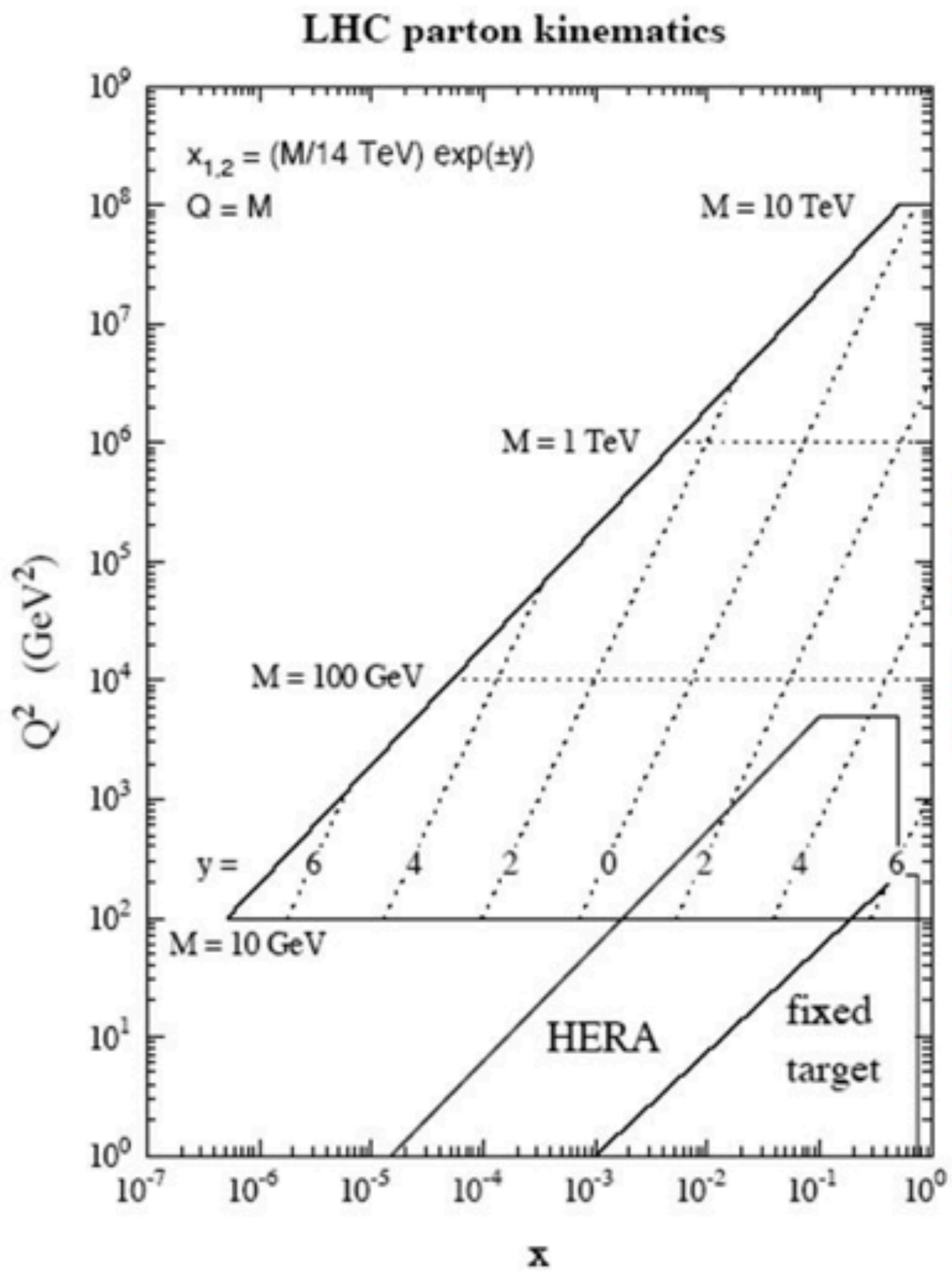
Partons and their evolution



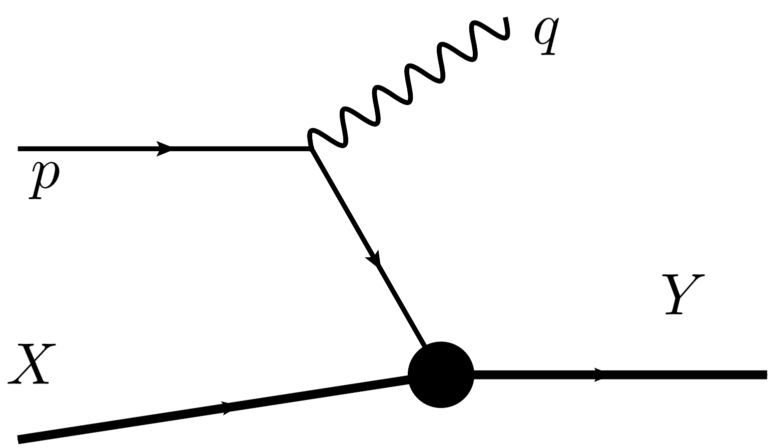
LHC Parton Kinematics

To accurately predict pdf's for the relevant kinematics, we depend on QCD evolution of the structure functions.

So how well do we predict Parton-parton lumis at LHC?

Partons and their evolution

To understand the concept of partons and their evolution we will again consider Quantum Electrodynamics. We will study the following process $e + X \to Y + \gamma$ and assume that we can focus on the photon emitted by an electron. As a consequence of the collinear singularity, photons are predominantly emitted along the direction of the incoming electron. To describe this situation, we use the collinear approximation for matrix elements that we already employed when discussing the NLO corrections to the Drell-Yan process in Lecture 1.



$$q = (1-z)p + \beta\bar{p} + q_\perp, \quad q_\perp \sim \theta_\gamma \ll 1, \quad \beta \sim \theta_\gamma^2$$

$$\sum_{\rm pol} |\mathcal{M}_{e(p)+X\to\gamma+Y}|^2 \approx \frac{-2e^2}{(p-q)^2}\frac{1+z^2}{1-z}\sum_{\rm pol}|\mathcal{M}_{e(zp)+X\to Y}|^2$$

$$\mathrm{d}\sigma_{e+X\to\gamma+Y} = \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_\gamma}\frac{\mathrm{d}^3 P_\gamma}{(2\pi)2E_Y}(2\pi)^4\delta^{(4)}(p+p_X-q-P_Y)\,\frac{1}{4(p_X p)}\frac{1}{2}\sum_{\rm pol}|\mathcal{M}_{eX}|^2$$

$$\mathrm{d}\sigma_{e+X\to\gamma+Y} = \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_\gamma}\frac{\mathrm{d}^3 P_\gamma}{(2\pi)2E_Y}\frac{-2e^2}{(p-q)^2}\frac{1+z^2}{1-z}$$

$$\times\,(2\pi)^4\delta^{(4)}(pz+p_X-P_Y)\,\frac{1}{4z(p_X p)}\frac{1}{2}\sum_{\rm pol}|\mathcal{M}_{e(zp)+X\to Y}|^2$$

$$= \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_\gamma}\frac{-2e^2}{(p-q)^2}\frac{1+z^2}{1-z}\;\mathrm{d}\sigma(zp+X\to Y).$$

Partons and their evolution

We simplify the phase-space and obtain the approximate result for the cross section. To make it well-defined, we need to specify the integration boundaries. The upper limit on the integration over the transverse momentum follows from the kinematics; physically, it indicates the value of the transverse momentum for which the collinear approximation (or neglecting recoil of the final state Y) becomes invalid.

Integration over z is regulated by requiring that soft singularities cancel. The contribution of virtual corrections is fixed by requiring that inclusive quantities are insensitive to collinear logarithms (same idea as in the case of the resummations).

$$\frac{\mathrm{d}^3 q_\gamma}{2(2\pi)^3 E_\gamma} \times \frac{(-2e^2)}{(p-q)^2} = \frac{\alpha}{2\pi}\frac{\mathrm{d}q_\perp^2}{q_\perp^2}\,\mathrm{d}z \qquad \mathrm{d}\sigma_{e+X\to\gamma+Y} = \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}z \frac{1+z^2}{1-z}\int_{m_e^2}^{s}\frac{\mathrm{d}q_\perp^2}{q_\perp^2} \times \mathrm{d}\sigma_{zp+X\to Y}$$

$$\mathrm{d}\sigma^{\rm incl}_{e+X\to Y} = \int_0^1 \mathrm{d}z\;\; \mathrm{d}\sigma_{zp+X\to Y}\left[\delta(1-z) + \frac{\alpha}{2\pi}\left(\left[\frac{1+z^2}{1-z}\right]_+ + V\delta(1-z)\right)\log\frac{s}{m_e^2}\right]$$

$$\mathrm{d}\sigma^{\rm incl}_{e+X\to Y} = \int_0^z \mathrm{d}z\; f_{e/e}(z,s)\;\mathrm{d}\sigma_{zp+X\to Y}$$

$$\int_0^1 \mathrm{d}z f_{e/e}(z,s) = 1 \quad\Rightarrow\quad V=0 \quad\Rightarrow f_{e/e}(z,s) = \delta(1-z) + \frac{\alpha}{2\pi}\left[\frac{1+z^2}{1-z}\right]_+ \ln\frac{s}{m_e^2}$$

Partons and their evolution

The distribution function of an electron in an electron has some instructive properties.

It is a simple delta-function at a low scale (this means that at a low scale there are no emissions and no modification of the spectrum, i.e. the original electron carries all the momentum). As the scale grows, emissions happen and the spectrum gets modified.

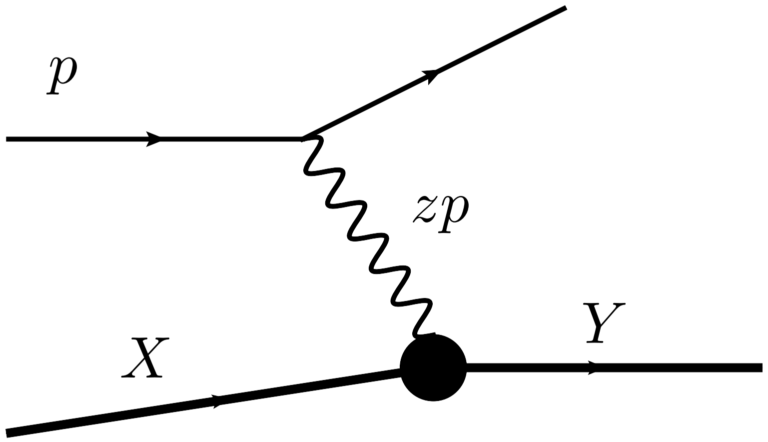
An average momentum carried by an electron is different from the original one because some of its momentum is transferred to the (emitted) photon. If an emitted photon momentum is tagged, we can also talk about the photon distribution function in an electron.

$$f_{e/e}(z,s) = \delta(1-z) + \frac{\alpha}{2\pi}\left[\frac{1+z^2}{1-z}\right]_+ \ln\frac{s}{m_e^2} \qquad P_{ee}(z) = \left[\frac{1+z^2}{1-z}\right]_+ \qquad f_{e/e}(z,m_e^2) = \delta(1-z)$$

$$\int_0^1 \mathrm{d}z\, z f_{e/e}(z) = 1 + \frac{\alpha}{2\pi}\ln\frac{s}{m_e^2}\int_0^1 \mathrm{d}z \frac{1+z^2}{1-z}(z-1) = 1 - \frac{2\alpha}{3\pi}\ln\frac{s}{m_e^2}$$

$$f_{\gamma/e}(z,s) = \frac{\alpha}{2\pi}\frac{1+(1-z)^2}{z}\ln\frac{s}{m_e^2},$$

$$f_{\gamma/e}(z,m_e^2) = 0, \quad \int_0^1 \mathrm{d}z\, z f_{\gamma/e}(z) = \frac{2\alpha}{3\pi}\ln\frac{s}{m_e^2}.$$

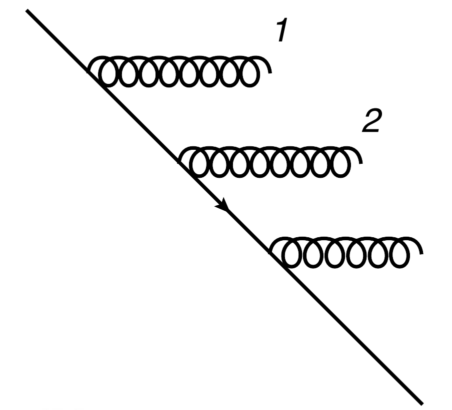


Partons and their evolution

As the previous analysis shows, emission of a single photon leads to the contribution to the cross section that is suppressed by the fine structure constant but, at the same time, enhanced by a large collinear logarithm.

To obtain the same enhancement for multiple photon emissions, we must have emissions ordered in p_T, i.e. transverse momenta must increase towards the hard process. The resulting formula for the cross section reads

$$p_{1\perp} \ll p_{2\perp} \ll p_{3\perp}$$



$$\int_0^1 \mathrm{d}z f_e(z,s) \mathrm{d}\sigma(zp) = \int_0^1 \mathrm{d}z \mathrm{d}\sigma(zp) \Bigg[\delta(1-z) + \frac{\alpha}{2\pi} \int \mathrm{d}z_1 P_{ee}(z_1) \int_{m_e^2}^{s} \frac{\mathrm{d}p_{1,\perp}^2}{p_{1,\perp}^2} \delta(z_1 - z)$$

$$+ \left(\frac{\alpha}{2\pi}\right)^n \int \mathrm{d}z_2 P_{ee}(z_2) \int_{m_e^2}^{s} \frac{\mathrm{d}p_{2,\perp}^2}{p_{2,\perp}^2} \int \mathrm{d}z_1 P_{ee}(z_1) \int_{m_e^2}^{p_{2,\perp}^2} \frac{\mathrm{d}p_{1,\perp}^2}{p_{1,\perp}^2} \delta(z_1 z_2 - z)$$

$$+ \left(\frac{\alpha}{2\pi}\right)^n \int \mathrm{d}z_n P_{ee}(z_n) \int_{m_e^2}^{s} \frac{\mathrm{d}p_{n,\perp}^2}{p_{n,\perp}^2} \int \mathrm{d}z_{n-1} P_{ee}(z_{n-1}) \int_{m_e^2}^{p_{n,\perp}^2} \frac{\mathrm{d}p_{n-1,\perp}^2}{p_{n1,\perp}^2} \ldots . \delta(z_1 z_2 \ldots z_n - z) + .. \Bigg]$$

This formula looks sufficiently complicated but it can be turned into an integro-differential (evolution) equation for the distribution function.

Partons and their evolution

$$f_{e/e}(z,s) = \delta(1-z) + \sum_{n=1}^{\infty} \left(\frac{\alpha}{2\pi}\right)^n \int_0^1 \mathrm{d}z_n P_{ee}(z_n) \int_{m_e^2}^{s} \frac{\mathrm{d}p_{n,\perp}^2}{p_{n,\perp}^2} \times \int_0^1 \mathrm{d}z_{n-1} P_{ee}(z_{n-1}) \int_{m_e^2}^{p_n^2} \frac{\mathrm{d}p_{n-1,\perp}^2}{p_{n-1,\perp}^2} \times \ldots\ldots$$

$$\times \int_0^1 \mathrm{d}z_1 P_{ee}(z_1) \int_{m_e^2}^{p_2^2} \frac{\mathrm{d}p_{1,\perp}^2}{p_{1,\perp}^2} \;\; \delta(z - z_1 \ldots z_n)$$

Taking a derivative with respect to the log(s), we obtain the celebrated DGLAP (Dokshitzer - Gribov - Lipatov - Altarelli-Parisi) evolution equation for the splitting function of an electron in an electron. In the derivation, we have neglected a possibility that an electron can transform into a photon; this introduces a mixing into an evolution.

$$s\frac{\partial f_{e/e}(z,s)}{\partial s} = \frac{\alpha}{2\pi} \sum_{n=0}^{\infty} \int_0^1 \mathrm{d}z_1 P_{ee}(z_1) \left[\delta(z-z_1) + \int_0^1 \mathrm{d}z_2 P_{ee}(z_2) \int_{m_e^2}^{s} \frac{\mathrm{d}p_{2,\perp}^2}{p_{2,\perp}^2} \delta(z - z_1 z_2) + \ldots \right]$$

$$\delta(z - z_1 X) = \frac{1}{z_1}\delta(z/z_1 - X), \quad \Rightarrow \quad \boxed{s\frac{\partial f_{e/e}(z,s)}{\partial s} = \frac{\alpha}{2\pi} \int_0^1 \frac{\mathrm{d}z_1}{z_1} P_{ee}(z_1) f_{e/e}(z/z_1, s)}$$

Partons and their evolution

Generalization to QCD is, in principle, straightforward (there are subtleties). Quarks and gluons are ``partons”; partons can ``split” to each other. Splitting is described in a collinear kinematics. The coupling constant becomes scale-dependent. Which quark flavors can be considered as ``part of the proton” depends on the hardness of the process (eventually, all of them). The initial condition for the evolution at low scales is not known (in contrast to our QED example, this is a non-perturbative problem).

$$\frac{\partial q_i(z,s)}{\partial s} = \frac{\alpha_s(s)}{2\pi} \int\limits_0^1 \frac{\mathrm{d}\xi}{\xi} \Bigg[P_{q\to q}(\xi) q_i(\xi/z) + P_{\bar{q}\to q}(\xi) \bar{q}_i(z/\xi, s)$$

$$+ P_{q'\to q}(\xi) \sum_{j\neq i} \left(q_j(z/\xi, s) + \bar{q}_j(z/\xi, s) \right) + P_{g\to q}(\xi)\; g(z/\xi, s) \Bigg]$$

$$\frac{\partial \bar{q}_i(z,s)}{\partial s} = \frac{\alpha_s(s)}{2\pi} \int\limits_0^1 \frac{\mathrm{d}\xi}{\xi} \Bigg[P_{q\to q}(\xi) \bar{q}_i(\xi/z) + P_{\bar{q}\to q}(\xi) q_i(z/\xi, s)$$

$$+ P_{q'\to q}(\xi) \sum_{j\neq i} \left(q_j(z/\xi, s) + \bar{q}_j(z/\xi, s) \right) + P_{g\to q}(\xi)\; g(z/\xi, s) \Bigg].$$

$$\frac{\partial g(z,s)}{\partial s} = \frac{\alpha_s(s)}{2\pi} \int\limits_0^1 \frac{\mathrm{d}\xi}{\xi} \left[P_{q\to g}(\xi)\; \sum \left(q_j(z/\xi, s) + \bar{q}_j(z/,\xi) \right) + P_{g\to g}(\xi)\; g(z/\xi, s) \right]$$

Partons and their evolution

Altarelli-Parisi splitting kernels are known to third order in QCD perturbation theory. I will show here the results for the first order.

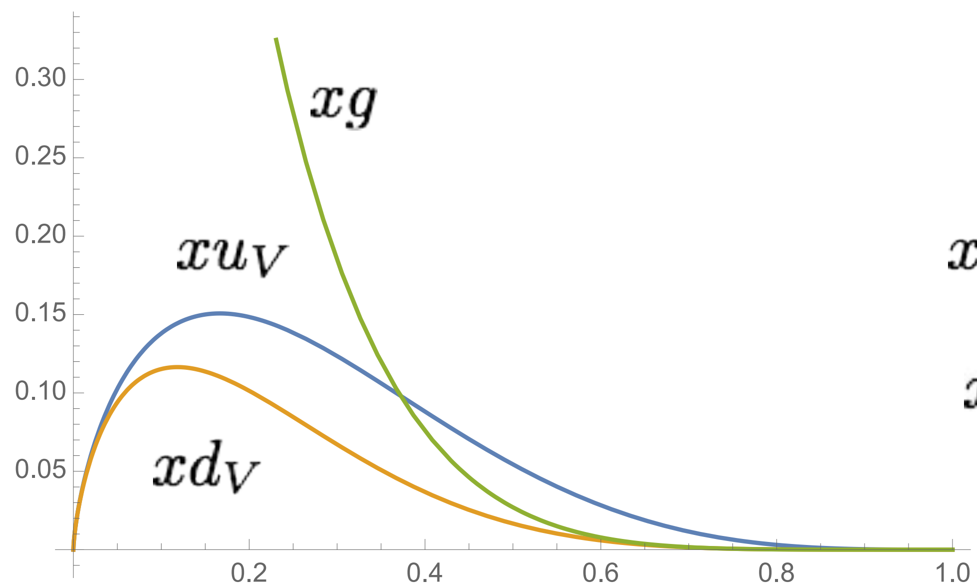
$$P_{q\to q}(z) = C_F \left[\frac{1+z^2}{1-z}\right]_+ \qquad\qquad P_{q\to g}(z) = C_F \left(\frac{1+(1-z)^2}{z}\right)$$

$$P_{g\to q}(z) = T_R \left(z^2 + (1-z)^2\right)$$

$$P_{g\to g}(z) = 2C_A \left[\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z) + \left(\frac{11}{6}C_A - \frac{2n_f T_R}{3}\right)\delta(1-z)\right]$$

Partons and their evolution

The DGLAP equations imply that parton distribution functions at any scale can be determined if they are known at some scale. So, the strategy is to parametrize PDFs at a relatively low scale and then use evolution and various data to constrain (determine) PDFs. Propagation of errors is an important question that is being constantly discussed and refined.



$$Q_0 = 3 \text{ GeV}$$

$$xu_V \sim x^{0.7}(1-x)^{3.5}, \quad xd_v \sim x^{0.7}(1-x)^{5.2}$$

$$xu_s \sim xd_s \sim x^{-0.2}(1-x)^7, \quad xg \sim x^{-0.2}(1-x)^{5.4}$$

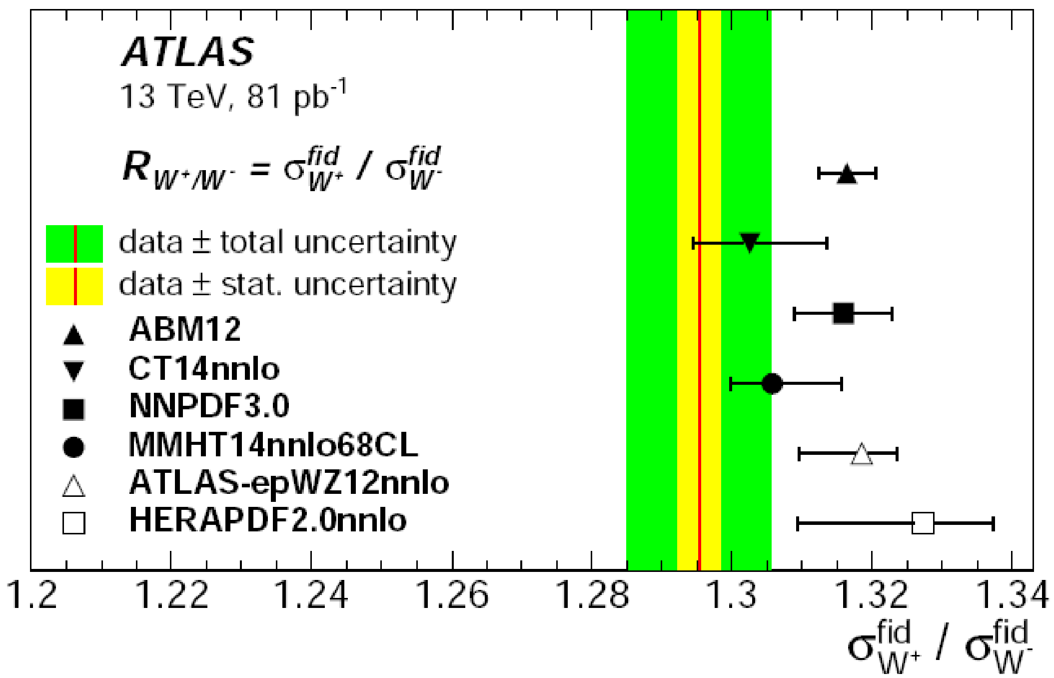
Another complication is that extraction of PDFs involves fixed order cross section computations. When fixed order results change (i.e. by accounting for higher order corrections), the PDFs change as well (provided, of course, that data does not). It is therefore customary to extract PDFs using fixed order cross sections of certain accuracy (LO, NLO, NNLO). These (LO, NLO, NNLO) PDFs sets should be used to predict physical observables using matching orders in computed partonic cross sections.

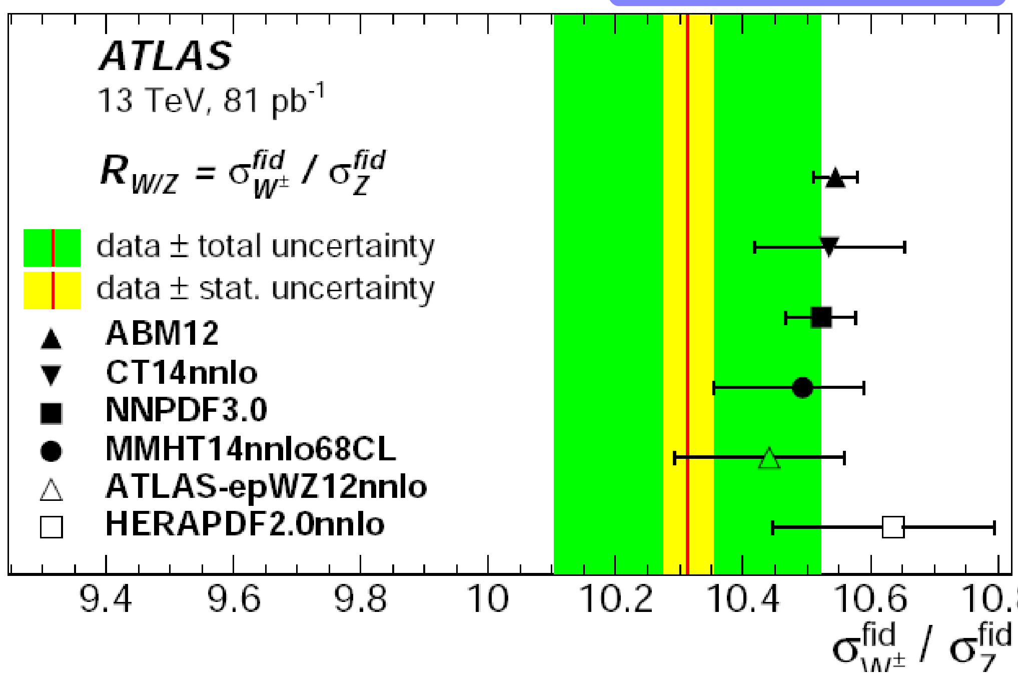
$$\sigma_{\rm exp} = \sigma^{\rm F.O.} \times {\rm PDF} \qquad\qquad {\rm PDF} \to {\rm PDF}_{\rm F.O.}$$

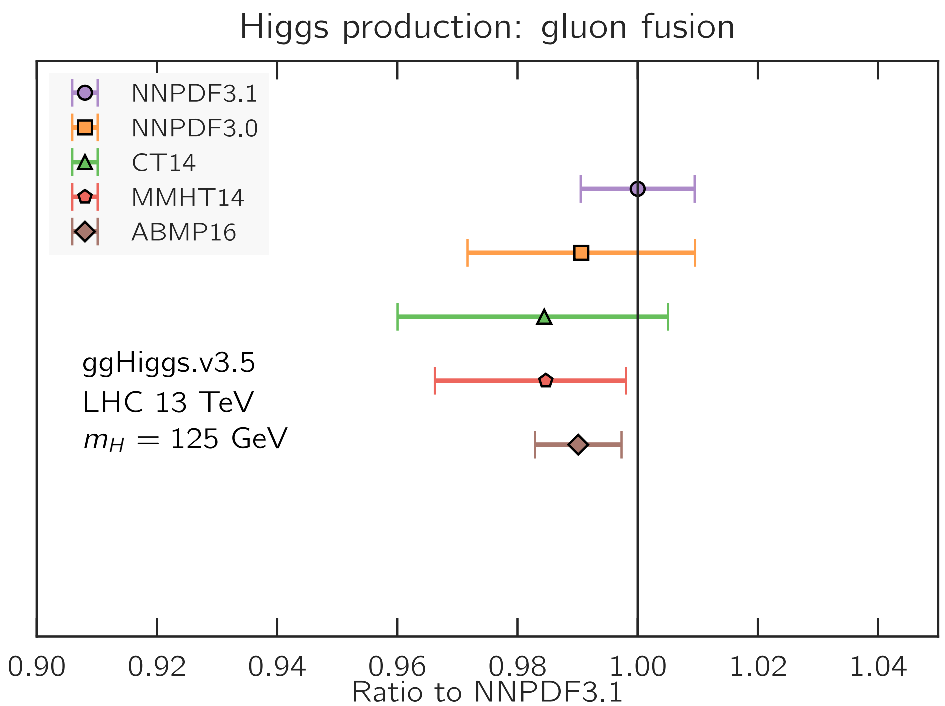
Parton distribution functions

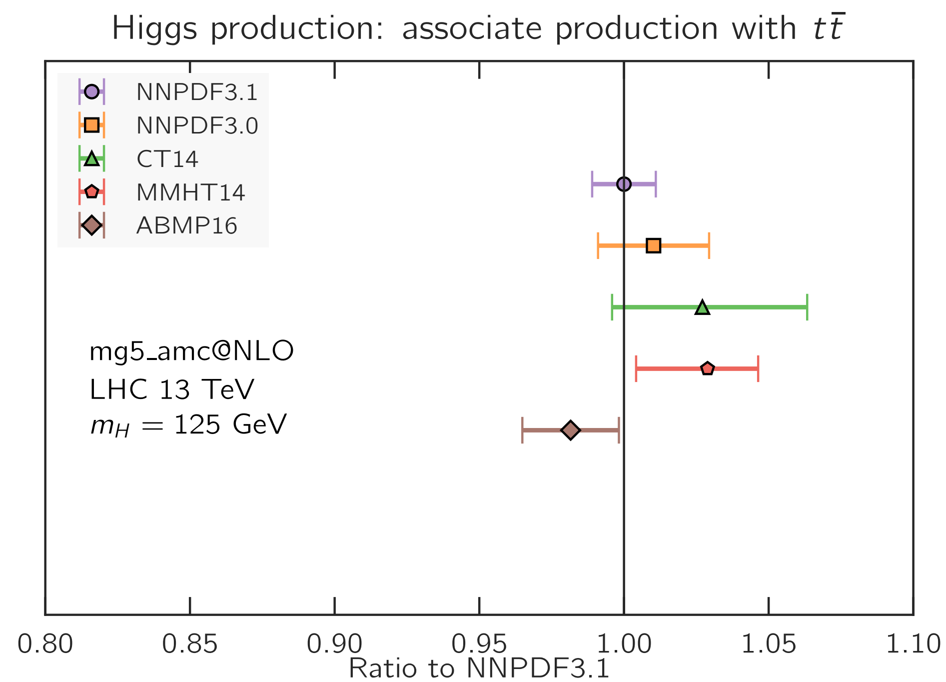
Below are some phenomenological results that demonstrate the current state-of-the-art in PDF determination for both Drell-Yan and Higgs boson production. In general, the situation appears to be quite reasonable, with results of different PDF-fitting groups showing signs of convergence (was not always the case).

arXiv:1603.09222

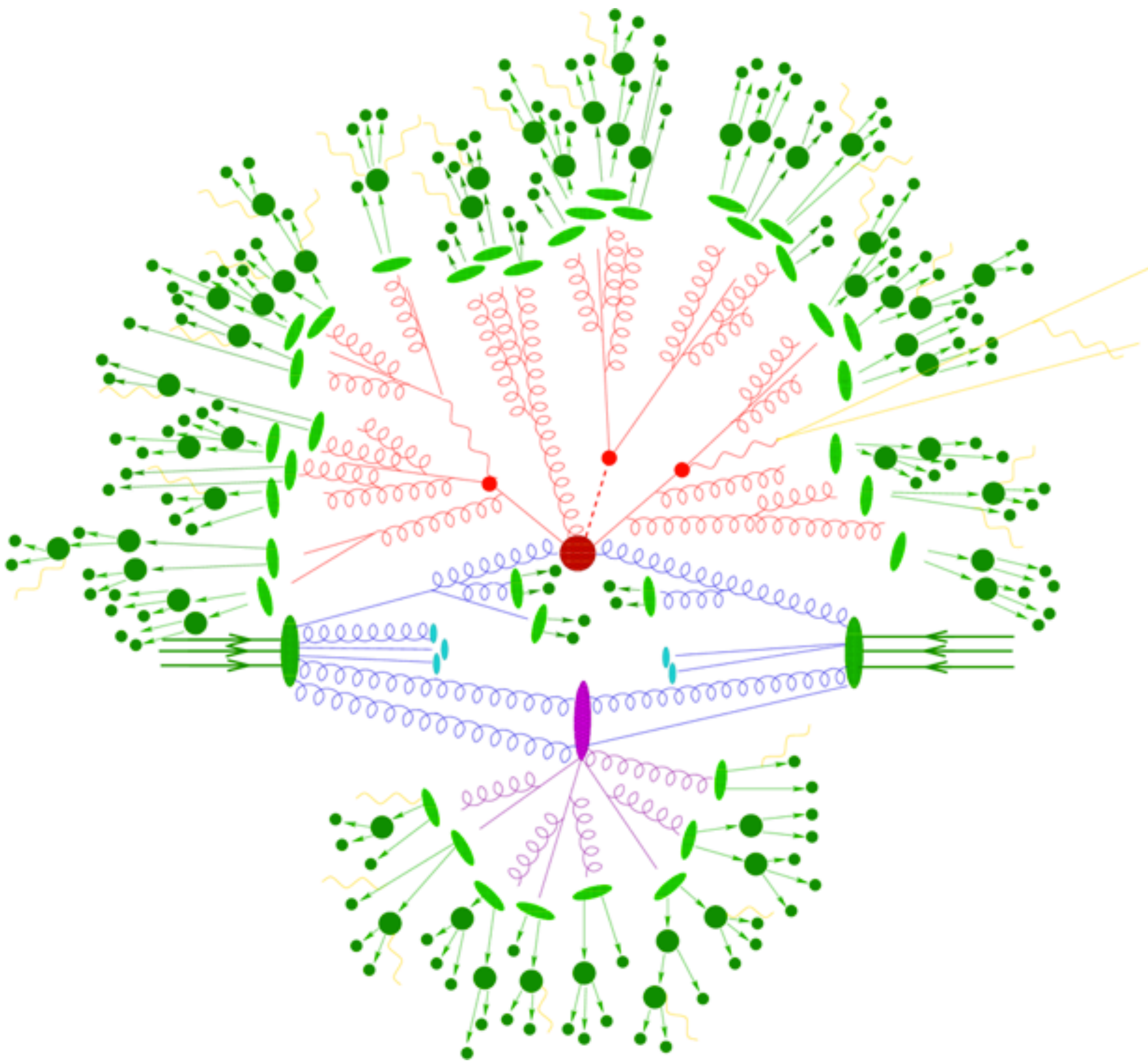








Parton showers



Parton showers

Parton showers are real workhorses among simulation tools for hadron colliders. They are used everywhere. Therefore, it is important to understand which physics is caputed by the parton showers and which one is not. This is quite a non-trivial issue that seems to cause some confusion even among the experts.

I will start this discussion with a toy model of soft emissions in QED, ignoring all the directional information about emitted photons.

$$\mathcal{M}_n = e^n \mathcal{M}_0 \prod_1^n \epsilon_i^\mu \left[\frac{p_\mu}{pk_i} - \frac{\bar{p}_\mu}{\bar{p}k_i} \right] \Rightarrow e^n M_0 \prod_1^n \frac{1}{\omega_i}$$

$$\mathrm{d}\sigma_n = \frac{1}{n!}\alpha^n \mathrm{d}\sigma_0 \mathrm{d}E \prod_{i=1}^n \frac{\mathrm{d}\omega_i}{\omega_i} \, \delta(E_t - E - \sum_{i=1}^n \omega_i) \qquad \mathrm{d}\sigma_n = \frac{\alpha^n}{n!} \mathrm{d}\sigma_0 \prod_{i=1}^n \frac{\mathrm{d}\omega_i}{\omega_i} \, \theta(E_t - \sum_{i=1}^n \omega_i)$$

$$\sigma_n = \frac{\alpha^n \sigma_0}{n!} \int_\lambda^{E_t} \frac{\mathrm{d}\omega_1}{\omega_1} \int_\lambda^{E_t-\omega_1} \frac{\mathrm{d}\omega_2}{\omega_2} \cdots \int_\lambda^{E_t-\omega_1 \ldots \omega_{n-1}} \frac{\mathrm{d}\omega_n}{\omega_n} \Rightarrow \sigma_n = \frac{\alpha^n \sigma_0}{n!} \int_\lambda^{E_t} \frac{\mathrm{d}\omega_1}{\omega_1} \int_\lambda^{E_t} \frac{\mathrm{d}\omega_2}{\omega_2} \cdots \int_\lambda^{E_t} \frac{\mathrm{d}\omega_n}{\omega_n}$$

$$\int_\lambda^{E_t} \frac{\mathrm{d}\omega_1}{\omega_1} \int_\lambda^{E_t-\omega_1} \frac{\mathrm{d}\omega_2}{\omega_2} = \int_\lambda^{E_t} \frac{\mathrm{d}\omega_1}{\omega_1} \log \frac{E_T - \omega_1}{\lambda} = \log^2 \frac{E_t}{\lambda} - \frac{\pi^2}{6} + \mathcal{O}(\lambda/E_t)$$

$$\sigma_n = \frac{\alpha^n \sigma_0}{n!} \left[\log \frac{E_t}{\lambda} \right]^n \qquad \mathrm{d}\sigma_{\mathrm{tot}} = \sum_{n=0}^\infty \sigma_n = \sigma_0 e^{\alpha \log \frac{E_t}{\lambda}} = \sigma_0 e^{\alpha \int_\lambda^{E_t} \frac{\mathrm{d}\omega}{\omega}}$$

Parton showers

The inclusive cross section can not depend on the infra-red regulator or contain large logarithmic corrections (the KLN theorem). The virtual corrections are included by requiring that the total cross section equals to the Born cross section (a statement known as the unitarity of a parton shower). As we have seen, the same idea is used in resummations and in deriving the DGLAP evolution equation.

$$\sigma_n = \frac{\alpha^n \sigma_0}{n!}\left[\log\frac{E_t}{\lambda}\right]^n \qquad d\sigma_{\rm tot} = \sum_{n=0}^{\infty}\sigma_n = \sigma_0 e^{\alpha\log\frac{E_t}{\lambda}} = \sigma_0 e^{\alpha\int_\lambda^{E_t}\frac{d\omega}{\omega}}$$

$$\sigma_{\rm tot} = \sigma_0 = \sigma_0 e^{-\alpha\int_\lambda^{E_t}\frac{d\omega}{\omega}} e^{\alpha\int_\lambda^{E_t}\frac{d\omega}{\omega}} = \sigma_0\sum_{n=0}^{\infty}\mathcal{P}_n \qquad 1 = \sum_n \mathcal{P}_n$$

$$\mathcal{P}_n = \alpha^n e^{-\phi(E_T,\lambda)}\int_\lambda^{E_t}\frac{d\omega_1}{\omega_1}\int_\lambda^{\omega_1}\frac{d\omega_1}{\omega_1}\ldots\int_\lambda^{\omega_{n-1}}\frac{d\omega_n}{\omega_n}, \qquad \boxed{\phi(\omega_1,\omega_2) = \alpha\int_{\omega_1}^{\omega_2}\frac{d\omega}{\omega}}$$

$$\mathcal{P}_n = \alpha^n\int_\lambda^{E_T}\frac{d\omega_1}{\omega_1}e^{-\phi(E_t,\omega_1)}\int_\lambda^{\omega_1}\frac{d\omega_2}{\omega_2}e^{-\phi(E_t,\omega_2)}\ldots\ldots\int_\lambda^{\omega_{n-1}}\frac{d\omega_n}{\omega_n}e^{-\phi(\omega_{n-1},\omega_n)}e^{-\phi(\omega_n,\lambda)}$$

$$\omega_i \to r_i = e^{-\phi(\omega_{i-1},\omega_i)}, \qquad \omega_0 = E_t.$$

$$dr_i = \alpha\frac{d\omega_i}{\omega_i}e^{-\phi(\omega_{i-1},\omega_i)}, \qquad r_i^{\rm min} < r_i < 1, \quad r_i^{\rm min} = e^{-\phi(\omega_{i-1},\lambda)}$$

Parton showers

We find that the probability to emit exactly n photons is given by the following expression

$$\mathcal{P}_n = \int\limits_{r^{\min}(\omega_0)}^{1} \mathrm{d}r_1 \int\limits_{r^{\mathrm{in}}(\omega_1)}^{1} \mathrm{d}r_2 \ldots . \int\limits_{r^{\min}(\omega_{n-1})} \mathrm{d}r_n e^{-\phi(\omega_n,\lambda)}$$

and the inclusive n-photon emission probability reads

$$\mathcal{P}_{n+X} = \int\limits_{r^{\min}(\omega_0)}^{1} \mathrm{d}r_1 \int\limits_{r^{\mathrm{in}}(\omega_1)}^{1} \mathrm{d}r_2 \ldots . \int\limits_{r^{\min}(\omega_{n-1})} \mathrm{d}r_n$$

$$\times \left[e^{-\phi(\omega_n,\lambda)} + \int\limits_{r^{\min}(\omega_n)}^{1} \mathrm{d}r_{n+1} + \int\limits_{r^{\min}(\omega_n)}^{1} \mathrm{d}r_{n+1} \int\limits_{r^{\min}(\omega_{n+1})}^{1} \mathrm{d}r_{n+2} + \ldots . \right]$$

$$= \int\limits_{r^{\min}(\omega_0)}^{1} \mathrm{d}r_1 \int\limits_{r^{\mathrm{in}}(\omega_1)}^{1} \mathrm{d}r_2 \ldots . \int\limits_{r^{\min}(\omega_{n-1})} \mathrm{d}r_n$$

$$\mathcal{P}_0 = e^{-\phi(E_T,\lambda)}, \quad \mathcal{P}_{1+X} = \int\limits_{r^{\min}(\omega_0)}^{1} \mathrm{d}r_1, \quad \mathcal{P}_{2+X} = \int\limits_{r^{\min}(\omega_0)}^{1} \mathrm{d}r_1 \int\limits_{r^{\min}(\omega_1)}^{1} \mathrm{d}r_2, \ldots \ldots$$

Parton showers

Parton showers are used to generate events that occur with certain probabilities. The above results suggest a simple way to do that.

Indeed, the probability to have an event with no photons is $\mathcal{P}_0 = e^{-\phi(E_t,\lambda)}$.
The inclusive probability to emit one photon

$$\mathcal{P}_{1+X} = \int_{r^{\min}(E_t)}^{1} \mathrm{d}r_1 = 1 - \mathcal{P}_0$$

The energy of the emitted photon is described by the variable r_1 that is uniformly distributed. Therefore, by generating a random number between zero and one, we can tell if the emission happened and -- if it did happen -- what is the energy of the emitted photon.

If the first photon was emitted, the probability to emit the second is described by a similar formula, where the only change is the energy in the no-emission probability.

$$\mathcal{P}_{2+X} = \int_{r^{\min}(\omega_0)} \mathrm{d}r_1 \int_{r^{\min}(\omega_1)}^{1} \mathrm{d}r_2$$

This is the key for the algorithm that is employed in constructing the parton showers.

Parton showers

It is now straightforward to provide the precise definition of the algorithm that allows us to generate events iteratively.

1. Decide if at least one emission happened. To this end, choose a random number between 0 and 1 and solve the equation for the photon energy.

$$\xi = e^{-\phi(E_T,\omega)} \Rightarrow \omega = E_T \, e^{\frac{1}{\alpha} \ln \xi}.$$

2. If the photon energy is smaller than the photon mass, no emissions happen; this is an ``elastic" event. Store the event. Start over.

3. If the photon energy exceeds the photon mass, the emission did happen and its energy (ω_1) has been determined. Decide if the second emission happened. To this end, go back to point 1 and replace $E_t \to \omega_1$. Continue as long as emissions keep happening. Once no emission occurs, store the full event. Start over if another event is needed.

This procedure allows us to generate unweighted events and then use them in exactly the same way as experimental events are used for physical analyses. Key requirement is an ability to describe growing multiplicities as a probabilistic process without ``a long-distance" memory.

Parton showers

Having discussed the toy model, we have to understand how to generalize it to the case of real gauge theories. I will again start with the QED and make use of the discussion that we had about parton distribution functions.

We need to find a ``conserved" quantity -- a total probability-- that is connected to sequences of emissions in some way. To this end, recall that the integral of the electron distribution function in an electron is scale-independent and can be chosen to be one. Recall that we arrived at the concept of the distribution function considering arbitrary number of collinear emissions. We now put the two remarks together and we are ready to go.

$$s\frac{\partial f_{e/e}(z,s)}{\partial s} = \frac{\alpha}{2\pi}\int\limits_0^1 \frac{\mathrm{d}z_1}{z_1} P_{ee}(z) f_{e/e}(z/z_1,s)$$

$$s\frac{\partial f_{e/e}(z,s)}{\partial s} = \frac{\alpha}{2\pi}\int\limits_0^1 \frac{\mathrm{d}\xi}{\xi}\, P_{ee}(\xi)\, f_{e/e}(z/\xi,s)$$

$$= \frac{\alpha}{2\pi}\int\limits_0^1 \mathrm{d}\xi \mathrm{d}\eta P_{ee}(\xi)\, f_{e/e}(\eta,s)\delta(z-\xi\eta)$$

$$s\frac{\partial}{\partial s}\int\limits_0^1 \mathrm{d}z f_{e/e}(z,s) = \frac{\alpha}{2\pi}\int\limits_0^1 \mathrm{d}\xi P_{ee}(\xi)\int\limits_0^1 \mathrm{d}\eta f_{e/e}(\eta,s) = 0$$

$$\int\limits_0^1 \mathrm{d}z f_{e/e}(z,s) = \int\limits_0^1 f_{e/e}(z,m_e^2) = 1.$$

Parton showers

$$P_{ee}(z) = \left[\frac{1+z^2}{1-z}\right]_+ \Rightarrow \quad s\frac{\partial f_{e/e}(z,s)}{\partial s} = \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}\xi \frac{1+\xi^2}{1-\xi}\left[f_{e/e}(z/\xi,s) - f_{e/e}(z,s)\right].$$

$$\bar{P}_{ee}(z) = \frac{1+z^2}{1-z} \quad \Rightarrow \boxed{s\frac{\partial f_{e/e}(z,s)}{\partial s} + \frac{\alpha}{2\pi}\int_0^{1-\lambda} \mathrm{d}\xi \bar{P}_{ee}(\xi) f(z,s) = \frac{\alpha}{2\pi}\int_0^{1-\lambda} \mathrm{d}\xi \bar{P}_{ee}(\xi) f_{e/e}(z/\xi,s)}$$

$$s\frac{\partial f_{e/e}(z,s)}{\partial s} + \frac{\alpha}{2\pi}\int_0^{1-\lambda} \mathrm{d}\xi \bar{P}_{ee}(\xi) f_{e/e}(z,s) = \frac{\alpha}{2\pi}\int_0^{1-\lambda} \frac{\mathrm{d}\xi}{\xi} \bar{P}_{ee}(\xi) f_{e/e}(z/\xi,s).$$

$$f_{e/e}(z,s) = \Delta(s)g(s,z) \quad \Rightarrow \quad s\frac{\partial \Delta(s)}{\partial s} + \frac{\alpha}{2\pi}\int_0^{1-\lambda} \mathrm{d}\xi \bar{P}_{ee}(\xi)\Delta(s) = 0$$

$$\Delta(s,s_0) = \exp\left[-\frac{\alpha}{2\pi}\int_{s_0}^{s}\frac{\mathrm{d}t}{t}\int_0^{1-\lambda}\mathrm{d}\xi \bar{P}_{ee}(\xi)\right] \qquad g(s,z) = g(s_0,z) + \frac{\alpha}{2\pi}\int_{s_0}^{s}\frac{\mathrm{d}t}{t}\int_0^{1-\lambda}\frac{\mathrm{d}\xi}{\xi}\bar{P}_{ee}(\xi) g(z/\xi,t)$$

$$\boxed{f_{e/e}(z,s) = \Delta(s,s_0) f_{e/e}(z,s_0) + \frac{\alpha}{2\pi}\int_{s_0}^{s}\frac{\mathrm{d}t}{t}\Delta(s,t)\int_0^{1-\lambda}\frac{\mathrm{d}\xi}{\xi}\bar{P}_{ee}(\xi) f_{e/e}(z/\xi,t)}$$

We have introduced the so-called Sudakov form factor; as we will now show, it describes the no-emission probability.

Parton showers

We can solve the modified evolution equation iteratively, choosing the mass of the electron as the initial scale. The resulting equation is structurally identical to the probability conservation equation where contributions of states with fixed number of photons are well-defined. We also see immediately the physical meaning of the Sudakov form factor -- it describes virtual corrections and summarizes their impact on the probability of a no-emission process.

$$f(s,z) = \Delta(s,s_0) f(s_0,z) + \frac{\alpha}{2\pi} \int_{s_0}^{s} \frac{\mathrm{d}t}{t} \Delta(s,t) \int_0^{1-\lambda} \frac{\mathrm{d}\xi}{\xi} P_{ee}(\xi) \; f(z/\xi, t)$$

$$\Delta(s,s_0) = \exp\left[-\frac{\alpha}{2\pi} \int_{s_0}^{s} \frac{\mathrm{d}t}{t} \int_0^{1-\lambda} \mathrm{d}\xi P_{ee}(\xi) \right] \qquad P_{ee}(z) = \frac{1+z^2}{1-z}$$

$$\begin{aligned} f(s,z) &= \Delta(s,s_0)\delta(1-z) + \frac{\alpha}{2\pi} \int_{s_0}^{s} \frac{\mathrm{d}t_1}{t_1} \Delta(s,t_1) \int_0^{1-\lambda} \mathrm{d}\xi_1 P_{ee}(\xi_1) \Delta(t_1,s_0) \delta(z-\xi_1) \\ &+ \left(\frac{\alpha}{2\pi}\right)^2 \int_{s_0}^{s} \frac{\mathrm{d}t_1}{t_1} \Delta(s,t_1) \int_0^{1-\lambda} \mathrm{d}\xi_1 P_{ee}(\xi_1) \int_{s_0}^{t_1} \frac{\mathrm{d}t_2}{t_2} \Delta(t_1,t_2) \int_0^{1-\lambda} \mathrm{d}\xi_2 P_{ee}(\xi_2) \Delta(t_2,s_0) \delta(z-\xi_1\xi_2) \\ &+ \ldots . \end{aligned}$$

Parton showers

Integrating the last equation over z, we obtain the ``probability conservation" equation. We can re-write it as we did in case of the toy model if we change variables similar to what we did before. The probability to emit a photon with the particular transverse momentum and particular energy is flatly distributed in appropriate variables. This allows us to generate unweighted events in a way that is similar to what was discussed in the context of the toy model.

$$1 = \Delta(s,s_0) + \frac{\alpha}{2\pi}\int\limits_{s_0}^{s}\frac{\mathrm{d}t_1}{t_1}\Delta(s,t_1)\int\limits_{0}^{1-\lambda}\mathrm{d}\xi_1 P_{ee}(\xi_1)\Delta(t_1,s_0)$$

$$+\left(\frac{\alpha}{2\pi}\right)^2\int\limits_{s_0}^{s}\frac{\mathrm{d}t_1}{t_1}\Delta(s,t_1)\int\limits_{0}^{1-\lambda}\mathrm{d}\xi_1 P_{ee}(\xi_1)\int\limits_{s_0}^{t_1}\frac{\mathrm{d}t_2}{t_2}\Delta(t_1,t_2)\int\limits_{0}^{1-\lambda}\mathrm{d}\xi_2 P_{ee}(\xi_2)\Delta(t_2,s_0)+\ldots.$$

$$t_n \to r_n, \quad \xi_m \to \eta_m, \quad r_n = \Delta(t_{n-1},t_n), \quad \eta_n = \frac{\int\limits_0^{\xi_n}\mathrm{d}\xi P_{ee}(\xi)}{\int\limits_0^{1-\lambda}\mathrm{d}\xi P_{ee}(\xi)}$$

$$\frac{\alpha}{2\pi}\frac{\mathrm{d}t_n}{t_n}\,\Delta(t_{n-1},t_n)\;\mathrm{d}\xi_n\;P_{ee}(\xi_n) = \mathrm{d}r_n\,\mathrm{d}\eta_n$$

$$1 = r^{\min}(t_0) + \int\limits_{r^{\min}(t_0)}^{1}\mathrm{d}r_1\int\limits_0^1\mathrm{d}\eta_1 + \int\limits_{r^{\min}(t_0)}^{1}\mathrm{d}r_1\int\limits_0^1\mathrm{d}\eta_1\int\limits_{r^{\min}(t_1)}^{1}\mathrm{d}r_2\int\limits_0^1\mathrm{d}\eta_2+\ldots$$

Parton showers

Here is an algorithm to generate events in the context of QED collinear splittings.

1) Start by generating a random variable r. Solve the equation $r = \Delta(s, t_1)$ for t_1. If $t_1 < m_e^2$, no emission happened; exit and start over if needed.

2) If $t_1 > m_e^2$, generate another random variable and solve the equation $\eta = \frac{\int_0^z \mathrm{d}\xi \bar{P}_{ee}(\xi)}{\int_0^1 \mathrm{d}\xi \bar{P}_{ee}(\xi)}$ for the variable z.

3) Record a photon with momentum (randomize the direction of the transverse momentum)

$$k_1 = (1 - z_1)p + \frac{t_1}{(1 - z_1)2p\bar{p}}\bar{p} + \sqrt{t_1} n_\perp^\mu$$

4) Go back to the point one. Replace s with t_1 and t_1 with t_2 and proceed with the generation of the kinematics of the second photon. If successful, record an additional photon and continue until no photon emission is generated.

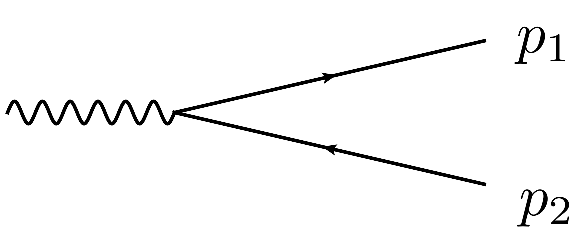
$$k_2 = (1 - z_2)z_1 p + \frac{t_2}{(1 - z_2)z_1 2p\bar{p}}\bar{p} + \sqrt{t_2} n_\perp^\mu$$

Parton showers

We have seen that collinear emissions can be described using parton showers. However, collinear emissions are not the only contributions in pQCD that leads to enhanced corrections; the other potentially large corrections are caused by soft emissions.

However, soft emissions are different from the collinear ones since interferences appear naturally. Dealing with an interference within a parton shower framework is difficult because identifying positive-definite ``probabilities" becomes obscure. There is an interesting way to deal with this problem which we will now describe.

Consider a situation where a virtual photon splits into an electron-positron pair. We assume that the photon is boosted, so that electron-positron pair has some (smallish) opening angle. We will describe an emission of soft photons from this system.



$$\mathcal{M} \approx e \left(\frac{p_1 \epsilon}{p_1 k} - \frac{p_2 \epsilon}{p_2 k} \right) \mathcal{M}_0 \qquad \sum_{\rm pol} |\mathcal{M}|^2 = \frac{e^2 2 p_1 p_2}{(p_1 k)(p_2 k)} |\mathcal{M}_0|^2.$$

$$\mathrm{d}\sigma = \mathrm{d}\sigma_0 \frac{\alpha}{2\pi} \frac{\mathrm{d}\omega_k}{\omega_k} \frac{\mathrm{d}\Omega_k}{(2\pi)} \left[\frac{2 p_1 p_2 \omega_k^2}{(p_1 k)(p_2 k)} \right] \qquad p_i p_j = E_i E_j \xi_{ij} \Rightarrow \frac{2 p_1 p_2 \omega_k^2}{(p_1 k)(p_2 k)} = \frac{2\xi_{12}}{\xi_{1k}\xi_{2k}} = 2W(1,2;k)$$

Parton showers

$$W(1,2;k) = \frac{2\xi_{12}}{\xi_{1k}\xi_{2k}} = \frac{1}{2}\left(\frac{\xi_{12}}{\xi_{1k}\xi_{2k}} - \frac{1}{\xi_{2k}} + \frac{1}{\xi_{1k}}\right) + (1 \Leftrightarrow 2)$$

$$W(1,2;k) = W_1(1,2;k) + W_2(1,2;k)$$

We want to interpret the first term on the right hand side as an emission off the particle 1 and the second contribution as an emission off the particle 2. To see how this interpretation comes about, it is important to average $W_{1,2}$ over carefully chosen angles.

Consider W_1 and choose the reference vectors in the following way

$$\vec{n}_1 = (0,0,1), \quad \vec{n}_2 = (\sin\theta_{12}, 0, \cos\theta_{12}), \quad \vec{n}_k = (\sin\theta\cos\phi, \sin\sin\phi, \cos\theta)$$

$$\xi_{1k} = 1 - \cos\theta, \quad \xi_{2k} = 1 - \sin\theta_{12}\sin\theta\cos\phi - \cos\theta_{12}\cos\theta$$

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \frac{1}{a + b\cos\phi} = \frac{1}{\sqrt{a^2 - b^2}} \qquad \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{1}{\xi_{2k}} = \frac{1}{|\xi_{1k} - \xi_{12}|}$$

$$W_1(1,2;k) = \frac{1}{2\xi_{1k}}\left(\frac{\xi_{12} - \xi_{1k}}{\xi_{2k}} + 1\right) \qquad \int_0^{2\pi} \frac{d\phi}{2\pi} W_1(1,2;k) = \frac{\theta(\xi_{12} - \xi_{1k})}{\xi_{1k}}$$

An average over a different azimuthal angle gives a similar result for $W_2(1,2;k)$.

Parton showers

Putting the two functions together, we obtain the following result for the radiation function and for the cross section

$$W(1,2;k) = \frac{\theta(\xi_{1k} - \xi_{12})}{\xi_{1k}} + \frac{\theta(\xi_{2k} - \xi_{12})}{\xi_{2k}}$$

$$\mathrm{d}\sigma = \mathrm{d}\sigma_0 \frac{\alpha}{2\pi} \frac{\mathrm{d}\omega_k}{\omega_k} \sum_{i=1}^{2} \frac{\mathrm{d}\xi_{ik}}{\xi_{ik}} \theta(\xi_{12} - \xi_{ik})$$

The cross section formula has now the singularities separated; we see that partons 1 and 2 radiate independently provided that they radiate within an opening angle between partons 1 and 2. If, on the other hand, the emission outside this open angle happens, the interference shuts down the radiation completely.

What makes this construction attractive for parton showers is that we seem to be capable to describe soft interferences by making educated choice of the evolution variable -- the opening angle. The evolution variables are largely arbitrary. By choosing angles rather than off-shellnesses or transverse momenta, we can re-use the previous construction of the parton shower and yet account for both soft and collinear enhancement.

$$\frac{\mathrm{d}p_\perp^2}{p_\perp^2} = \frac{\mathrm{d}\theta^2}{\theta^2}$$

Parton showers

Consider a process when a virtual photon splits into three partons with charges Q_1,Q_2,Q_3 . We assume that the sum of the three charges vanishes and that none of them is zero. This is not the case in QED but we can use this as a model for color charges.

$$\mathcal{M} \sim \sum_{i=1}^{3} Q_i \frac{p_i \epsilon}{p_i k} \mathcal{M}_0 \quad \Rightarrow \quad |\mathcal{M}|^2 \sim -\sum_{ij} Q_i Q_i \frac{p_i p_j}{(p_i k)(p_j k)} |\mathcal{M}_0|^2.$$

$$W = -Q_1 Q_2 W_{12} - Q_1 Q_3 W_{13} - Q_2 Q_3 W_{23}$$

$$W = \frac{1}{2}\left[Q_1^2 \left(W_{12} + W_{13} - W_{23}\right) + Q_2^2 \left(W_{12} + W_{23} - W_{13}\right) + Q_3^2 \left(W_{13} + W_{23} - W_{12}\right)\right]$$

Now consider the kinematic situation when the opening angle of 1 and 2 is small compared to the opening angle of 1(or 2) and 3. In these cases, some of the contributions to the above equation can be simplified.

Parton showers

Consider a contribution proportional to Q_1^2; split each term into a pair of relevant radiator functions and average over the respective azimuthal angles. We find

$$W_{12}+W_{13}-W_{23} \to W_{12}^{[1]}+W_{12}^{[2]}+W_{13}^{[1]}+W_{13}^{[3]}-W_{23}^{[2]}-W_{23}^{[3]}$$
$$= 2W_{12}^{[1]}+\left\{[W_{13}^{[1]}-W_{12}^{[1]}]+[W_{12}^{[1]}-W_{23}^{[2]}]\right\}+\left\{W_{13}^{[3]}-W_{23}^{[3]}\right\}$$

$$\left\{[W_{13}^{[1]}-W_{12}^{[1]}]+[W_{12}^{[1]}-W_{23}^{[2]}]\right\}$$
$$=\frac{(\theta(\xi_{13}-\xi_{1k})-\theta(\xi_{12}-\xi_{1k}))}{\xi_{1k}}-\frac{(\theta(\xi_{23}-\xi_{2k})-\theta(\xi_{12}-\xi_{2k}))}{\xi_{2k}} \to 0|_{1\to 2}$$

$$W_{13}^{[3]}-W_{23}^{[3]}=\frac{1}{\xi_{3k}}(\theta(\xi_{13}-\xi_{3k})-\theta(\xi_{23}-\xi_{3k})) \to 0$$

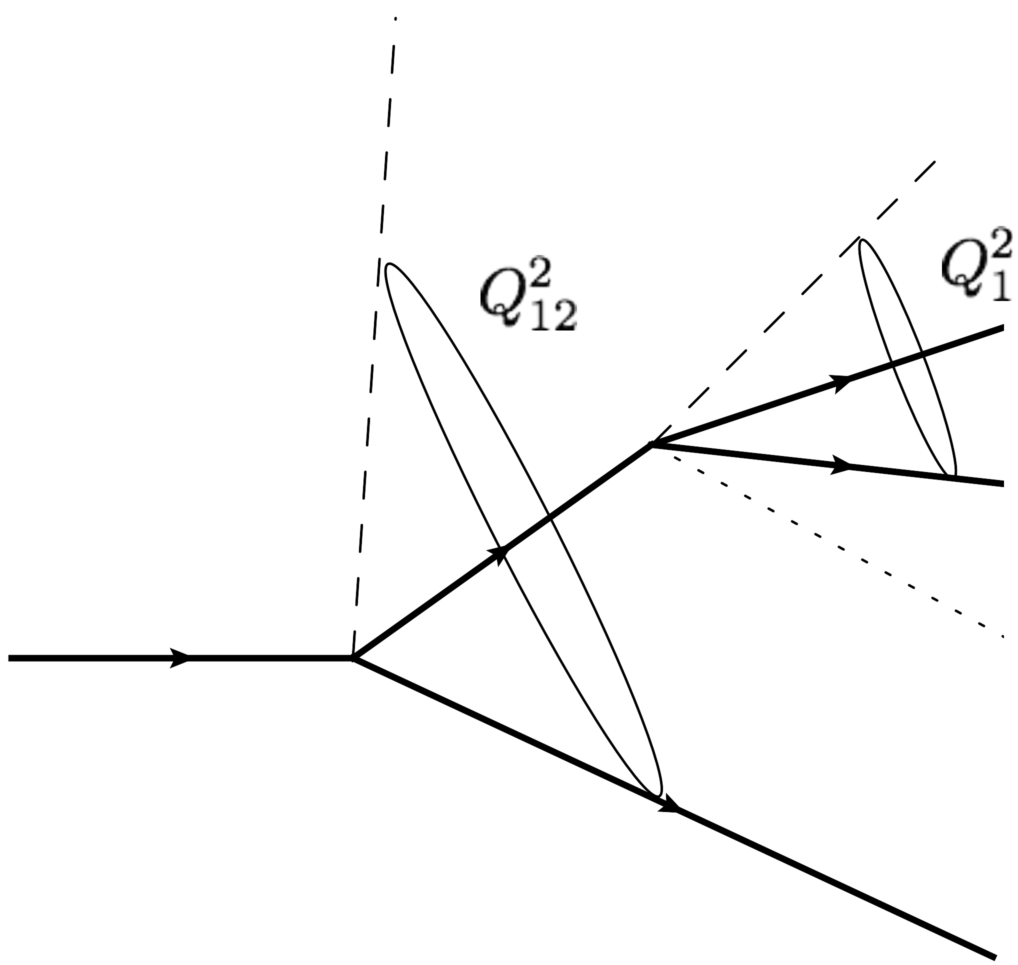
$$\frac{Q_1^2}{2}(W_{12}+W_{13}-W_{23}) \to Q_1^2\, W_{12}^{[1]} \qquad \frac{Q_2^2}{2}(W_{12}+W_{23}-W_{13}) \to Q_2^2\, W_{12}^{[2]}$$

$$\frac{Q_3^2}{2}(W_{13}+W_{23}-W_{12}) \to Q_3^2\, \tilde{W}_{12,3}^{[12]}+Q_3^2\, W_{12,3}^{[3]} \qquad \tilde{W}_{12,3}^{[12]}=\frac{\theta(\xi_{12,3}-\xi_{12,k})\theta(\xi_{12,k}-\xi_{12})}{\xi_{12,k}}$$

Parton showers

The final formula that we can write down is, therefore and it is susceptible to the description using the relative angle as ordered variable for events generation. Even if opening angles are not chosen to be primary variables, one can check at every step that the energy ordering is respected. Again, the fact that simple ordering allows us to account for the interferences makes this construction very important for practical parton showers.

$$W \approx Q_1^2\, W_{12}^{[1]} + Q_2^2\, W_{12}^{[2]} + Q_{12}^2\, \tilde{W}_{12,3}^{[12]} + Q_3^2\, W_{12,3}^{[3]}$$



Parton showers

Generalization of our discussion to QCD is straightforward; there are a few issues that one has to address but most of them do not require significant changes in physics or the algorithm. Here is the list of the important points.

1) The running coupling constant;

2) Many different options for partons to branch; the different probabilities have to be generated properly.

3) Secondary branchings -- radiated partons are not on the mass-shell;

4) Space-like and time-like showers (forward and backward evolution algorithms are different);

5) Alternative ways to combine soft and collinear emissions -- large-N_c approximation in the so-called ``dipole showers".

6) Modeling of non-perturbative physics and parton-to-hadron transitions (double-parton scattering, string fragmentation, cluster models).