Chiral dynamics in QCD

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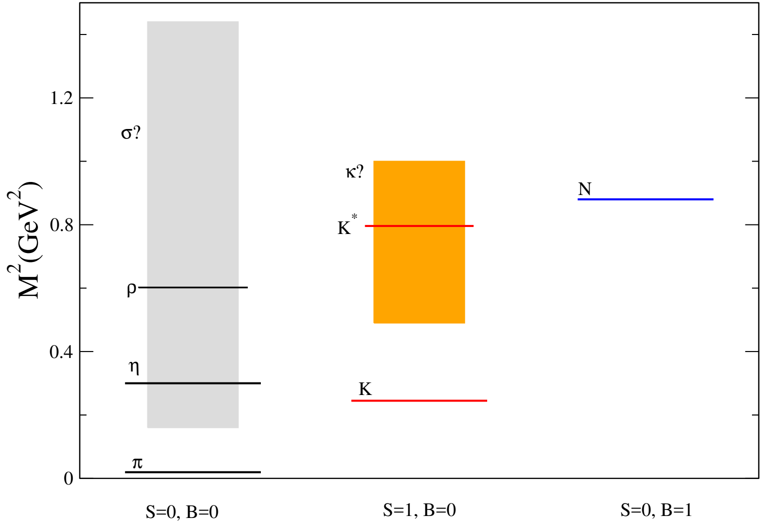
Lecture I: Foundations of chiral perturbation theory

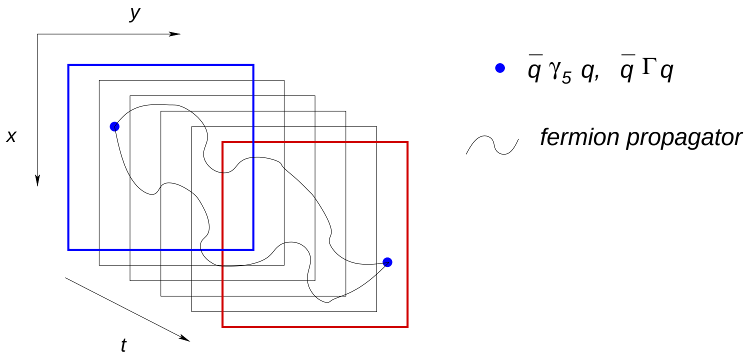
The QCD spectrum



The QCD spectrum – on the lattice

$$C(t) = \int d^3x \, \langle [\bar{q}\gamma_5 q(x)] \, [\bar{q}\gamma_5 q(0)] \rangle \, e^{i\mathbf{px}} \xrightarrow{t\to\infty} \sum_{n=0}^{\infty} c_n \, e^{-E_n t}$$

$$M_\pi = \lim_{\substack{L\to\infty \\ a\to 0}} M_\pi(L,a) \qquad M_\pi(L,a) = E_n(\mathbf{p}=0)$$



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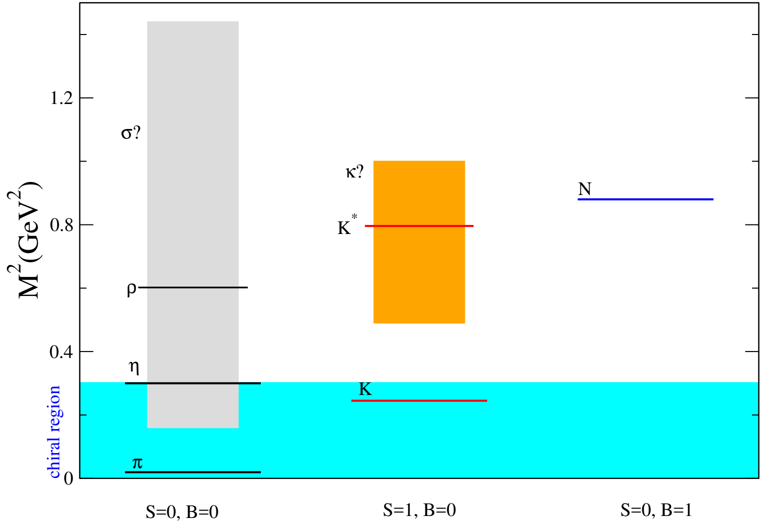

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- they set the limit of validity of the chiral expansion

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- effective Lagrangian: systematic method to construct this expansion, respecting symmetry and all the general principles of quantum field theory Weinberg (79)
- The method leads to predictions – even very sharp ones

Quantum Chromodynamics in the chiral limit

$$\mathcal{L}^{(0)}_{\rm QCD} = \bar{q}_L i \not{D} q_L + \bar{q}_R i \not{D} q_R - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \qquad q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

Large global symmetry group:

$$SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$$

1. $U(1)_V \Rightarrow$ baryonic number
2. $U(1)_A$ is anomalous
3. $$SU(3)_L \times SU(3)_R \Rightarrow SU(3)_V$$

   $\Rightarrow$ Goldstone bosons with the quantum numbers of pseudoscalar mesons will be generated

Quark masses, chiral expansion

In the real world quarks are not massless:

$$\mathcal{L}_{QCD} = \mathcal{L}^{(0)}_{QCD} + \mathcal{L}_m, \qquad \mathcal{L}_m := -\bar{q}\mathcal{M}q$$

$$\mathcal{M} = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

the mass term $\mathcal{L}_m$ can be considered as a small perturbation $\Rightarrow$
Expand around $\mathcal{L}^{(0)}_{QCD}$ $\equiv$ Expand in powers of m_q

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Chiral perturbation theory, the low-energy effective theory of QCD, is a simultaneous expansion in powers of momenta and quark masses

Quark mass expansion of meson masses

General quark mass expansion for the P particle:

$$M_P^2 = M_0^2 + \langle P|\bar{q}\mathcal{M}q|P\rangle + O(m_q^2)$$

For the pion $M_0^2 = 0$:

$$M_\pi^2 = -(m_u + m_d)\frac{1}{F_\pi^2}\langle 0|\bar{q}q|0\rangle + O(m_q^2)$$

where we have used a Ward identity:

$$\langle \pi|\bar{q}q|\pi\rangle = -\frac{1}{F_\pi^2}\langle 0|\bar{q}q|0\rangle =: B_0$$

$\langle 0|\bar{q}q|0\rangle$ is an order parameter for the chiral spontaneous symmetry breaking

Gell-Mann, Oakes and Renner (68)

Quark mass expansion of meson masses

Consider the whole pseudoscalar octet:

$$
\begin{aligned}
M_\pi^2 &= (m_u + m_d)B_0 + O(m_q^2) \\
M_{K^+}^2 &= (m_u + m_s)B_0 + O(m_q^2) \\
M_{K^0}^2 &= (m_d + m_s)B_0 + O(m_q^2) \\
M_\eta^2 &= \frac{1}{3}(m_u + m_d + 4m_s)B_0 + O(m_q^2)
\end{aligned}
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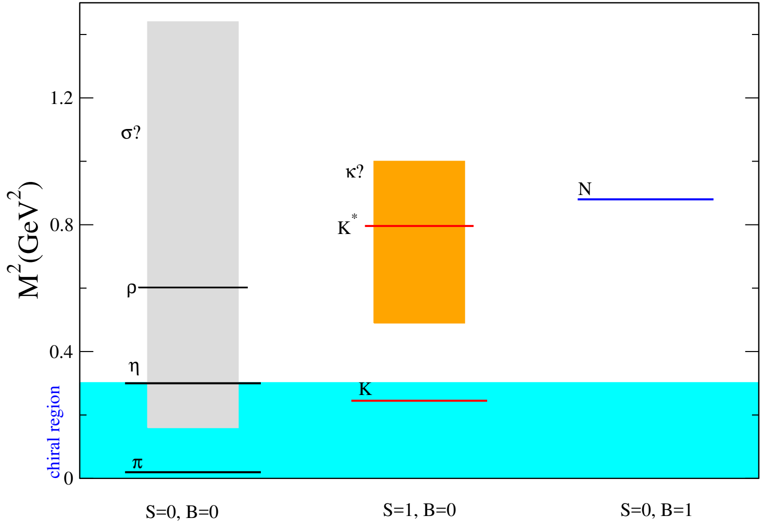
Consequences: $(\hat{m} = (m_u + m_d)/2)$

$$
\begin{aligned}
M_K^2/M_\pi^2 &= (m_s + \hat{m})/2\hat{m} &&\Rightarrow m_s/\hat{m} = 25.9 \\
M_\eta^2/M_\pi^2 &= (2m_s + \hat{m})/3\hat{m} &&\Rightarrow m_s/\hat{m} = 24.3 \\
3M_\eta^2 &= 4M_K^2 - M_\pi^2 &&\text{Gell-Mann–Okubo (62)} \\
(0.899 &= 0.960)\ \text{GeV}^2
\end{aligned}
$$

Quark mass expansion of meson masses



Goldstone theorem

Hamiltonian $\mathcal{H}$ symmetric under the group of transformations G: [Q_i are the generators of G]

$$[Q_i, \mathcal{H}] = 0 \qquad\qquad i = 1, \ldots n_G$$

Ground state not invariant under G, i.e. for some generators X_i

$$X_i|0\rangle \neq 0$$

$$\{Q_1, \ldots, Q_{n_G}\} = \{H_1, \ldots, H_{n_H}, X_1, \ldots, X_{n_G - n_H}\}$$

Goldstone theorem

$$[Q_i, \mathcal{H}] = 0 \qquad i = 1, \ldots n_G \ , \qquad X_i|0\rangle \neq 0 \ , \qquad H_i|0\rangle = 0$$

1. The subset of generators H_i which annihilate the vacuum forms a subalgebra

$$[H_i, H_k]|0\rangle = 0 \qquad\qquad i, k = 1, \ldots n_H$$

2. The spectrum of the theory contains $n_G - n_H$ massless excitations

$$X_i|0\rangle \qquad\qquad i = 1, \ldots n_G - n_H$$

from $[X_i, \mathcal{H}] = 0$ follows that $X_i|0\rangle$ is an eigenstate of the Hamiltonian with the same eigenvalue as the vacuum

Goldstone theorem

$$[Q_i, \mathcal{H}] = 0 \qquad i = 1, \dots n_G \;, \qquad X_i|0\rangle \neq 0 \;, \qquad H_i|0\rangle = 0$$

- $X_i|0\rangle$ are the Goldstone boson states
- the X_i are generators of the quotient space G/H
- the Goldstone fields are elements of the space G/H
- their transformation properties under G are fully dictated: they transform nonlinearly
- the dynamics of the Goldstone bosons at low energy is strongly constrained by symmetry

Matrix elements of conserved currents

Goldstone's theorem also asserts that:

the transition matrix elements between the conserved currents associated with the generators Q_i and the pions*

$$\langle 0|J_i^\mu|\pi^a(p)\rangle = iF_i^a p^\mu$$

is an $n_G \times (n_G - n_H)$ matrix F_i^a of rank $N_{GB} = n_G - n_H$

*We have introduced the symbol π for the Goldstone boson fields, and will call them "pions", as in strong interactions. The discussion however, remains completely general

Pions do not interact at low energy

Current conservation implies

$$p_\mu \langle \pi^{a_1}(p_1)\pi^{a_2}(p_2)\ldots \text{out}|J_i^\mu|0\rangle = 0 \qquad p^\mu = p_1^\mu + p_2^\mu + \ldots$$

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Consider the amplitude for pair creation

$$\langle \pi^{a_1}(p_1)\pi^{a_2}(p_2)\text{out}|J_i^\mu|0\rangle = \frac{p_3^\mu}{p_3^2}\sum_{a_3} F_i^{a_3} v_{a_1 a_2 a_3}(p_i) + \ldots$$

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Current conserv. $\Rightarrow \sum_{a_3} F_i^{a_3} v_{a_1 a_2 a_3}(0) = 0 \Rightarrow v_{a_1 a_2 a_3}(0) = 0$

Lorentz invariance $\Rightarrow$ $v_{a_1 a_2 a_3}(p_1, p_2, p_3)$ can only depend on $p_1^2,\ p_2^2,\ p_3^2$: on the mass shell it is always zero

Pions do not interact at low energy

Amplitude for three–pion creation from a conserved current

$$\langle \pi^{a_1}\pi^{a_2}\pi^{a_3}\text{out}|J_i^\mu|0\rangle = \frac{p_4^\mu}{p_4^2}\sum_{a_4} F_i^{a_4} v_{a_1 a_2 a_3 a_4}(p_i) + \ldots$$

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Current conservation $\Rightarrow$

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In this case the vertex function can depend on two Lorentz scalars, s and t, and we can do a Taylor expansion:

$$v_{a_1 a_2 a_3 a_4}(p_1, p_2, p_3, p_4) = c^1_{a_1 a_2 a_3 a_4} s + c^2_{a_1 a_2 a_3 a_4} t + \dots$$

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- Effective Lagrangian for Goldstone Bosons = χPT

Weinberg (79)

Transformation properties of the pions

The pion fields transform according to a representation of G

$$g \in G : \vec{\pi} \to \vec{\pi}' = \vec{f}(g, \vec{\pi})$$

where f has to obey the composition law

$$\vec{f}(g_1, \vec{f}(g_2, \vec{\pi})) = \vec{f}(g_1 g_2, \vec{\pi})$$

$\vec{f}(g, 0) =$ image of the origin : the elements which leave the origin invariant form a subgroup – the conserved subgroup H

$\vec{f}(gh, 0)$ coincides with $\vec{f}(g, 0)$ for each $g \in G$ and $h \in H \Rightarrow$ the function $\vec{f}$ maps elements of G/H onto the space of pion fields

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The mapping is invertible: $\vec{f}(g_1, 0) = \vec{f}(g_2, 0)$ implies $g_1 g_2^{-1} \in H$
$\Rightarrow$ pions can be identified with elements of G/H

Action of G on G/H

Two elements of G, $g_{1,2}$ are identified with the same element of G/H if

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The transformation properties of the coordinates of G/H under the action of G are nonlinear (h is in general a nonlinear function of q_1 and g)

The space G/H for QCD

The choice of a representative element inside each equivalence class is arbitrary. For example

$$g = (g_L, g_R) = (1, g_R g_L^{-1}) \cdot (g_L, g_L) =: q \cdot h$$

but also $$g = (g_L, g_R) = (g_L g_R^{-1}, 1) \cdot (g_R, g_R) =: q' \cdot h'$$

where $$q, q' \in G/H \quad \text{and} \quad h, h' \in H$$

Action of G on G/H

$$\begin{aligned}(V_L, V_R) \cdot (1, g_R g_L^{-1}) &= (V_L, V_R g_R g_L^{-1}) \\ &= (1, V_R g_R g_L^{-1} V_L^{-1}) \cdot (V_L, V_L)\end{aligned}$$

The space G/H for QCD

The pion fields are usually collected in a matrix–valued field U, which transforms like

$$U \xrightarrow{G} U' = V_R U V_L^{-1}$$

U is a shorthand notation for $(1, g_R g_L^{-1})$, or its nontrivial part $g_R g_L^{-1}$

As a matrix U is a member of $SU(3) \Rightarrow$ it can be written as

$$U = e^{i\phi^a \lambda_a}$$

where ϕ^a are the eight pion fields

Construction of the effective Lagrangian

Goal: reproduce the low–energy structure of QCD
$\Rightarrow$ construct an effective Lagrangian which:

- contains the pion fields as the only degrees of freedom
- is invariant under G
- and expand it in powers of momenta

$$\begin{aligned}\mathcal{L}_{eff} &= f_1(U) + f_2(U)\langle U^+\Box U\rangle \\ &+ f_3(U)\langle \partial_\mu U^+ \partial^\mu U\rangle + O(p^4)\end{aligned}$$

The invariance under transformations $U \xrightarrow{G} U' = V_R U V_L^{-1}$ implies that $f_{1,2,3}(U)$ do not depend on U
$\Rightarrow$ f_1 is an irrelevant constant and can simply be dropped

Construction of the effective Lagrangian

Partial integration $\Rightarrow$

$$\mathcal{L}_{eff} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \dots \qquad \mathcal{L}_2 = \frac{F^2}{4}\langle \partial_\mu U^+ \partial^\mu U \rangle$$

the constant in front of the trace fixed by looking at the Noether currents of the G symmetry:

$$V_i^\mu = i\frac{F^2}{4}\langle \lambda_i [\partial^\mu U, U^+] \rangle \qquad A_i^\mu = i\frac{F^2}{4}\langle \lambda_i \{\partial^\mu U, U^+\} \rangle$$

and comparing the result of the matrix element with

$$\langle 0 | A_i^\mu | \pi^k(p) \rangle = i p^\mu \delta_{ik} F$$

Some technical details

The matrix field U is an exponential of the pion fields π. If we want fields π of canonical dimension, we have to introduce a dimensional constant in the definition of U:

$$U = \exp\left\{\frac{i}{F'}\pi^k\lambda_k\right\}$$

The requirement that the kinetic term of the pion fields is standard:

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}\partial_\mu\pi^i\partial^\mu\pi^i \qquad \text{implies:} \qquad F = F'$$

The Lagrangian contains only one coupling constant which is the pion decay constant

The first prediction: $\pi\pi$ scattering

Isospin invariant amplitude:

$$M(\pi^a\pi^b \to \pi^c\pi^d) = \delta_{ab}\delta_{cd}A(s,t,u) + \delta_{ac}\delta_{bd}A(t,u,s) + \delta_{ad}\delta_{bc}A(u,s,t)$$

Using the effective Lagrangian above

$$A(s,t,u) = \frac{s}{F^2}$$

Exercise: calculate it!

χPT and explicit symmetry breaking?

- The effective Lagrangian was constructed in order to systematically account for symmetry relations.
  What if the symmetry is explicitly broken?
- If the symmetry breaking is weak $\Rightarrow$ perturbative expansion: matrix elements of the symmetry breaking Lagrangian (or of powers thereof) will appear
- Once the transformation properties of the symmetry breaking term are known: use symmetry to constrain its matrix elements
- Effective Lagrangian $=$ appropriate tool to derive systematically all symmetry relations

Effective Lagrangian with ESB

$$\mathcal{L}^{\text{QCD}} = \mathcal{L}_0^{\text{QCD}} - \bar{q}\mathcal{M}q$$

The symmetry breaking term

$$\bar{q}\mathcal{M}q = \bar{q}_R\mathcal{M}q_L + \text{h.c.}$$

becomes also chiral invariant if we impose that $\mathcal{M}$ transforms according to

$$\mathcal{M} \to \mathcal{M}' = V_R\mathcal{M}V_L^+$$

Proceed to construct a chiral invariant effective Lagrangian that includes explicitly the matrix $\mathcal{M}$:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(U, \partial U, \partial^2 U, \ldots, \mathcal{M})$$

Effective Lagrangian with ESB

To first order in $\mathcal{M}$ there is only one chiral invariant term:

$$\mathcal{L}^{(1)}_{\mathcal{M}} = \frac{F^2}{2}\left[B\langle \mathcal{M} U^+\rangle + B^*\langle \mathcal{M}^+ U\rangle\right]$$

Strong interactions respect parity $\Rightarrow$ B must be real:

$$\mathcal{L}^{(1)}_{\mathcal{M}} = \frac{F^2 B}{2}\langle \mathcal{M}\left(U + U^+\right)\rangle$$

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Before using this Lagrangian, pin down the constant B:

$$B = -\frac{1}{F^2}\langle 0|\bar{q}q|0\rangle \qquad\qquad M_\pi^2 = 2B\hat{m}$$

Leading order effective Lagrangian

The complete leading order effective Lagrangian of QCD reads:

$$\mathcal{L}_2 = \frac{F^2}{4}\left[\langle \partial_\mu U^+ \partial^\mu U \rangle + \langle 2B\mathcal{M}\left(U + U^+\right)\rangle\right]$$

F is the pion decay constant in the chiral limit

B is related to the $\bar{q}q$–condensate and to the pion mass

$$M_\pi^2 = 2B\hat{m} + O(\hat{m}^2)$$

$\pi\pi$ scattering to leading order

In the presence of quark masses the $\pi\pi$ scattering amplitude becomes

$$A(s,t,u) = \frac{s - M_\pi^2}{F_\pi^2}$$ Weinberg (66)

The two S–wave scattering lengths read

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16 \qquad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} = -0.045$$

External fields

QCD coupled to external fields ($\mathcal{M} \to s$):

$$\mathcal{L} = \mathcal{L}^{(0)}_{\rm QCD} + \bar{q}\gamma^\mu(v_\mu + \gamma_5 a_\mu)q - \bar{q}(s - i\gamma_5 p)q$$

Generating functional of Green functions of quark bilinears

$$\langle 0| T e^{\mathrm{i}\int d^4x \mathcal{L}} |0\rangle = e^{\mathrm{i}Z[v,a,s,p]}$$

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Generating functional of Green functions of quark bilinears

$$\langle 0|Te^{\mathrm{i}\int d^4x\mathcal{L}}|0\rangle = e^{\mathrm{i}Z[v,a,s,p]} = \mathcal{N}^{-1}\int [dU]e^{\mathrm{i}\int d^4x\mathcal{L}_{\rm eff}}$$

External fields in $\mathcal{L}_{\rm eff} = \mathcal{L}_2(U,v,a,s,p) + \mathcal{L}_4(U,v,a,s,p) + \ldots$

$$\mathcal{L}_2 = \frac{F^2}{4}\left[\langle D_\mu U^\dagger D^\mu U\rangle + \langle U\chi^\dagger + \chi U^\dagger\rangle\right]$$

$$D_\mu U = \partial_\mu U - ir_\mu U + iUl_\mu \qquad \chi = 2B(s+ip) \quad (r_\mu, l_\mu) = v_\mu \pm a_\mu$$

Gasser, Leutwyler (84)

The chiral Lagrangian to higher orders

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \dots$$

$\mathcal{L}_2$	contains	$(2,2)$	constants	
$\mathcal{L}_4$	contains	$(7,10)$	constants	Gasser, Leutwyler (84)
$\mathcal{L}_6$	contains	$(53,90)$	constants	Bijnens, GC, Ecker (99)

The number in parentheses are for an $SU(N)$ theory with $N = (2,3)$

The $\mathcal{L}_4$ Lagrangian

$$
\begin{aligned}
\mathcal{L}_4 = {} & L_1 \langle D_\mu U^\dagger D^\mu U \rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U \rangle \langle D^\mu U^\dagger D^\nu U \rangle \\
& + L_3 \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U \rangle + L_4 \langle D_\mu U^\dagger D^\mu U \rangle \langle \chi^\dagger U + \chi U^\dagger \rangle \\
& + L_5 \langle D_\mu U^\dagger D^\mu U (\chi^\dagger U + U^\dagger \chi) \rangle + L_6 \langle \chi^\dagger U + \chi U^\dagger \rangle^2 \\
& + L_7 \langle \chi^\dagger U - \chi U^\dagger \rangle^2 + L_8 \langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger \rangle \\
& - i L_9 \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U \rangle \\
& + L_{10} \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu} \rangle
\end{aligned}
$$

$$
\begin{aligned}
D_\mu U &= \partial_\mu U - i r_\mu U + i U l_\mu \qquad \chi = 2B(s + ip) \\
F_R^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu] \\
r_\mu &= v_\mu + a_\mu \qquad l_\mu = v_\mu - a_\mu
\end{aligned}
$$

Why go beyond $O(p^2)$? Why loops?

$$\langle 0|Te^{\mathrm{i}\int d^4x\mathcal{L}}|0\rangle = e^{\mathrm{i}Z[v,a,s,p]} = \mathcal{N}^{-1}\int [dU]e^{\mathrm{i}\int d^4x\mathcal{L}_{\mathrm{eff}}}$$

Why go beyond $O(p^2)$? Why loops?

- Why not? Chiral Symmetry forbids $O(p^0)$ interactions between pions, but allows all higher orders
- Unitarity: if an amplitude at order p^2 is purely real, at order p^4 its imaginary part is nonzero.
  Take the $\pi\pi$ scattering amplitude. Elastic unitarity relation for the partial waves t^I_ℓ of isospin I and angular momentum ℓ:

$$Im\, t^I_\ell = \sqrt{1 - \frac{4M_\pi^2}{s}}\,|t^I_\ell|^2$$

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- *The correct imaginary parts are generated automatically by loops*
- *The divergences occuring in the loops can be disposed of just like in a renormalizable field theory*

Effective quantum field theory

The method of effective quantum field theory provides a rigorous framework to compute Green functions that respect:

symmetry, analyticity, unitarity

The method yields a systematic expansion of the Green functions in powers of momenta and quark masses

In the following I will discuss in detail how this works when you consider loops:

- I will consider the finite, analytically nontrivial part of the loops and discuss in detail its physical meaning
- I will consider the divergent part of the loops and discuss how the renormalization program works

Scalar form factor of the pion

$$\langle \pi^i(p_1)\pi^j(p_2)|\hat{m}(\bar{u}u+\bar{d}d)|0\rangle =: \delta^{ij}\Gamma(t) \quad , \qquad t=(p_1+p_2)^2 \quad ,$$

At tree level:

$$\Gamma(t) = 2\hat{m}B = M_\pi^2 + O(p^4) \quad ,$$

in agreement with the Feynman–Hellman theorem:
the expectation value of the perturbation in an eigenstate of the total Hamiltonian determines the derivative of the energy level with respect to the strength of the perturbation:

$$\hat{m}\frac{\partial M_\pi^2}{\partial \hat{m}} = \langle \pi|\hat{m}\bar{q}q|\pi\rangle = \Gamma(0) \quad .$$

This matrix element is relevant for the decay $h \to \pi\pi$, which, for $m_H \sim 1$ GeV would have been the main decay mode

Dispersion relation for $\Gamma(t)$

For $t \geq 4M_\pi^2$ $\mathrm{Im}\,\Gamma(t) \neq 0$. $\Gamma(t)$ is analytic everywhere else in the complex t plane, and obeys the following dispersion relation: $\bar{\Gamma}(t) = \Gamma(t)/\Gamma(0)$

$$\bar{\Gamma}(t) = 1 + bt + \frac{t^2}{\pi} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{\mathrm{Im}\,\bar{\Gamma}(t')}{t' - t}$$

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$$\bar{\Gamma}(t) = 1 + bt + \frac{t^2}{\pi}\int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2}\frac{\mathrm{Im}\,\bar{\Gamma}(t')}{t'-t}$$

Unitarity implies $[\sigma(t) = \sqrt{1 - 4M_\pi^2/t}]$

$$\mathrm{Im}\,\bar{\Gamma}(t) = \sigma(t)\bar{\Gamma}(t)t_0^{0*}(t) = \bar{\Gamma}(t)e^{-i\delta_0^0}\sin\delta_0^0 = |\bar{\Gamma}(t)|\sin\delta_0^0$$

where t_0^0 is the S–wave, $I = 0$ $\pi\pi$ scattering amplitude

Strictly speaking, the above unitarity relation is valid only for $t \leq 16M_\pi^2$. To a good approximation, however, it holds up to the $K\bar{K}$ threshold

Dispersion relation and chiral counting

$$\begin{aligned}\bar{\Gamma}(t) &= 1 + bt + \frac{t^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{dt'}{t'^2}\frac{|\bar{\Gamma}(t')|\sin\delta_0^0(t')}{t'-t} \\ b &\sim O(1)\left(1+O(M_\pi^2)\right) \\ \delta_0^0 &\sim O(p^2)\left(1+O(p^2)\right)\end{aligned}$$

There are two $O(p^2)$ correction to $\bar{\Gamma}$:

1. $O(1)$ contribution to b;
2. the dispersive integral containing the $O(p^2)$ phase δ_0^0.

Notice that the latter is fixed by unitarity and analyticity

Are these respected by the one loop calculation?

Dispersion relation and one–loop CHPT

The full one–loop expression of $\bar{\Gamma}(t)$ reads as follows:

$$\bar{\Gamma}(t) = 1 + \frac{t}{16\pi^2 F_\pi^2}(\bar{\ell}_4 - 1) + \frac{2t - M_\pi^2}{2F_\pi^2}\bar{J}(t)$$

$$\bar{J}(t) = \frac{1}{16\pi^2}\left[\sigma(t)\ln\frac{\sigma(t)-1}{\sigma(t)+1} + 2\right]$$

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To prove that unitarity and analyticity are respected at this order is sufficient to add:

$$\delta_0^0(t) = \sigma(t)\frac{2t - M_\pi^2}{32\pi F_\pi^2} + O(p^4) \qquad \bar{J}(t) = \frac{t}{16\pi^2}\int_{4M_\pi^2}^{\infty}\frac{dt'}{t'}\frac{\sigma(t')}{t'-t}$$

Can you prove it?

Hints:

- Subtract $\bar{J}(t)$ once more

$$\bar{J}(t) = \frac{t}{96\pi^2 M_\pi^2} + \frac{t^2}{16\pi^2} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{\sigma(t')}{t' - t}$$

- Trick to pull out a linear term from the dispersive integral:

$$\int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{t'\sigma(t')}{t' - t} = t \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \frac{\sigma(t')}{t' - t} + \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2} \sigma(t')$$

High-energy contributions

The dispersive integral goes up to $s' = \infty$, but the integrand is correct only at low energy!

$$
\begin{aligned}
\bar{\Gamma}(t)_{h.e.} &= \frac{t^2}{\pi}\int_{\Lambda^2}^{\infty}\frac{dt'}{t'^2}\frac{|\bar{\Gamma}(t')|\sin\delta_0^0(t')}{t'-t} \\
&\sim \frac{t^2}{\pi}\int_{\Lambda^2}^{\infty}\frac{dt'}{t'^2}|\bar{\Gamma}(t')|\sin\delta_0^0(t')\frac{1}{t'}\left(1+\frac{t}{t'}+\dots\right) \\
&\sim ct^2+\mathcal{O}(t^3)
\end{aligned}
$$

The contributions from the high-energy region of the dispersive integral are formally of higher order – introducing a cut-off to remove them would only make the formulae more cumbersome

Renormalization at one loop

$$\int \frac{d^4 l}{(2\pi)^4} \frac{\{p^2, p\cdot l, l^2\}}{(l^2 - M^2)((p-l)^2 - M^2)} \;, \qquad p = p_1 + p_2$$

$$\sim \underbrace{\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - M^2)}}_{T(M^2)} + p^2 \underbrace{\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - M^2)((p-l)^2 - M^2)}}_{J(p^2)}$$

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$$T(M^2) = a + bM^2 + \bar{T}(M^2) \qquad J(t) = J(0) + \bar{J}(t)$$

$\bar{T}(M^2)$ and $\bar{J}(t)$ are finite

$$\Gamma(t) \sim M^2 \left[1 + \underbrace{bM^2 + tJ(0)}_{\text{divergent part}} + \bar{T}(M^2) + \bar{J}(t) \right]$$

Counterterms

$$\mathcal{L}_2 \quad \Rightarrow \quad \Gamma^{(2)}(t) \sim M^2$$

$$\mathcal{L}_4 \quad \Rightarrow \quad \Gamma^{(4)}(t) \sim \ell_3 M^4 + \ell_4 M^2 t$$

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To remove the divergences need to properly define the couplings ($\ell_{3,4}$) in the Lagrangian at order $O(p^4)$

Quote from Weinberg's book on QFT, vol. I: "(...) as long as we include every one of the infinite number of interactions allowed by symmetries, the so–called non–renormalizable theories are actually just as renormalizable as renormalizable theories."

Chiral logarithms

Scalar radius of the pion

$$\Gamma(t) = \Gamma(0)\left[1 + \frac{1}{6}\langle r^2\rangle^\pi_S t + O(t^2)\right]$$

$$\langle r^2\rangle^\pi_S \sim J(0) = \int \frac{d^4 l}{(2\pi)^4}\frac{1}{(l^2 - M^2)^2} \sim \ln\frac{M^2}{\Lambda^2}$$

The integral is UV divergent, but also IR divergent if $M \to 0$:

$$\lim_{M^2\to 0}\langle r^2\rangle^\pi_S \sim \ln M^2 \quad ,$$

The extension of the cloud of pions surrounding a pion (or any other hadron) goes to infinity if pions become massless (Li and Pagels '72)

Chiral symmetry and renormalization

To remove the divergent part in $\Gamma(t)$ must fix the divergent part of chiral–invariant operators of order $O(p^4)$

e.g. $\langle \partial_\mu U^\dagger \partial^\mu U \rangle \langle B\mathcal{M}(U + U^\dagger) \rangle \sim \ldots + M^2 \phi^2 \partial_\mu \phi^4 \partial^\mu \phi^6 + \ldots$

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Chiral symmetry implies that after calculating the divergent part of $\Gamma(s)$ I also know the divergent part of the $6\pi \to 6\pi$ scattering amplitude

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1. Do we have a proof that quantum effects do not introduce violations of the chiral symmetry? Or that one can build a chiral invariant generating functional only with a path integral over a chiral invariant classical action?

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1. Do we have a proof that quantum effects do not introduce violations of the chiral symmetry? Or that one can build a chiral invariant generating functional only with a path integral over a chiral invariant classical action?
2. Is there a tool that allows one to calculate the divergences keeping chiral invariance explicit in every step of the calculation?

Leutwyler's theorem

What is the most general way of constructing a chiral-invariant generating functional out of a path integral over the Goldstone boson degrees of freedom?

$$Z[v', a', s', p'] = Z[v, a, s, p] \Leftrightarrow \mathcal{L}_{\rm eff}[v', a', s', p'] = \mathcal{L}_{\rm eff}[v, a, s, p]?$$

For Lorentz–invariant theories in 4 dimensions, a path integral constructed with gauge–invariant lagrangians is a necessary and sufficient condition to obtain a gauge–invariant generating functional

The theorem also includes the case in which the symmetry is anomalous and the case in which the symmetry is explicitly broken

Chiral invariant renormalization

- Gasser & Leutwyler (84): the calculation of the divergences at one loop – and the corresponding renormalization – can be performed in an explicitly chiral invariant manner
- The method has been extended and applied to two loops (Bijnens, GC & Ecker 98). After a long and tedious calculation, the divergent parts of all the counterterms at $\mathcal{O}(p^6)$ has been provided
- Renormalization of CHPT up to two loops has been performed explicitly: the calculation of any two-loop amplitude can be immediately checked by comparing the divergent part of Feynman diagrams to the divergent parts of the relevant counterterms

Chiral perturbation theory

- Chiral perturbation theory provides a rigorous framework to compute Green functions that respect all the good properties we require:
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  symmetry, analyticity, unitarity
- The method yields a systematic expansion of the Green functions in powers of momenta and quark masses
- The method has been rigorously established and can be formulated as a set of calculational rules:

LO	tree level diagrams with $\mathcal{L}_2$
NLO	tree level diagrams with $\mathcal{L}_4$
	1-loop diagrams with $\mathcal{L}_2$
NNLO	tree level diagrams with $\mathcal{L}_6$
	2-loop diagrams with $\mathcal{L}_2$
	1-loop diagrams with one vertex from $\mathcal{L}_4$

Summary

- Goldstone's theorem has physical implications at low energy
- Effective Lagrangian for Goldstone bosons
  $=$ tool to systematically derive symmetry constraints on their interactions
- I have discussed how to construct the effective Lagrangian, even in the presence of a (small) symmetry breaking
- Anatomy of loop contributions:
  - the analytically nontrivial part of loop integrals automatically yields the correct imaginary parts (unitarity)
  - IR singular behaviour of loop integrals ($=$ chiral logs) is a physical effect expected in a system with massless particles
  - UV divergences encountered in loop integrals can be removed according to standard renormalization methods