Inverse Mathematics for QCD diffraction

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Proton-proton diffraction

Pomeron physics.

You can think also in terms of wee partons, soft color dipoles, pomeron parton (ladder) structure etc., unfortunately there are yet no truly solid experimental constraints from the LHC data for inclusive inelastic diffraction. Basic Regge domain features, however, are observed in data.

Essential fluctuating degrees of freedom: rapidity (predominantly low x), p_t, multiplicity and multidimensional correlations over the full range of acceptance.

$\Rightarrow$ N-dimensional observables

Basic questions of soft diffraction

- Unitarity, asymptotic energy behavior of total cross sections
- Transition between "different" Pomerons: soft ... hard $\rightarrow$ Pomeron intercept $1 + \Delta_P$ ($\rightsquigarrow s$ evolution) and slope $\alpha'_P \sim$ "t-cone behavior" functional behavior
- $p \rightarrow N^*$ Good-Walker spectrum of low-mass dissociation, relativistic wavefunction and "atmosphere" of proton
- Gluonia/glueballs/soft central diffractive production
- Regge/QCD factorization properties
- Pomeron via AdS space ...

- + Correlations and fluctuations

⋮

Ultimately, the goal here is have a "unified" approach for interpreting the data.

Vector space view to the soft *pp* Diffraction

So, usually the experimental definition when talking about soft diffraction goes through large rapidity gaps $\Delta y \gtrsim 3$ and

$$\sigma_{inel}^{pp} \equiv \sigma_{SDL} + \sigma_{SDR} + \sigma_{DD} + \sigma_{CD} + \sigma_{ND}$$

The decomposition above is experimentally well posed only in limited phase space.

So, instead, let us start with $n = 2^N - 1$ partial cross sections

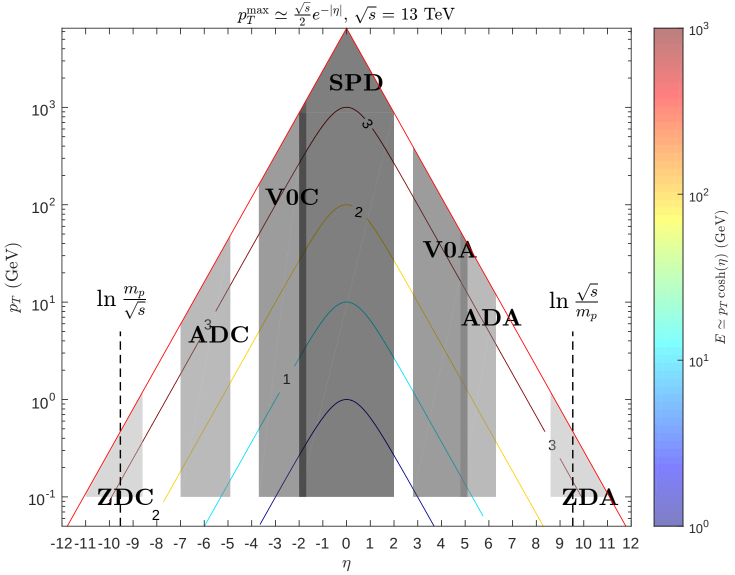
$$\sigma_{inel}^{pp} \equiv \sigma_1 + \sigma_2 + \sigma_3 + \cdots + \sigma_n, \quad (1)$$

where each subcomponent corresponds to one particular final state topology class over rapidity.

"Slice the (pseudo)rapidity space into N intervals"

Example: Geom.-kinem. ALICE phase-space span at Run 2

Not all subdetectors shown (~ #20). Very good ($\eta, p_\perp$) coverage for diffractive physics.



Vector valued partial cross sections

Partial cross sections ($\#2^N$) $\sim$

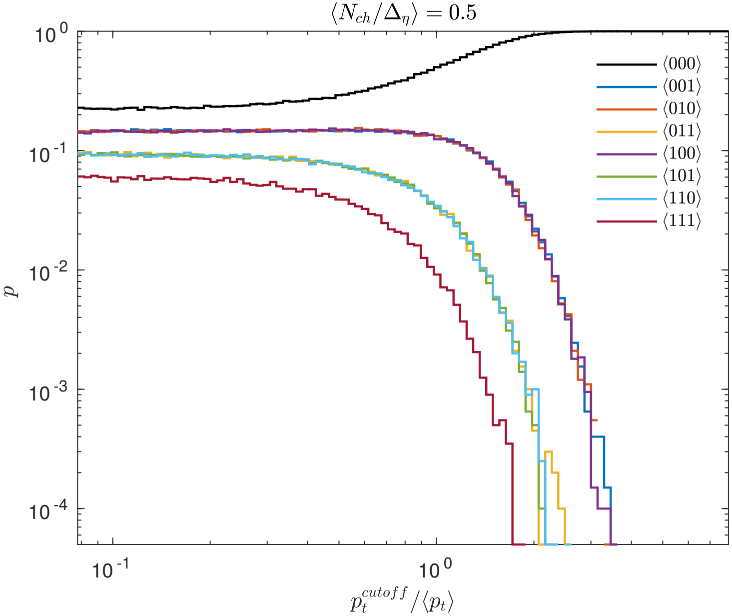
$$\frac{1}{2s}\sum_M \int_{\Omega_M} d\Pi_M \delta^{(4)}\left(p_1+p_2-\sum_M p_i\right)|\mathcal{M}_{2\to M}|^2 \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}^{\otimes N} \begin{pmatrix}1\\ \mathcal{I}\{\Pi_M;\Xi_1\}\end{pmatrix}\otimes \begin{pmatrix}1\\ \mathcal{I}\{\Pi_M;\Xi_2\}\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}1\\ \mathcal{I}\{\Pi_M;\Xi_N\}\end{pmatrix},$$

where the acceptance function $\mathcal{I}:\Pi_M\to\{0,1\}$, Π_M is a set of final state kinematical variables and Ξ_i is the i-th fiducial acceptance domain parametrization. The expression above is a 2^N-vector.

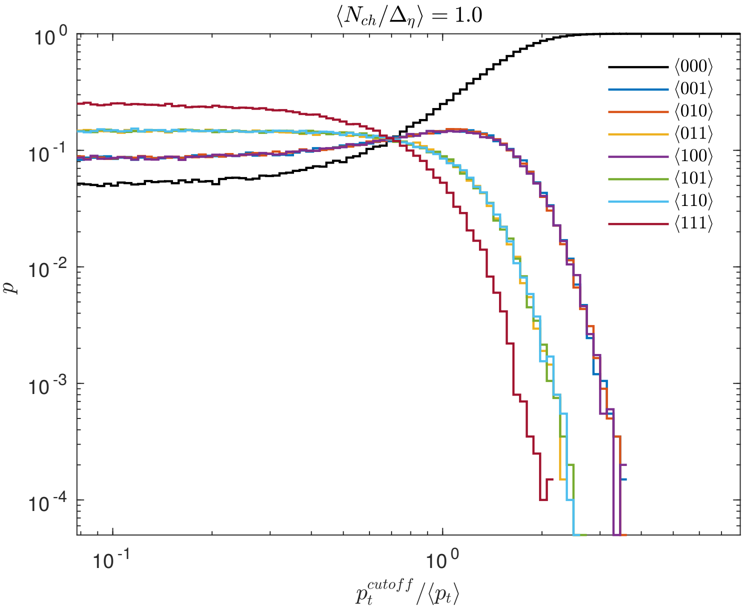
Synthetic Monte Carlo example

3 rapidity slices giving us Bernoulli combinations:
$\langle 000 \rangle, \langle 001 \rangle, \langle 010 \rangle, \ldots, \langle 111 \rangle$

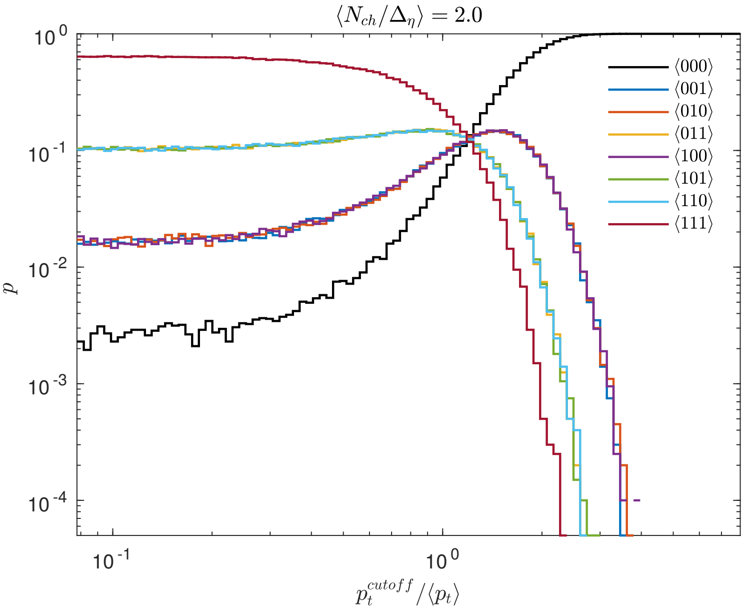
Particles drawn uniformly over rapidity, with fluctuating number of particles per interval $\sim$ Poisson($\langle N_{ch}/\Delta_\eta \rangle$) with transverse momentum $p_t \sim p_t \exp(-b p_t^2)$. Varying smoothly the p_t cutoff (normalized by $\langle p_t \rangle$) for four different particle densities per rapidity interval Δ_η.



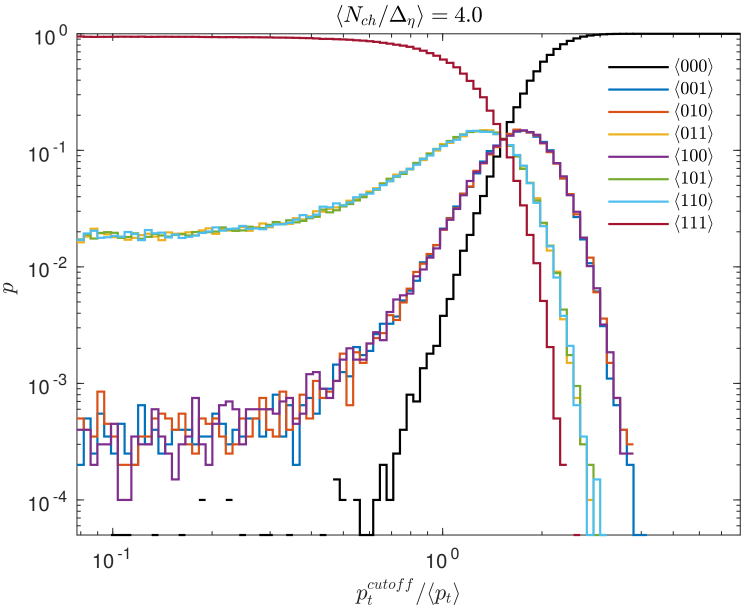
$\langle N_{ch}/\Delta_\eta \rangle = 0.5$
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
p
$\langle 000 \rangle$
$\langle 001 \rangle$
$\langle 010 \rangle$
$\langle 011 \rangle$
$\langle 100 \rangle$
$\langle 101 \rangle$
$\langle 110 \rangle$
$\langle 111 \rangle$
10^{-1}
10^0
$p_t^{cutoff}/\langle p_t \rangle$



$\langle N_{ch}/\Delta_\eta \rangle = 1.0$
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
p
$\langle 000 \rangle$
$\langle 001 \rangle$
$\langle 010 \rangle$
$\langle 011 \rangle$
$\langle 100 \rangle$
$\langle 101 \rangle$
$\langle 110 \rangle$
$\langle 111 \rangle$
10^{-1}
10^0
$p_t^{cutoff}/\langle p_t \rangle$



$\langle N_{ch}/\Delta_\eta \rangle = 2.0$
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
p
10^{-1}
10^0
$p_t^{cutoff}/\langle p_t \rangle$
⟨000⟩
⟨001⟩
⟨010⟩
⟨011⟩
⟨100⟩
⟨101⟩
⟨110⟩
⟨111⟩


$\langle N_{ch}/\Delta_\eta \rangle = 4.0$
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
p
10^{-1}
10^0
$p_t^{cutoff}/\langle p_t \rangle$
$\langle 000 \rangle$
$\langle 001 \rangle$
$\langle 010 \rangle$
$\langle 011 \rangle$
$\langle 100 \rangle$
$\langle 101 \rangle$
$\langle 110 \rangle$
$\langle 111 \rangle$

A short lesson from above

Without characterizing p_T (and η) acceptance $\rightarrow$ measurements of soft inclusive diffraction unstable $\rightarrow$ can easily explain all "discrepancies" between LHC experiments. Actually, ultimate measurement would be as a function of $p_t \ldots$

Thus open problem: how do you characterize (p_t, η) acceptance of forward scintillators and other low granularity counters without relying on MC generator $\circledast$ GEANT?

Applications

★ A machinery for the (multi)-rapidity gap measurements and correlation structure

★ A framework for generalized studies of Regge factorization at the LHC. Not just simplified SD,DD type, but more general

★ Framework to study AGK type shadowing, and beyond, by comparing the differential distributions within each vector combination

★ An attempt to re-define the soft diffraction observables more precisely, also introducing a hierachy of vector observables for minbias Monte Carlo tuning

★ A new framework for extracting single diffraction (SD), double diffraction (DD) . . . type component cross sections using N-dimensional Monte Carlo model "templates", which can be tuned to data

With connections to

[E. Onofri, G. Veneziano, J. Wosiek, Commun. Math. Phys. (2007)], *"We show how a recently proposed supersymmetric quantum mechanics model leads to non-trivial results/conjectures on the combinatorics of binary necklaces and linear-feedback shift- registers."*

[H. Fu, R. Sasaki, J. Math. Phys. 38 (1997)], *"Following the relationship between probability distribution and coherent states, for example the well known Poisson distribution and the ordinary coherent states and relatively less known one of the binomial distribution and the su(2) coherent states."*

[D. Spector, Commun. Math. Phys. (1990)], *"We show that the Möbius inversion function of number theory can be interpreted as the operator* $(-1)^F$ *in quantum field theory."*

Algebraic representations

The *probability vector* $\mathbf{p}$ (2^N-dim), the components of *ordinary moments* m_k and the components of *central moments* δ_k below are defined using the Kronecker (tensor) products

$$\mathbf{p} = \left\langle \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{\otimes N} \begin{pmatrix} 1 \\ X_N \end{pmatrix} \otimes \begin{pmatrix} 1 \\ X_{N\text{-}1} \end{pmatrix} \otimes \cdots \begin{pmatrix} 1 \\ X_1 \end{pmatrix} \right\rangle$$

$$m_k = \left\langle \prod_{i=1}^{N} X_i^{k_i} \right\rangle = \left\langle \begin{pmatrix} 1 \\ X_N \end{pmatrix} \otimes \begin{pmatrix} 1 \\ X_{N\text{-}1} \end{pmatrix} \otimes \cdots \begin{pmatrix} 1 \\ X_1 \end{pmatrix} \right\rangle_k$$

$$\delta_k = \left\langle \prod_{i=1}^{N} (X_i - \langle X_i \rangle)^{k_i} \right\rangle = \left\langle \begin{pmatrix} 1 \\ X_N - \langle X_N \rangle \end{pmatrix} \otimes \begin{pmatrix} 1 \\ X_{N\text{-}1} - \langle X_{N\text{-}1} \rangle \end{pmatrix} \otimes \cdots \begin{pmatrix} 1 \\ X_1 - \langle X_1 \rangle \end{pmatrix} \right\rangle_k,$$

where we use $k = 1 + \sum_{i=1}^{N} k_i 2^{i-1}$ (little endian binary expansion), $1 \leq k \leq 2^N$ and $k_i \in \{0, 1\}$. The central moments describe the correlations ($\# \, 2^N - N - 1$) between any 2 or more subspaces (rapidity slices). X_i are the corresponding random variables.

[Teugels, Jozef L. "Some representations of the multivariate Bernoulli and binomial distributions." Journal of multivariate analysis 32.2 (1990): 256-268.]

Diffraction analysis technique++

To summarize, we utilize different detector combinations over $\eta \to$ vector signals $\to$ partial cross sections + multidimensional model fitting to extract σ_{SD}, σ_{DD} etc.

This latest vector space combinatorial construction goes beyond multidimensional fitting, and is compatible with discussion about multigaps, gap destruction and rescattering and short/long range y-correlations:

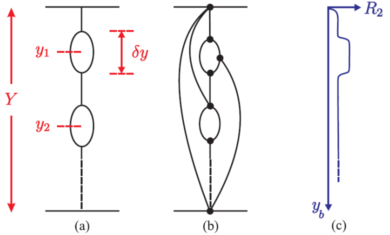


Figure: (a) Multigap event, (b) Gap destruction, (c) Correlation coeff. R_2
Figure from: [Khoze, Martin, Ryskin, Shuvaev, J. Phys. G: Nucl. Part. Phys. 36 (2009) 093001]

AGK Cutting Rules

Field theory Combinatorics

The total cross section for exchange of μ Pomerons, σ_μ^{tot}, partial cross section $\sigma_\mu^{(\nu)}$ of a final state with a number of ν cut Pomerons and their ratio

$$\frac{\sigma_\mu^{(\nu)}}{\sigma_\mu^{tot}} = (-1)^{\mu-\nu} \frac{\mu!}{\nu!(\mu-\nu)!} (2^{\mu-1} - \delta_{0\nu}), \qquad (2)$$

[Abramovski, Gribov, Kancheli, Sov. J. Nucl. Phys. 18, 308 (1974)], [E. Levin, hep-ph/9503399]

$\mu \diagdown \nu$	0	1	2	3	4	5	6
1	0	1	0	0	0	0	0
2	1	-4	2	0	0	0	0
3	-3	12	-12	4	0	0	0
4	7	-32	48	-32	8	0	0
5	-15	80	-160	160	-80	16	0
6	31	-192	480	-640	480	-192	32

Table: AGK factors for $\mu = 1, 2, \ldots, 6$ exchanged Pomerons. Summing over μ requires some explicit (Regge/Eikonal etc.) model in addition to these.

"Super-Eikonals"

Combinatorial (de)-compounding or pileup inversion

Poisson ⊗ Multinomial Vector Model

$$\hat{y}_i = \frac{1}{1-e^{-\mu}} \sum_{k=1}^{\infty} \frac{\mu^k}{k!} e^{-\mu} W_{ik}, \quad i = 1, \ldots, 2^N - 1 = n$$

$$= \frac{e^{-\mu}}{1-e^{-\mu}} \sum_{k=1}^{\infty} \frac{\mu^k}{k!} \left\{ \sum_{\Omega_{ik}} \frac{k!}{\prod_{j=1}^{n} x_j!} \prod_{j=1}^{n} p_j^{x_j} \right\} \tag{3}$$

The multinomial term and its values of $x_j \in \mathbb{N}$ are evaluated over all valid combinations for probabilities y_i from the set of n-tuples Ω_{ik}, that is, those which are allowed by poset combinatorics:

$$\Omega_{ik} = \left\{ (x_1, \ldots, x_j, \ldots, x_n) \mid \bigvee_j x_j \mathbf{c}_j = \mathbf{c}_i \text{ and } \sum_j x_j = k \right\}, \tag{4}$$

where $\bigvee$ operator takes care of "summing" the binary vectors $\mathbf{c}_j$ of multiplicity x_j and thus evaluating the "pileup" compositions.

The idea in a nutshell: We measure probabilities $\mathbf{y}$, and want to solve $\mathbf{p}$

Solution based on the principle of inclusion-exclusion

General math framework: Incidence algebras [Gian-Carlo Rota, MIT, 60's]

The principle of inclusion-exclusion is the Möbius inversion for subsets. Now let different rapidity slices and their signals be represented with subsets $D_1, D_2, \ldots, D_N \subset D$. Then

$$P(\bigcup_{i=1}^{N} D_i) = \sum_{k=1}^{N} \left((-1)^{k-1} \sum_{I \subset \{1,\ldots,N\}, |I|=k} P(D_I) \right). \tag{5}$$

One can wrap that thing above into a matrix. Notice the $(-1)^{k-1}$ factor, that gives the essential structure.

Uniform (max entropy) input $\mathbf{p} = \mathbf{1}$ case, $N = 3$

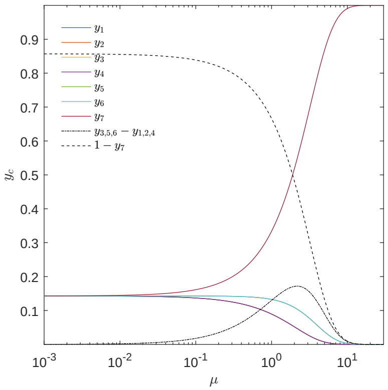


Figure: A solution. On x-axis the Poisson μ and on y-axis the components of the vector $\mathbf{y}$.

Starting with very elementary definitions, interesting distributions emerge from combinatorics.

Alternating sign inverse solution for $N = 3$

$$\mathbf{p} = \frac{1}{\mu}\begin{pmatrix} \ln(e_-^\mu y_1 + 1) \\ \ln(e_-^\mu y_2 + 1) \\ -\sum_{c=1,2} \ln(e_-^\mu y_c + 1) + \ln(1 + \sum_{c=1,2,3} e_-^\mu y_c) \\ \ln(e_-^\mu y_4 + 1) \\ -\sum_{c=1,4} \ln(e_-^\mu y_c + 1) + \ln(1 + \sum_{c=1,4,5} e_-^\mu y_c) \\ -\sum_{c=2,4} \ln(e_-^\mu y_c + 1) + \ln(1 + \sum_{c=2,4,6} e_-^\mu y_c) \\ \mu + \sum_{c=1,2,4} \ln(e_-^\mu y_c + 1) - \ln(1 + \sum_{c=1,2,3} e_-^\mu y_c) \qquad \ldots \\ - \ln(1 + \sum_{c=1,4,5} e_-^\mu y_c) - \ln(1 + \sum_{c=2,4,6} e_-^\mu y_c) \end{pmatrix},$$

where by conservation of probability we chose to fix $y_7 = 1 - \sum_{c=1}^{6} y_c$ and for saving ink we set $e_-^\mu \equiv e^\mu - 1$.

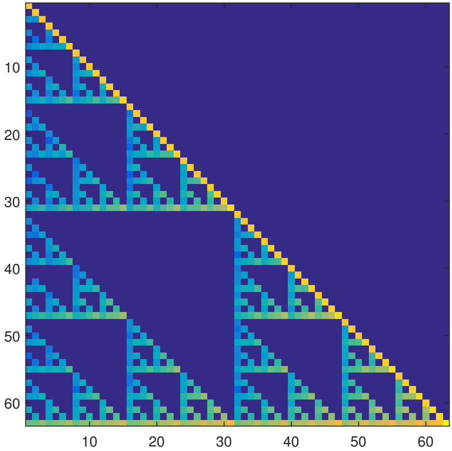


Figure: Poisson model $\otimes$ Dirichlet distribution drawn probabilities as a statistical mixing operator (matrix) $S : \mathbf{p} \mapsto \mathbf{y}$, $N = 6$. Fractal structure, due to the Boolean vector space, is the Sierpinski triangle. (Dark blue $=$ $0 \ldots$ Yellow $= 1$)

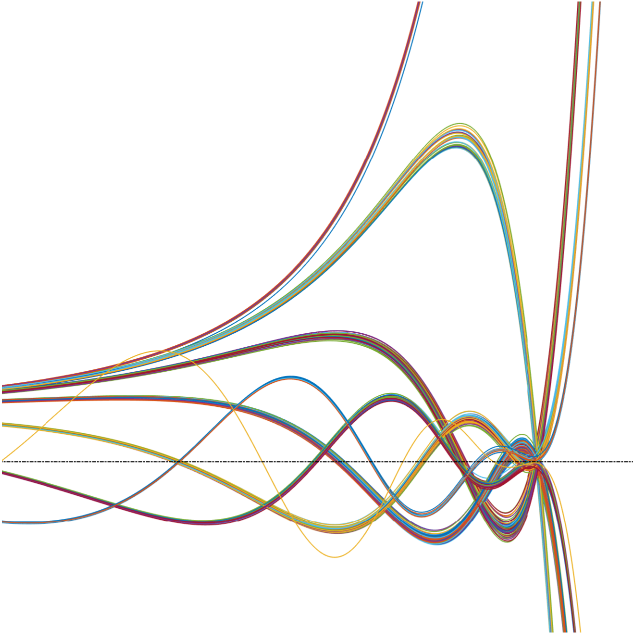


Figure: Hidden polynomial structure, $N = 8$.

Conclusions

The vector space measurement model allows a mathematically self consistent way to do combinatorial analysis of soft diffraction, plus also to extract $\sigma_{SD}, \sigma_{DD}, \sigma_{ND}$ etc. via multidimensional Bayesian/Frequentist fitting (given the MC model).

AGK cutting rules can be incorporated into the combinatorics inversion framework. Leading the way to completely new analyses of, e.g., gap survival $S^2(\Omega)$ discussion. This framework works directly for pile-up inversion of gap topologies (multiple *pp* interactions per bunch crossing).

The vector space itself can be studied in the context of kinematics, diffraction models and Regge theory, together with tools from combinatorics and algebraic geometry (technically the structure is Grassmannian).

Recursive Inverse of Stochastic Autoconvolution

The first solution with fully non-linear uncertainty estimation

Recursion, M.C. Escher

The problem?

Think about having a superposition of final state multiplicities (= autoconvolution[1]), let's say, in proton-proton collisions

Main problem is limited statistics in steeply falling tails $\rightarrow$ huge oscillations, naive (textbook[2]) solutions fail miserably

[1]sum of random variables is equivalent to a convolution of their densities
[2]inverting stochastic autoconvolution is not usual textbook material

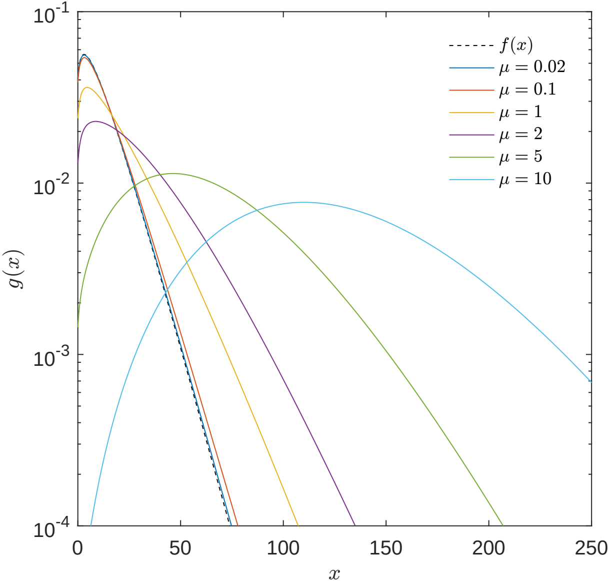


Figure: Poissonian superposition with different Poisson mean values μ, with x a random variable $\sim$ Negative Binomial Distribution.

Forward problem

The autoconvoluted distribution of $Y \sim g_Y$ is now written formally as a Poisson probabilities weighted infinite series[3]

$$\begin{aligned} g_Y(y) &= P_1 f_X(y) + P_2[f_X \circledast f_X](y) + P_3[[f_X \circledast f_X] \circledast f_X](y) + \dots \\ &= \frac{1}{1-e^{-\mu}} \sum_{K=1}^{\infty} \frac{\mu^K}{K!} e^{-\mu} f_X^{\circledast^K}(y) \end{aligned} \quad (6)$$

where the *convolution power* $\circledast^K$ is defined recursively as $f^{\circledast^K} = f^{\circledast^{(K-1)}} \circledast f$ and $f^{\circledast 1} = f$.

We do need not to limit ourself to the Poisson compound sum, but take that as an example

[3]We have removed the unobservable case $K = 0$ which gives $Y = 0$ and renormalized the remaining Poisson probabilities $P_K, K = 1, 2, 3, \dots$ to sum to one.

A spectral solution to the forward problem via the characteristic function

In the spectral domain, the characteristic function (CHF) φ_X is defined as

$$\varphi_X(t) = \mathbb{E}[e^{itX}] = \int_{\mathbb{R}} e^{itX} f_X(x)\, dx \tag{7}$$

and for the compound Poisson case you end up with

$$\varphi_g(t) \equiv \varphi_{Y|K>0}(t) = \frac{e^{-\mu}\left(e^{\mu\varphi_f(t)} - 1\right)}{1 - e^{-\mu}} = \frac{1}{e^{\mu} - 1}(e^{\mu\varphi_f(t)} - 1).$$

The main thing is that you want to find out $\varphi_f(t)$.

Inverse solution in a nutshell

To find out $\hat{f}(x)$, use recursion. First estimate $\hat{f}^0 = g(x)$.

Take Fast Fourier Transform (FFT) of $\hat{f}^k(x)$ to get $\hat{\varphi}_f^k(t)$, use the spectral map to get $\hat{\varphi}_g(t)$ and construct corresponding AC operator, take IFFT of AC operator, map $g(x) \to \hat{f}^{k+1}(x)$ in original domain with Max Entropy inversion + regularization, use Efron's statistical Bootstrap to estimate uncertainty, and add one so-called bias substraction iteration around it:

"Bias substraction" $\curvearrowright$

"Daughter Bootstrap" $\curvearrowright$

Fast Fourier Transform & Max Entropy recursion $\circlearrowleft$

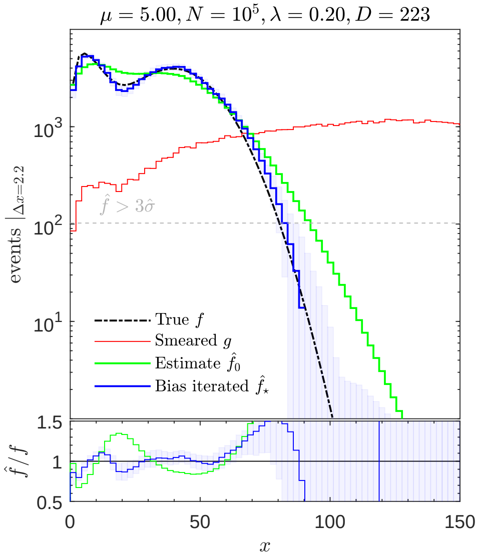


Figure: Inverse solution with algorithmic uncertainty estimation (blue band 95CL).

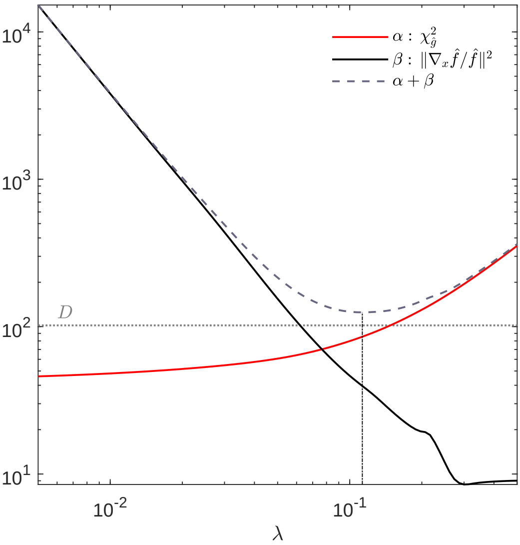


Figure: Data driven regularization parameter λ selection as an equilibrium between "backprojection" error $\chi^2_{\hat{g}}$ and smoothness $\|\nabla_x \hat{f}/\hat{f}\|^2$.

Work in progress

Algorithms will be available online at:
github.com/mieskolainen