# Chiral dynamics in QCD

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# Lecture II: two applications

$\pi\pi$ scattering beyond LO
- Experimental tests

$\eta \to 3\pi$ and $Q \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$
- How to determine $m_u - m_d$

Summary

# Outline

$\pi\pi$ scattering beyond LO

Experimental tests

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How to determine $m_u - m_d$

Summary

# $\pi\pi$ scattering at NLO

$$
\begin{aligned}
a_0^0 &= \frac{7M_\pi^2}{32\pi F_\pi^2}\left[1 + \frac{M_\pi^2}{3}\langle r^2\rangle_S^\pi + \frac{200\pi F_\pi^2 M_\pi^2}{7}(a_2^0 + 2a_2^2)\right. \\
&- \left.\frac{M_\pi^2}{672\pi^2 F_\pi^2}(15\bar{\ell}_3 - 353)\right] = 0.16 \cdot 1.25 = 0.20 \\
2a_0^0 - 5a_0^2 &= \frac{3M_\pi^2}{4\pi F_\pi^2}\left[1 + \frac{M_\pi^2}{3}\langle r^2\rangle_S^\pi + \frac{41 M_\pi^2}{192\pi^2 F_\pi^2}\right] = 0.624
\end{aligned}
$$

Gasser and Leutwyler (83)

# Higher orders

Higher order corrections are suppressed by $\mathcal{O}(m_q^2/\Lambda^2)$
$\Lambda \sim 1$ GeV $\Rightarrow$ expected to be a few percent

$$a_0^0 = 0.200 + \mathcal{O}(p^6) \qquad a_0^2 = -0.0445 + \mathcal{O}(p^6)$$

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The reason for the rather large correction in $a_0^0$ is a chiral log

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\left[1 + \frac{9}{2}\ell_\chi + \dots\right] \qquad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2}\left[1 - \frac{3}{2}\ell_\chi + \dots\right]$$

$$\ell_\chi = \frac{M_\pi^2}{16\pi^2 F_\pi^2} \ln \frac{\mu^2}{M_\pi^2}$$

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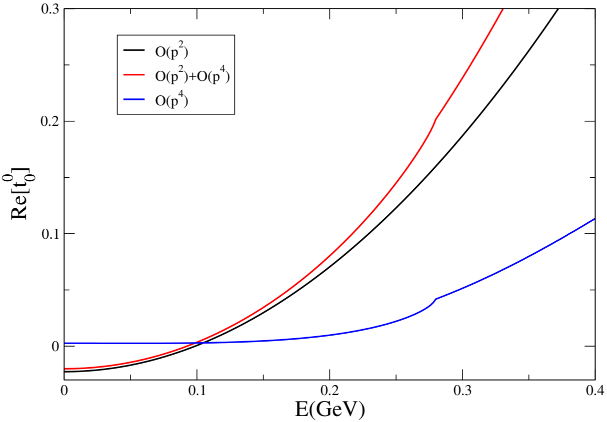

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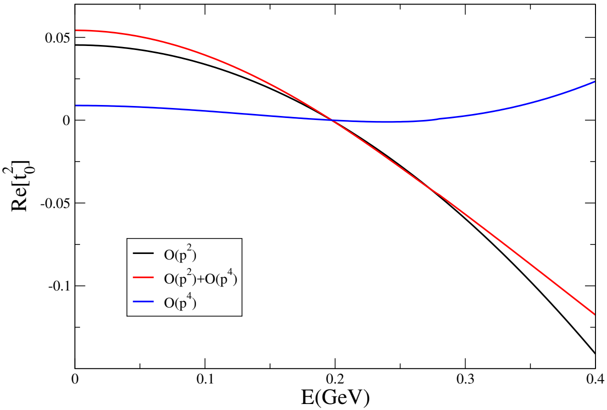

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How large are yet higher orders?
Is it at all possible to make a precise prediction?

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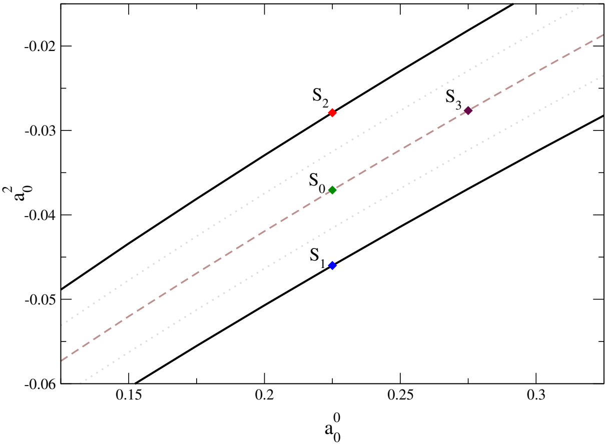
Numerical solutions of the Roy equations
Pennington-Protopopescu, Basdevant-Froggatt-Petersen (70s)
Ananthanarayan, GC, Gasser and Leutwyler (00)
Descotes-Genon, Fuchs, Girlanda and Stern (01)

# Numerical solutions



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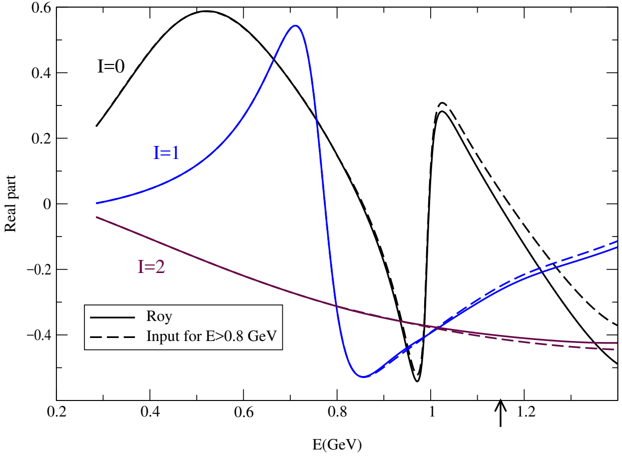

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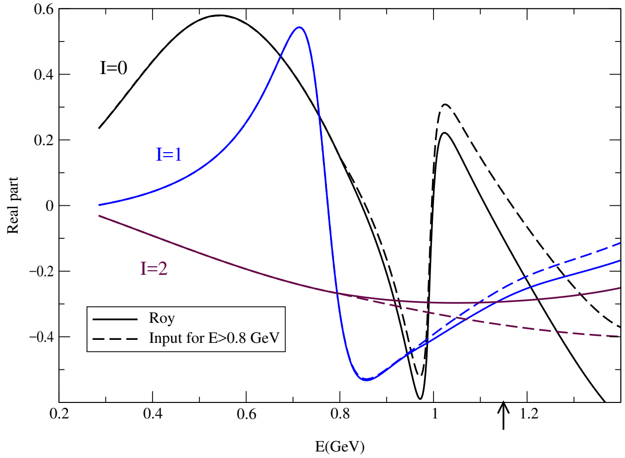

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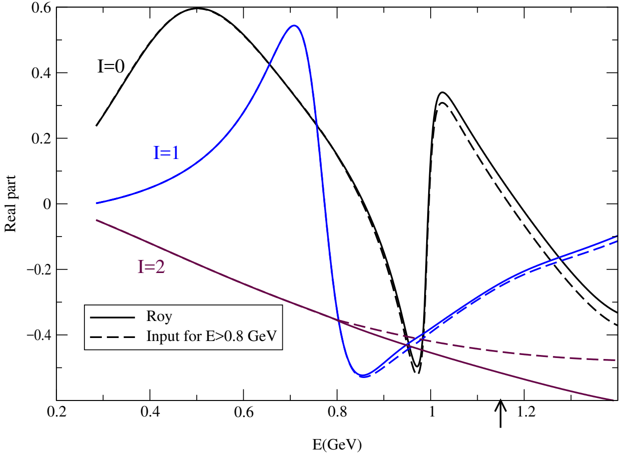

# Combining CHPT and dispersive methods

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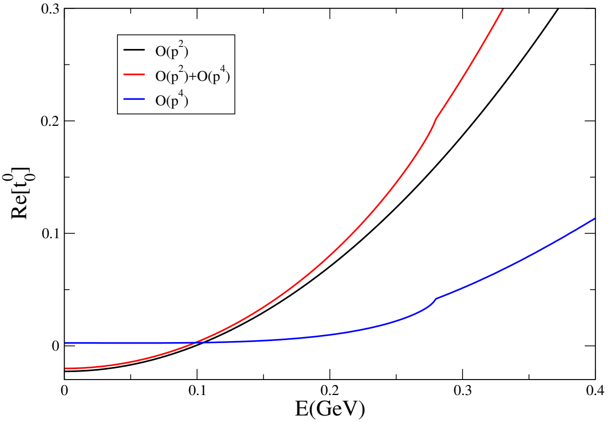
Subtracting the amplitude at threshold ($a_0^0, a_0^2$) is not mandatory

The freedom in the choice of the subtraction point can be exploited to use the chiral expansion where it converges best, *i.e.* below threshold

# Combining CHPT and dispersive methods



# Combining CHPT and dispersive methods

The convergence of the series at threshold is greatly improved if CHPT is used only below threshold

## CHPT at threshold

$$
\begin{aligned}
a_0^0 &= \quad 0.159 \to \quad 0.200 \to \quad 0.216 \\
10 \cdot a_0^2 &= -0.454 \to -0.445 \to -0.445 \\
& \qquad p^2 \qquad\qquad p^4 \qquad\qquad p^6
\end{aligned}
$$

GC, Gasser and Leutwyler (01)

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& \qquad p^2 \qquad\qquad p^4 \qquad\qquad p^6
\end{aligned}
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## CHPT below threshold + Roy

$$
\begin{aligned}
a_0^0 &= \quad 0.197 \to \quad 0.2195 \to \quad 0.220 \\
10 \cdot a_0^2 &= -0.402 \to -0.446 \to -0.444
\end{aligned}
$$

GC, Gasser and Leutwyler (01)

# Low-energy theorem for $\pi\pi$ scattering

$$\mathcal{M}(\pi^0\pi^0 \to \pi^+\pi^-) \equiv A(s,t,u) = \text{isospin invariant amplitude}$$

Low energy theorem: $$A(s,t,u) = \frac{s - M^2}{F^2} + \mathcal{O}(p^4)$$ Weinberg 1966

$$M^2 = B(m_u + m_d) \qquad M_\pi^2 = M^2 + O(m_q^2), \quad F_\pi = F + O(m_q)$$

All physical amplitudes can be expressed in terms of $A(s,t,u)$

$$T^{I=0} = 3A(s,t,u) + A(t,s,u) + A(u,t,s) \;\Rightarrow\; T^{I=0} = \frac{2s - M_\pi^2}{F_\pi^2}$$

S wave projection (I=0)

$$t_0^0(s) = \frac{2s - M_\pi^2}{32\pi F_\pi^2} \qquad a_0^0 = t_0^0(4M_\pi^2) = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16$$

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All physical amplitudes can be expressed in terms of $A(s,t,u)$

$$T^{I=2} = A(t,s,u) + A(u,t,s) \;\Rightarrow\; T^{I=2} = \frac{-s + 2M_\pi^2}{F_\pi^2}$$

S wave projection (I=2)

$$t_0^2(s) = \frac{2M_\pi^2 - s}{32\pi F_\pi^2} \qquad a_0^2 = t_0^2(4M_\pi^2) = \frac{-M_\pi^2}{16\pi F_\pi^2} = -0.045$$

# Chiral predictions for $a_0^0$ and $a_0^2$

Quark mass dependence of $M_\pi$ and $F_\pi$:

$$M_\pi^2 = M^2 \left(1 - \frac{M^2}{32\pi^2 F^2}\bar{\ell}_3 + O(p^4)\right)$$

$$M^2 \equiv -\frac{m_u + m_d}{F^2}\langle 0|\bar{q}q|0\rangle$$

Gell-Mann, Oakes, Renner (68)

$$F_\pi = F \left(1 + \frac{M^2}{16\pi^2 F^2}\bar{\ell}_4 + O(p^4)\right)$$

Phenomenological determinations (indirect):

$$\bar{\ell}_3 = 2.9 \pm 2.4$$

Gasser & Leutwyler (84)

$$\bar{\ell}_4 = 4.4 \pm 0.2$$

GC, Gasser & Leutwyler (01)

Lattice calculations determine these constants directly

# Chiral predictions for $a_0^0$ and $a_0^2$

$\chi$PT calculations at NLO (Gasser & Leutwyler 84)
and at NNLO (Bijnens, GC, Ecker, Gasser & Sainio, 95)

Prediction obtained matching $O(p^6)$ $\chi$PT to Roy equations (disp. relation): GC, Gasser & Leutwyler (01)

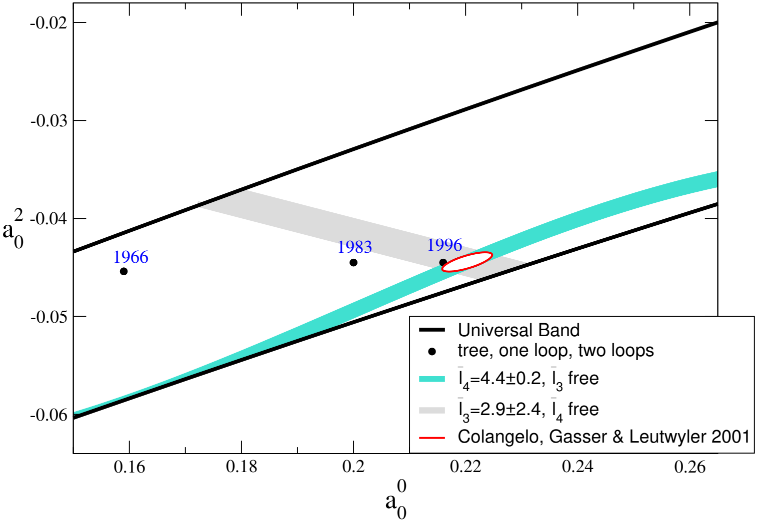
$$\begin{aligned} a_0^0 &= 0.220 \pm 0.001 + 0.009\Delta\ell_4 - 0.002\Delta\ell_3 \\ 10 \cdot a_0^2 &= -0.444 \pm 0.003 - 0.01\Delta\ell_4 - 0.004\Delta\ell_3 \end{aligned}$$

where $\bar{\ell}_4 = 4.4 + \Delta\ell_4$   $\bar{\ell}_3 = 2.9 + \Delta\ell_3$

Adding errors in quadrature   $[\Delta\ell_4 = 0.2,\ \Delta\ell_3 = 2.4]$

$$\begin{aligned} a_0^0 &= 0.220 \pm 0.005 \\ 10 \cdot a_0^2 &= -0.444 \pm 0.01 \\ a_0^0 - a_0^2 &= 0.265 \pm 0.004 \end{aligned}$$

# Chiral predictions for $a_0^0$ and $a_0^2$



# Sensitivity to the quark condensate

The constant $\bar{\ell}_3$ appears in the chiral expansion of the pion mass

$$M_\pi^2 = 2B\hat{m}\left[1 + \frac{2B\hat{m}}{16\pi F_\pi^2}\bar{\ell}_3 + \mathcal{O}(\hat{m}^2)\right]$$

$$\hat{m} = \frac{m_u + m_d}{2} \qquad B = -\frac{1}{F^2}\langle 0|\bar{q}q|0\rangle$$

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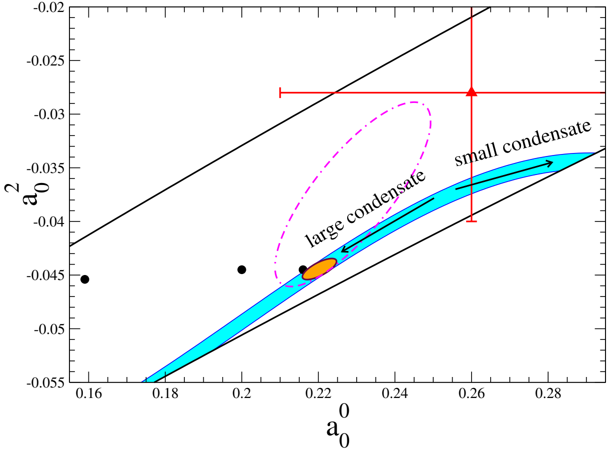
Its size tells us what fraction of the pion mass is given by the Gell-Mann–Oakes–Renner term

$$M^2_{\mathrm{GMOR}} \equiv 2B\hat{m}$$

# Sensitivity to the quark condensate

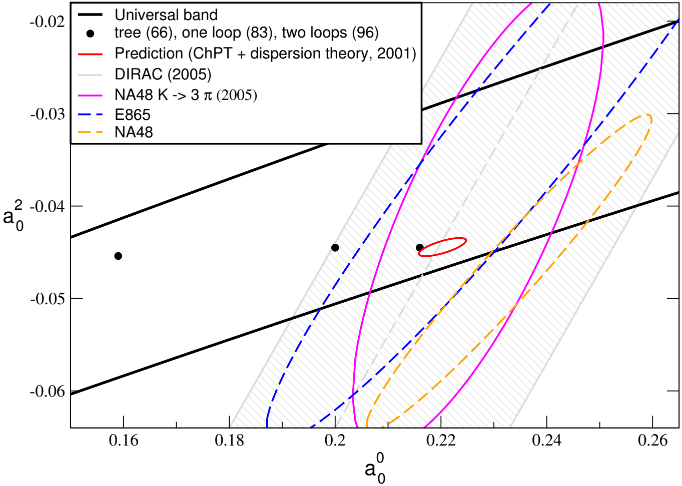



The E865 data on $K_{\ell 4}$ imply that

GC, Gasser and Leutwyler PRL (01)

$$M_{\mathrm{GMOR}} > 94\% M_\pi$$

# Experimental tests



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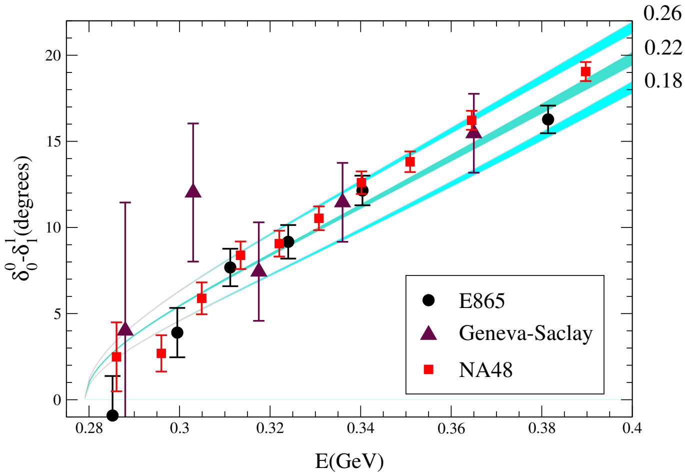

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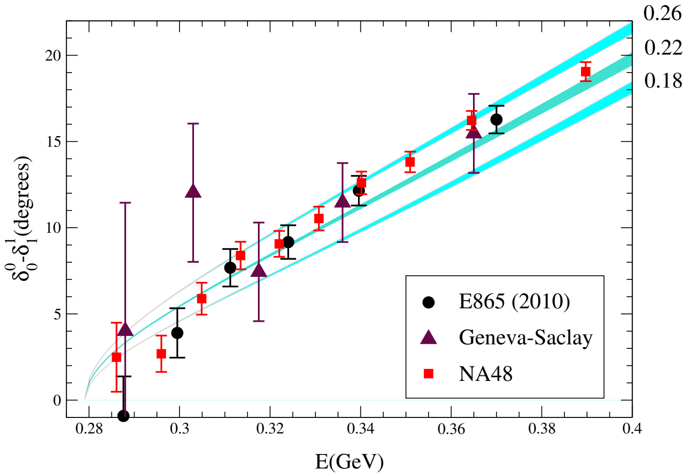


Recent update: E865 corrected their data

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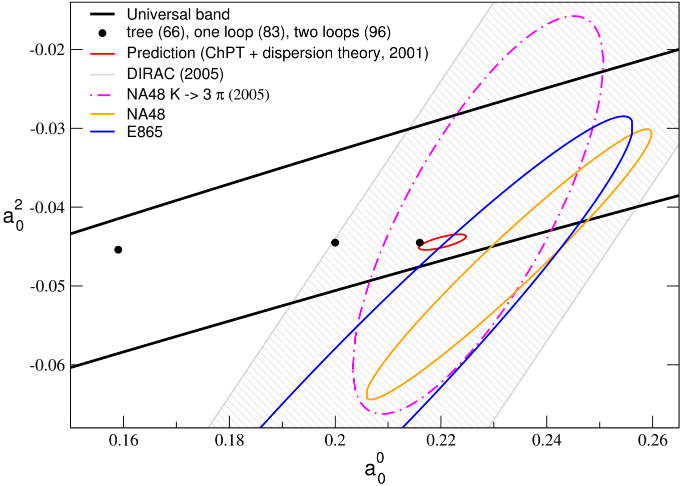


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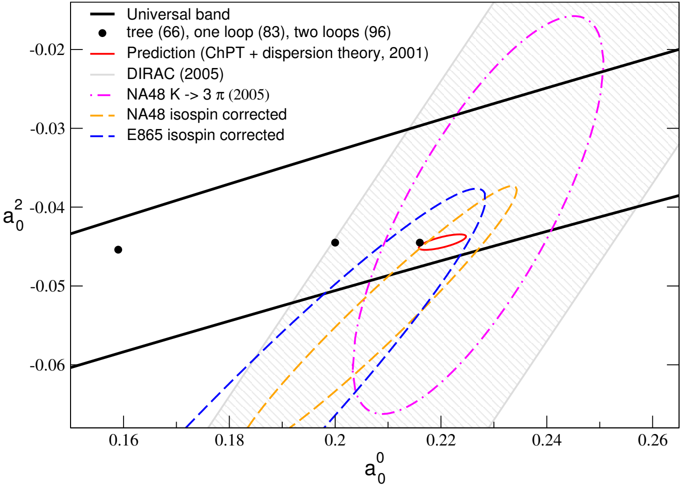


isospin breaking corrections recently calculated for $K_{e4}$ are essential at this level of precision    GC, Gasser, Rusetsky (09)

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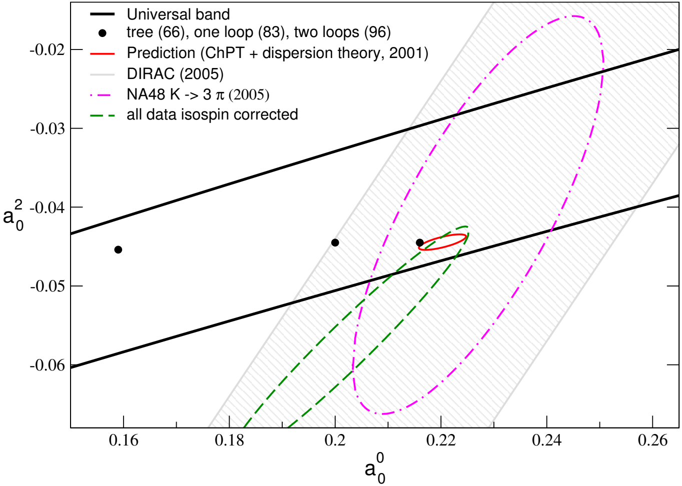


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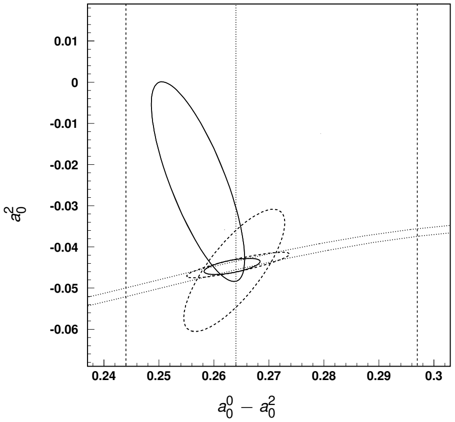
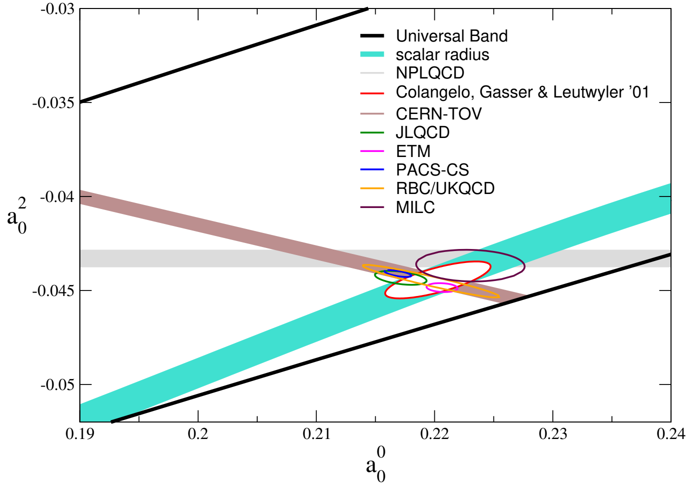



Figure from NA48/2 Eur.Phys.J.C64:589,2009

# Lattice input for $\bar{\ell}_3$ and $\bar{\ell}_4$



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$\pi\pi$ scattering beyond LO

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$\eta \to 3\pi$ and $Q \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$

How to determine $m_u - m_d$

Summary

# Quark masses

QCD Lagrangian:

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4g^2}\mathrm{Tr}\,G_{\mu\nu}G^{\mu\nu} + \sum_i \bar{q}_i(i\not{D} - m_{q_i})q_i + \sum_j \bar{Q}_j(i\not{D} - m_{Q_j})Q_j$$

- In the limit $m_{q_i} \to 0$ and $m_{Q_j} \to \infty$: $M_{\mathrm{hadrons}} \propto \Lambda$
- Observe that $m_{q_i} \ll \Lambda$ while $m_{Q_j} \gg \Lambda$ $[\Lambda \sim M_N]$
- Quarks do not propagate:
  quark masses are coupling constants! (not observables)

  they depend on the renormalization scale $\mu$ (like $\alpha_s$ )
  for light quarks by convention: $\mu = 2\ \mathrm{GeV}$

# How to determine quark masses

- From their influence on the spectrum  $\chi$PT, lattice
  - $m_Q \gg \Lambda$
    $$M_{\bar{Q}q_i} = m_Q + \mathcal{O}(\Lambda)$$
  - $m_q \ll \Lambda$
    $$M_{\bar{q}_i q_j} = M_{0\,ij} + \mathcal{O}(m_{q_i}, m_{q_j}) \qquad M_{0\,ij} = \mathcal{O}(\Lambda)$$

  In both cases need to understand the $\mathcal{O}(\Lambda)$ term

- From their influence on any other observable  $\chi$PT, sum rules

  Quark masses are coupling constants
  $\Rightarrow$ exploit the sensitivity to them of any observable
  [e.g. $\eta$ decays, spectral functions from $\tau$ decays, etc. ]

# $m_d + m_u$ is easier to get than $m_d - m_u$

$$m_d,\ m_u \ll \Lambda \quad \Rightarrow \quad \mathcal{L}_m = -m_u \bar{u}u - m_d \bar{d}d = \text{small perturbation}$$

However:

$$\begin{aligned} \mathcal{L}_m &= -\frac{m_d + m_u}{2}(\bar{u}u + \bar{d}d) - (m_d - m_u)\frac{\bar{u}u - \bar{d}d}{2} \\ &= -\hat{m}\,\underbrace{\bar{q}q}_{\mathcal{O}_{I=0}} + (m_d - m_u)\,\underbrace{\bar{q}\tau_3 q}_{\mathcal{O}_{I=1}} \end{aligned}$$

and selection rules make the effect of $\mathcal{O}_{I=1}$ well hidden

$\Rightarrow$ $\hat{m}$ responsible for the mass of pions
but $(m_d - m_u)$ only contributes at $\mathcal{O}(p^4)$ (a tiny $\delta M_{\pi^0}$)

better sensitivity in $K$ masses

# First estimates

Leading-order masses of $\pi$ and $K$:

$$M_\pi^2 = B_0(m_u + m_d) \quad M_{K^+}^2 = B_0(m_u + m_s) \quad M_{K^0}^2 = B_0(m_d + m_s)$$

Quark mass ratios:

$$\frac{m_u}{m_d} \simeq \frac{M_{\pi^+}^2 - M_{K^0}^2 + M_{K^+}^2}{M_{\pi^+}^2 + M_{K^0}^2 - M_{K^+}^2} \simeq 0.67$$

$$\frac{m_s}{m_d} \simeq \frac{M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \simeq 20$$

# Electromagnetic corrections to the masses

According to Dashen's theorem

$$\begin{aligned} M^2_{\pi^0} &= B_0(m_u + m_d) \\ M^2_{\pi^+} &= B_0(m_u + m_d) + \Delta_{\text{em}} \\ M^2_{K^0} &= B_0(m_d + m_s) \\ M^2_{K^+} &= B_0(m_u + m_s) + \Delta_{\text{em}} \end{aligned}$$

Extracting the quark mass ratios gives Weinberg (77)

$$\frac{m_u}{m_d} = \frac{M^2_{K^+} - M^2_{K^0} + 2M^2_{\pi^0} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}} = 0.56$$

$$\frac{m_s}{m_d} = \frac{M^2_{K^0} + M^2_{K^+} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}} = 20.1$$

# Higher order chiral corrections

Mass formulae to second order

Gasser-Leutwyler (85)

$$\frac{M_K^2}{M_\pi^2} = \frac{m_s + \hat{m}}{2\hat{m}}\left[1 + \Delta_M + \mathcal{O}(m^2)\right]$$

$$\frac{M_{K^0}^2 - M_{K^+}^2}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}}\left[1 + \Delta_M + \mathcal{O}(m^2)\right]$$

$$\Delta_M = \frac{8(M_K^2 - M_\pi^2)}{F_\pi^2}(2L_8 - L_5) + \chi\text{-logs}$$

The same $\mathcal{O}(m)$ correction appears in both ratios
$\Rightarrow$ this double ratio is free from $\mathcal{O}(m)$ corrections

$$Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} = \frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{M_{K^0}^2 - M_{K^+}^2}\left[1 + \mathcal{O}(m^2)\right]$$

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Mass formulae to second order    Gasser-Leutwyler (85)

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The same $\mathcal{O}(m)$ correction appears in both ratios
$\Rightarrow$ this double ratio is free from $\mathcal{O}(m)$ and em corrections

$$Q_D^2 \equiv \frac{(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 + M_{\pi^0}^2)(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 - M_{\pi^0}^2)}{4M_{\pi^0}^2(M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2 - M_{\pi^0}^2)} = 24.3$$

# Violation of Dashen's theorem

In pure QCD ($\hat{M}_P \equiv M_{P|_{\alpha_{em}=0}}$)

$$\begin{aligned}\hat{M}_{K^+} &= B_0(m_s + m_u) + \mathcal{O}(m_q^2)\\ \hat{M}_{K^0} &= B_0(m_s + m_d) + \mathcal{O}(m_q^2)\end{aligned}$$

$$\Rightarrow \quad \hat{M}_{K^+} - \hat{M}_{K^0} = B_0(m_u - m_d) + \mathcal{O}(m_q^2)$$

Define em contributions to masses

$$M_P^\gamma \equiv M_P - \hat{M}_P, \quad \Delta_P^\gamma \equiv M_P^2 - \hat{M}_P^2$$

Dashen's theorem: $\Delta_{K^+}^\gamma = \Delta_{\pi^+}^\gamma$

and its violation $[\Delta_\pi \equiv M_{\pi^+}^2 - M_{\pi^0}^2]$

$$\Delta_{K^+}^\gamma - \Delta_{K^0}^\gamma - \Delta_{\pi^+}^\gamma + \Delta_{\pi^0}^\gamma = \epsilon\Delta_\pi$$

# Estimates of the size of Dashen's theorem violation

$\chi$PT + model-based calculations:

$$\epsilon = \begin{cases} 0.8 & \text{Bijnens-Prades (97)} & Q = 22 \text{ (ENJL model)} \\ 1.0 & \text{Donoghue-Perez (97)} & Q = 21.5 \text{ (VMD)} \\ 1.5 & \text{Anant-Moussallam (04)} & Q = 20.7 \text{ (Sum rules)} \end{cases}$$

Lattice-based calculations

(the value of $Q$ is calculated in $\chi$PT at NLO)

$$\epsilon = \begin{cases} 0.50(8) & \text{Duncan et al. (96)} & Q = 22.9 \\ 0.5(1) & \text{RBC (07)} & Q = 22.9 \\ 0.78(6)(2)(9)(2) & \text{BMW (11)} & Q = 22.1 \\ 0.65(7)(14)(10) & \text{MILC (13)} & Q = 22.6 \\ 0.79(18)(18) & \text{RM123 (13)} & Q = 22.1 \\ 0.73(2)(5)(17) & \text{BMW (16)} & Q = 22.2 \\ 0.73(3)(13)(5) & \text{MILC (16)} & Q = 22.2 \end{cases}$$

Value quoted in FLAG-3: $\epsilon = 0.7(3)$

# FLAG-3 summary of the quark masses

all masses in MeV

| $N_F$ | $m_{ud}$ | $m_s$ | $m_s/m_{ud}$ |
|---|---|---|---|
| 2+1+1 | 3.70(17) | 93.9(1.1) | 27.30(34) |
| 2+1 | 3.373(80) | 92.0(2.1) | 27.43(31) |
| 2 | 3.6(2) | 101(3) | 27.3(9) |

| $N_F$ | $m_u$ | $m_d$ | $m_u/m_d$ | $R$ | $Q$ |
|---|---|---|---|---|---|
| 2+1+1 | 2.36(24) | 5.03(26) | 0.470(56) | 35.6(5.1) | 22.2 (1.6) |
| 2+1 | 2.16(9)(7) | 4.68(14)(7) | 0.46(2)(2) | 35.0(1.9)(1.8) | 22.5(6)(6) |
| 2 | 2.40(23) | 4.80(23) | 0.50(4) | 40.7(3.7)(2.2) | 24.3(1.4)(0.6) |

# $\chi$PT and $\eta \to 3\pi$

Lowest order chiral amplitude: Osborn, Wallace (70)

$$\mathcal{M}(\eta \to \pi^+\pi^-\pi^0) =: A(s,t,u) \qquad s = (p_{\pi^+} + p_{\pi^-})^2, \ldots$$

$$A(s,t,u) = \frac{B_0(m_u - m_d)}{3\sqrt{3}F_\pi^2}\left[1 + \frac{3(s - s_0)}{M_\eta^2 - M_\pi^2} + O(m)\right] + O(e^2 m)$$

Relate $m_u - m_d$ to meson masses Dashen (69)

$$B_0(m_u - m_d) = (M_{K^+}^2 - M_{K^0}^2) - (M_{\pi^+}^2 - M_{\pi^0}^2) + O(e^2 m)$$

LO chiral prediction

$$\Gamma(\eta \to \pi^+\pi^-\pi^0) \sim 70 \text{ eV} \qquad \ll \qquad \Gamma_{\text{exp}} = 295 \pm 20 \text{ eV}$$

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NLO chiral prediction Gasser-Leutwyler (85)

$$\Gamma(\eta \to \pi^+\pi^-\pi^0) \sim 70 \text{ eV} \to 160 \pm 50 \text{ eV} \quad \ll \quad \Gamma_{\text{exp}} = 295 \pm 20 \text{ eV}$$

# Dispersive approach

Isospin decomposition of $M(s,t,u)$

Stern, Sazdjian, Fuchs (93)

Anisovich, Leutwyler (96)

$$M(s,t,u) = M_0(s)+(s-u)M_1(t)+(s-t)M_1(u)+M_2(t)+M_2(u)-\frac{2}{3}M_2(s)$$

assumes only: $\mathrm{disc}[t_\ell^I(s)] = 0 \qquad \forall \ell \geq 2$ in all channels

Analytic properties of the $M_I(s)$ functions: $[s > 4M_\pi^2]$

$$\mathrm{disc}[M_I(s)] = \mathrm{disc}[t_\ell^I(s)] = t_\ell^I(s) e^{i\delta_\ell^I(s)} \sin \delta_\ell^I(s)$$

$t_\ell^I(s) =$ partial wave with isospin $I$ and angular momentum $\ell$

$$t_\ell^I(s) = M_I(s) + \hat{M}_I(s)$$

Dispersion relation for $M_0$

analogous ones for $M_1$ and $M_2$

$$M_0(s) = \Omega_0(s)\left[\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^2}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\hat{M}_0(s')\sin\delta_0^0(s')}{|\Omega_0(s')|s'^2(s'-s)}\right]$$

Anisovich, Leutwyler (96)

# How we fit the data

- our dispersive amplitude, linear in the subtraction constants $(\alpha_0, \beta_0, \ldots)$ (corrected for isospin breaking)
- (we use linear combinations of the $\alpha_0, \beta_0, \ldots \longrightarrow H_{0,1,\ldots,5}$ dividing by $H_0$ we get $h_i \equiv H_i/H_0, \quad i = 1, 2, \ldots, 5$ )
- the invariants $h_1$, $h_2$ and $h_3$ are constrained by the $\chi$PT NLO calculation: (theoretical $\chi^2$ added to the experimental)

$$h_1 = 4.52(36), \quad h_2 = 16.4(4.9), \quad h_3 = 6.3(1.9)$$

- the invariants $h_4$ and $h_5$ are treated as free parameters
- available data are from:
  - KLOE (2016)
  - WASA@COSY (2014)
  - Crystal Ball@MAMI (2007)
  - several values for $\alpha$ are in the PDG

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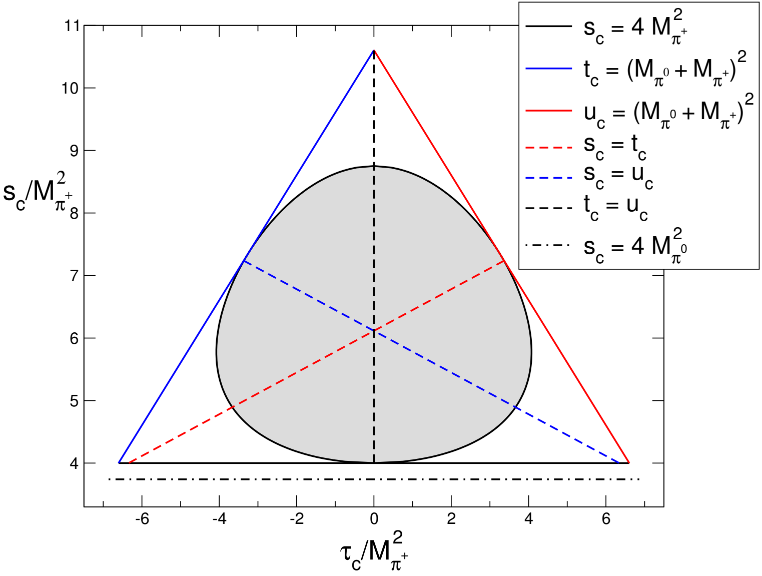
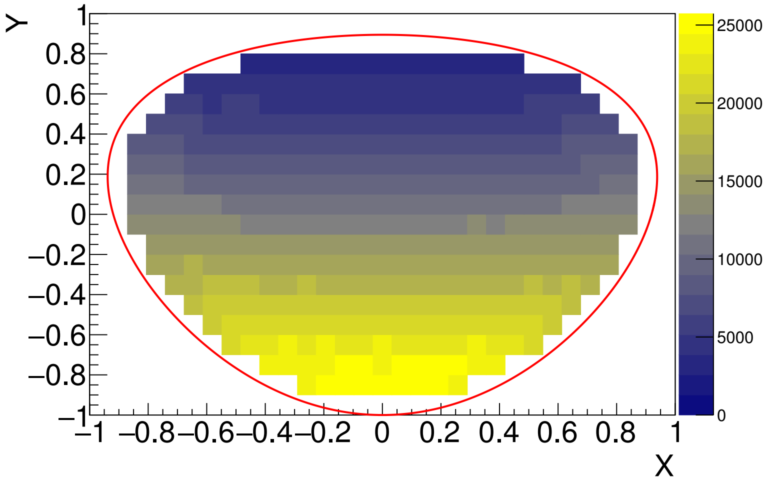
# KLOE data



11
10
9
8
7
6
5
4
-6
-4
-2
0
2
4
6
$s_c/M_{\pi^+}^2$
$\tau_c/M_{\pi^+}^2$
$s_c = 4\, M_{\pi^+}^2$
$t_c = (M_{\pi^0} + M_{\pi^+})^2$
$u_c = (M_{\pi^0} + M_{\pi^+})^2$
$s_c = t_c$
$s_c = u_c$
$t_c = u_c$
$s_c = 4\, M_{\pi^0}^2$

# KLOE data




KLOE JHEP collab. 2016

# Fit results

$\chi$PT:

$$h_1^{NLO} = 4.52(36), \quad h_2^{NLO} = 16.4(4.9), \quad h_3^{NLO} = 6.3(1.9)$$

$$s_A^{LO} = \frac{4}{3}M_\pi^2 \qquad s_A^{NLO} = 1.40 M_\pi^2$$

Fit outcomes:

3-parameter fit — w/o $\chi^2_{\rm th}$

$$\chi^2_{\rm exp} = 385.3 \quad \text{for} \quad 371 \quad \text{data points}$$

$$h_1 = 4.53,\ h_2 = 12.6,\ h_3 = 6.4,$$

Adler zero:

$$s_A = 1.43 M_\pi^2$$

# Fit results

$\chi$PT:

$$h_1^{NLO} = 4.52(36), \quad h_2^{NLO} = 16.4(4.9), \quad h_3^{NLO} = 6.3(1.9)$$

$$s_A^{LO} = \frac{4}{3} M_\pi^2 \qquad s_A^{NLO} = 1.40 M_\pi^2$$

Fit outcomes:

5-parameter fit — w/o $\chi^2_{\rm th}$

$$\chi^2_{\rm exp} = 370.3 \quad \text{for} \quad 371 \quad \text{data points}$$

$$h_1 = 0.93,\ h_2 = 16.3,\ h_3 = 52.0,$$
$$h_4 = 77.9,\ h_5 = -56.7$$

Adler zero: none!

# Fit results

$\chi$PT:

$$h_1^{NLO} = 4.52(36), \quad h_2^{NLO} = 16.4(4.9), \quad h_3^{NLO} = 6.3(1.9)$$

$$s_A^{LO} = \frac{4}{3} M_\pi^2 \qquad s_A^{NLO} = 1.40 M_\pi^2$$

Fit outcomes:

5-parameter fit — w/ $\chi^2_{\mathrm{th}}$

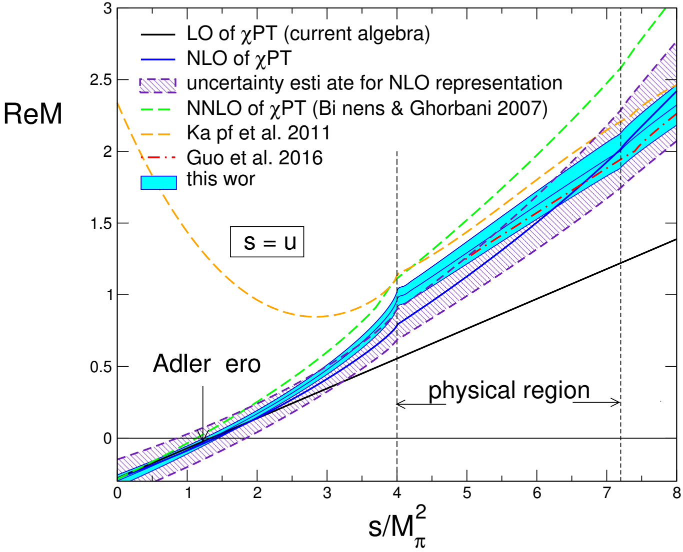
$$\chi^2_{\mathrm{exp}} = 380.2 \quad \text{for} \quad 371 \quad \text{data points}$$

$$h_1 = 4.49(14),\ h_2 = 21.2(4.3),\ h_3 = 7.1(1.7),$$
$$h_4 = 76.4(3.4),\ h_5 = 47.3(5.8)$$

Adler zero:

$$s_A = 1.34(10) M_\pi^2$$

# Momentum dependence



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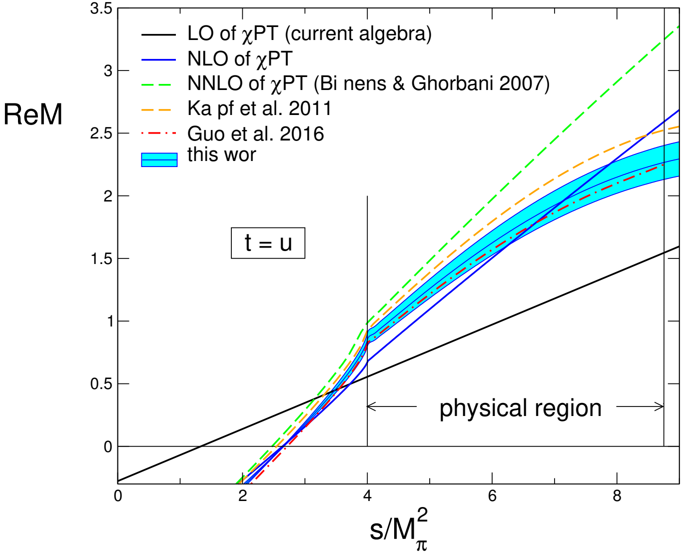
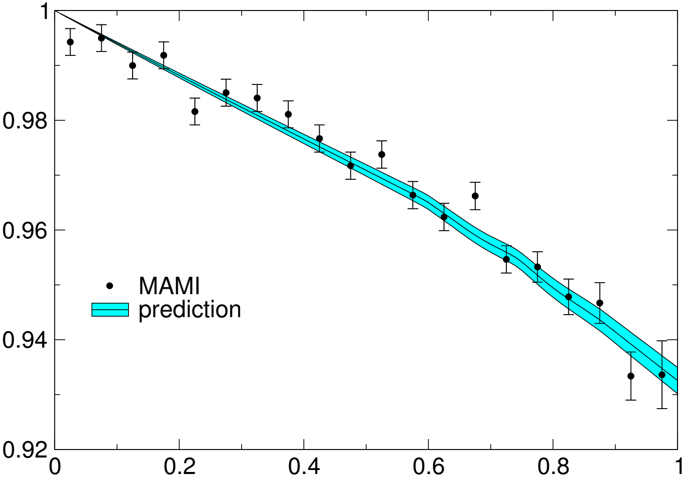
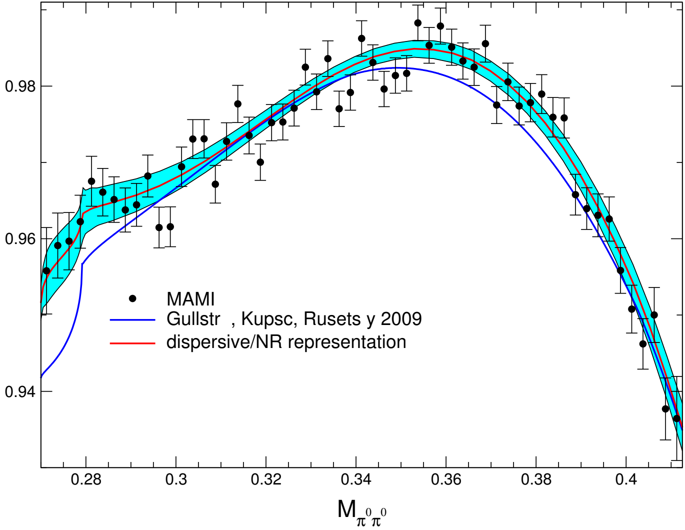


# Dalitz plot in the neutral channel

Having fixed the subtraction constants, the Dalitz plot in the neutral channel can be predicted: $\chi^2 = 22.5$

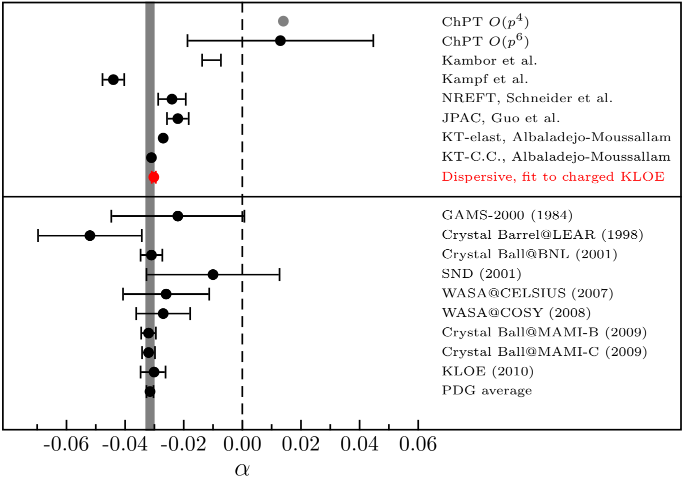


# Dalitz plot in the neutral channel



# Dalitz plot in the neutral channel: value of $\alpha$

Comparison with other determinations:



# Determination of $Q$

$H_0^{\rm NLO} = 1.176(53)$ and $\Gamma(\eta \to \pi^+\pi^-\pi^0) = (300 \pm 12)$ eV yield:

$$(\hat{M}^2_{K^0} - \hat{M}^2_{K^+}) = 6.27(38) 10^{-3}\,\mathrm{GeV}^2$$

which implies:

$$(M^2_{K^0} - M^2_{K^+})_{\rm QED} = -2.38(38) 10^{-3}\,\mathrm{GeV}^2$$

This corresponds to

$$\epsilon = 0.9(3)$$

in agreement with recent lattice determinations:

$$\epsilon = \begin{cases} 0.74(18) & \mathrm{BMW} \\ 0.73(14) & \mathrm{MILC} \\ 0.50(6) & \mathrm{QCDSF/UKQCD} \\ 0.801(110) & \mathrm{RM123} \end{cases}$$

BMW (16), MILC (16) somewhat lower than recent lattice determinations

# Determination of $Q$

$H_0^{\mathrm{NLO}} = 1.176(53)$ and $\Gamma(\eta \to \pi^+\pi^-\pi^0) = (300 \pm 12)$ eV yield:

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which implies:   (upon use of)

$$(\hat{M}^2_{K^0} - \hat{M}^2_{K^+}) = \frac{M_K^2(M_K^2 - M_\pi^2)}{Q^2 M_\pi^2}\left(1 + \mathcal{O}(m^2)\right)$$

$$Q = 22.0(7)$$

somewhat lower than recent lattice determinations

$$Q = \begin{cases} 23.40(64) & \mathrm{BMW} \\ 23.8(1.1) & \mathrm{RM123} \end{cases}$$

Unexpectedly large $\mathcal{O}(m^2)$ effects?

# Ratio of decay rates

The ratio of decay rates for the two channels can also be calculated and with remarkable accuracy Gasser-Leutwyler (85)

The normalization $H_0$ also drops out in this ratio

As it turns out most uncertainties cancel out, giving:

$$B \equiv \frac{\Gamma(\eta \to 3\pi^0)}{\Gamma(\eta \to \pi^+\pi^-\pi^0)} = 1.44(4)$$

which agrees perfectly with the measured value

$$B_{\rm PDG}(\text{our fit}) = 1.426(26), \qquad B_{\rm PDG}(\text{our average}) = 1.48(5)$$

# Outline

$\pi\pi$ scattering beyond LO

Experimental tests

$\eta \to 3\pi$ and $Q \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$

How to determine $m_u - m_d$

Summary

# Summary

As an illustration of the effective field theory method I have discussed a two applications:

- the analysis of the $\pi\pi$ scattering amplitude beyond one loop
- the determination of $Q$ from $\eta \to 3\pi$