



Full-color decompositions for loop amplitudes in Yang-Mills

based on work with Ben PAGE
arXiv:1612.04366 [hep-ph]

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Invitation

- ▶ HEP experiment wants cross-sections
- ▶ QFT analytics gives S-matrix (scattering amplitudes ♡)
- ▶ Amplitudes — one of many ingredients in HEP calculations
- ▶ On-shell techniques: simple tree amplitudes
- ▶ Unitarity methods: trees recycled into loops

Berends, Kleiss, De Causmaecker, Gastmans, Wu, Parke, Taylor, Xu, Zhang, Chang etc.

Bern, Dixon, Dunbar, Kosower (1994)

Britto, Cachazo, Feng (2004)

This talk is about

managing adjoint color at loop level

- ▶ systematically
- ▶ including nonplanar information
- ▶ consistently with unitarity methods

Badger, Mogull, AO, O'Connell (2015)
AO, Page (2016)

This talk: example-based

[1612.04366](#): general treatment

Ongoing work: extend to fundamental representation

also Johansson, AO (2015)

Reduction to traces

Any Feynman diagram $\Rightarrow \sum N_c^\# \text{Tr}(T^{a_1} \dots T^{a_\#}) \dots \text{Tr}(T^{a_\#} \dots T^{a_\#})$

review by Dixon (1996)

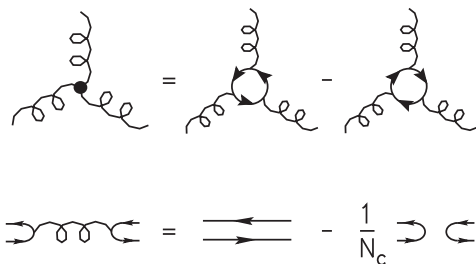


Figure 1: Diagrammatic equations for simplifying $SU(N_c)$ color algebra. Curly lines (“gluon propagators”) represent adjoint indices, oriented solid lines (“quark propagators”) represent fundamental indices, and “quark-gluon vertices” represent the generator matrices $(T^a)_i^{\bar{j}}$.

$$\tilde{f}^{abc} = \text{Tr}(T^a T^b T^c) - \text{Tr}(T^c T^b T^a)$$

$$T_{i\bar{j}}^a T_{k\bar{l}}^a = \delta_{i\bar{l}} \delta_{k\bar{j}} - \frac{1}{N_c} \delta_{i\bar{j}} \delta_{k\bar{l}}$$

Reduction to leading traces

Any Feynman diagram $\Rightarrow \sum N_c^{\#} \text{Tr}(T^{a_1} \dots T^{a_{\#}}) \dots \text{Tr}(T^{a_{\#}} \dots T^{a_{\#}})$

review by Dixon (1996)

The image shows two identities for Feynman diagrams involving gluon lines (represented by curly lines).
The first identity shows a vertex with a gluon line entering from the left and a gluon line exiting to the right, with a gluon loop attached to the vertex. This is equal to the difference of two diagrams: one with a gluon loop on the incoming line and one with a gluon loop on the outgoing line.
The second identity shows a gluon line with a loop (a bubble) on it. This is equal to the sum of two diagrams: one with a straight gluon line and one with a gluon line that has a loop on it, with a factor of $1/N_c$ in front of the second diagram.

Maximal power of $N_c = \text{Tr} 1$ in planar diagrams:

$$\mathcal{A}_n^{(L)} = \left\{ N_c^L \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n)}}) A(\sigma(1), \sigma(2), \dots, \sigma(n)) + O(N_c^{L-1}) \right\}$$

Trace-based tree decomposition

At tree level, leading color = full color:

$$\mathcal{A}_n^{(0)} = \sum_{\sigma \in S_n/D_n} T(\sigma(1), \sigma(2), \dots, \sigma(n)) A(\sigma(1), \sigma(2), \dots, \sigma(n))$$

sum is over $(n-1)!/2$ noncyclic
reflection-inequivalent permutations

$$T(1, 2, \dots, n) \equiv \text{Tr}(T^{a_1} T^{a_2} \dots T^{a_n}) + (-1)^n \text{Tr}(T^{a_n} \dots T^{a_2} T^{a_1})$$

$$2 \begin{array}{c} \dots \\ \diagup \quad \diagdown \\ 1 \quad n \\ \diagdown \quad \diagup \\ n-1 \end{array} = 2 \begin{array}{c} \dots \\ \curvearrowright \\ 1 \quad n \\ \curvearrowleft \end{array} + (-1)^n \begin{array}{c} \dots \\ \curvearrowright \\ n \quad 1 \\ \curvearrowleft \end{array}$$

Kleiss-Kuijf relations

Color ordering $\Rightarrow (n-1)!$ gluonic color-ordered amplitudes:

$$\mathcal{A}_n^{(0)} = \sum_{\sigma \in S_{n-1}(\{2, \dots, n\})} \text{Tr}(T^{a_1} T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n)}}) A(1, \sigma(2), \dots, \sigma(n))$$

KK relations:

Kleiss, Kuijf (1988)

$$A(1, \beta, 2, \alpha) = (-1)^{|\beta|} \sum_{\sigma \in \alpha \sqcup \beta^T} A(1, 2, \sigma)$$

\Rightarrow KK basis of $(n-2)!$ color-ordered amplitudes:

$$\{A(1, 2, \sigma) \mid \sigma \in S_{n-2}(\{3, \dots, n\})\}$$

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$$\{A(1, 2, \sigma) \mid \sigma \in S_{n-2}(\{3, \dots, n\})\}$$

\Rightarrow DDM decomposition:

Del Duca, Dixon, Maltoni (1999)

$$\mathcal{A}_n^{(0)} = \sum_{\sigma \in S_{n-2}(\{3, \dots, n\})} \tilde{f}^{a_2 a_{\sigma(3)} b_1} \tilde{f}^{b_1 a_{\sigma(4)} b_2} \dots \tilde{f}^{b_{n-3} a_{\sigma(n)} a_1} \\ \times A(1, 2, \sigma(3), \dots, \sigma(n))$$

DDM tree decomposition

Del Duca, Dixon, Maltoni (1999)

$$\begin{aligned} \mathcal{A}_n^{(0)} &= \sum_{\sigma \in S_{n-2}} \tilde{f}^{a_1 a_{\sigma(2)} b_1} \tilde{f}^{b_1 a_{\sigma(3)} b_2} \dots \tilde{f}^{b_{n-4} a_{\sigma(n-2)} b_{n-3}} \tilde{f}^{b_{n-3} a_{\sigma(n-1)} a_n} \\ &\quad \times A(1, \sigma(2), \dots, \sigma(n-1), n) \\ &= \sum_{\sigma \in S_{n-2}} C \left(\begin{array}{c} \sigma(2) \sigma(3) \quad \dots \quad \sigma(n-1) \\ | \quad | \quad | \quad \dots \quad | \quad | \\ \hline 1 \quad \quad \quad \quad \quad \quad \quad n \end{array} \right) A(1, \sigma(2), \dots, \sigma(n-1), n) \end{aligned}$$

DDM tree decomposition

Del Duca, Dixon, Maltoni (1999)

$$\begin{aligned}
 \mathcal{A}_n^{(0)} &= \sum_{\sigma \in S_{n-2}} \tilde{f}^{a_1 a_{\sigma(2)} b_1} \tilde{f}^{b_1 a_{\sigma(3)} b_2} \dots \tilde{f}^{b_{n-4} a_{\sigma(n-2)} b_{n-3}} \tilde{f}^{b_{n-3} a_{\sigma(n-1)} a_n} \\
 &\quad \times A(1, \sigma(2), \dots, \sigma(n-1), n) \\
 &= \sum_{\sigma \in S_{n-2}} C \left(\begin{array}{c} \sigma(2) \sigma(3) \quad \dots \quad \sigma(n-1) \\ | \quad | \quad | \quad \dots \quad | \quad | \\ \hline 1 \quad \quad \quad \quad \quad \quad \quad n \end{array} \right) A(1, \sigma(2), \dots, \sigma(n-1), n)
 \end{aligned}$$

“DDM stretch”

$$\begin{aligned}
 \sum_{\sigma \in S_n / D_n} \sigma(2) \begin{array}{c} \dots \\ \diagup \quad \diagdown \\ \sigma(1) \quad \sigma(n-1) \\ \diagdown \quad \diagup \\ \sigma(n) \end{array} &\longrightarrow \sum_{\sigma \in S_{n-2}} \leftarrow 1 \begin{array}{c} \sigma(2) \quad \sigma(3) \quad \dots \quad \sigma(n-1) \\ | \quad | \quad | \quad \dots \quad | \quad | \\ \hline \quad \quad \quad \quad \quad \quad \quad n \end{array} \rightarrow \\
 (n-1)!/2 &\quad \quad \quad (n-2)!
 \end{aligned}$$

Done with trees ✓

$$\sum_{\sigma \in S_n / D_n} \begin{array}{c} \dots \\ \sigma(2) \text{---} \text{---} \text{---} \sigma(n-1) \\ \diagup \quad \diagdown \\ \sigma(1) \quad \sigma(n) \end{array} \longrightarrow \sum_{\sigma \in S_{n-2}} \begin{array}{c} \sigma(2) \quad \sigma(3) \quad \dots \quad \sigma(n-1) \\ \leftarrow 1 \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} n \rightarrow \\ \uparrow \quad \uparrow \quad \uparrow \quad \dots \quad \uparrow \end{array}$$

cyclic-symmetric color-minimal

From trees to cut loops

Generic colored cut:

$$\begin{aligned}\mathcal{C}ut_{\mathcal{I}} &\equiv \prod_{j=1}^v \mathcal{A}^{(0)}(\tau_j), & \mathcal{I} &= \{\tau_1, \dots, \tau_v\} \\ &= \sum_{\substack{\sigma_1 \in S_{|\tau_1|}/D_{|\tau_1|} \\ \dots \\ \sigma_v \in S_{|\tau_v|}/D_{|\tau_v|}}} \prod_{j=1}^v T(\sigma_j(\tau_j))A(\sigma_j(\tau_j))\end{aligned}$$

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Generic color-ordered cut:

$$\text{Cut}_i \equiv \prod_{j=1}^v A(\sigma_j), \quad i = \{\sigma_1, \dots, \sigma_v\}$$

Its natural trace-based color factor:

$$T_i \equiv \prod_{j=1}^v T(\sigma_j), \quad i = \{\sigma_1, \dots, \sigma_v\}$$

From trees to loops

Bern, Dixon, Dunbar, Kosower (1994)

Britto, Cachazo, Feng (2004)

Ossola, Papadopoulos, Pittau (2006)

Mastrolia, Mirabella, Ossola, Peraro; Badger, Frellesvig, Zhang (2012)

Unitarity method (integrand reduction version):

bijection between each cut and its **irreducible numerator**:

$$\text{Cut}_{\mathcal{I}} = \prod_{j=1}^v \mathcal{A}^{(0)}(\tau_j) \quad \Leftrightarrow \quad \tilde{\Delta}_{\mathcal{I}}$$

$$\text{Cut}_i = \prod_{j=1}^v A(\sigma_j) \quad \Leftrightarrow \quad \Delta_i$$

such that

$$\mathcal{A}^{(L)}|_{\mathcal{I}} = \text{Cut}_{\mathcal{I}} \quad \Leftrightarrow \quad \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \frac{\Delta_i}{\prod_{k \in i} D_k} \in \mathcal{A}^{(L)}.$$

From trees to loops

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Puzzle: once Δ_i are computed, how to combine them into $\mathcal{A}^{(L)}$?

Trace-based loop decomposition

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Solution (trivial):

$$\mathcal{A}_n^{(L)} = \sum_{\mathcal{I} \in \text{1PI graphs}} \int \frac{d^{LD} \ell}{(2\pi)^{LD}} \frac{\tilde{\Delta}_{\mathcal{I}}}{S_{\mathcal{I}} \prod_{l \in \mathcal{I}} (-iD_l)},$$
$$\tilde{\Delta}_{\mathcal{I}} = \sum_{i = \left\{ \begin{array}{l} \sigma_1 \in S_{|\tau_1|} / D_{|\tau_1|} \\ \vdots \\ \sigma_v \in S_{|\tau_v|} / D_{|\tau_v|} \end{array} \right\}} T_i \Delta_i, \quad T_i = \prod_{j=1}^v T(\sigma_j)$$

subtleties discussed in AO, Page (2016)

Loop-level KK relations

Badger, Mogull, AO, O'Connell (2015)

AO, Page (2016)

Question: do irred. numerators Δ_i satisfy extra relations?

Answer: **yes**, they inherit KK relations from cuts.

Loop-level KK relations

Badger, Mogull, AO, O'Connell (2015)

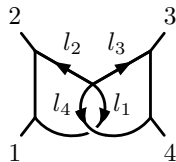
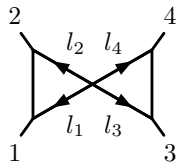
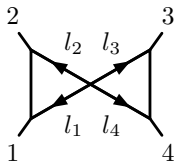
AO, Page (2016)

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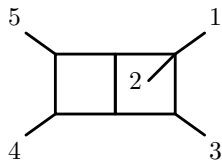
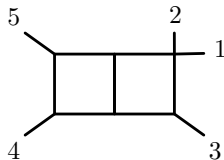
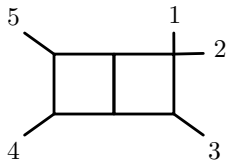
Answer: **yes**, they inherit KK relations from cuts.

$$A(1, 2, 3, 4) + A(1, 2, 4, 3) + A(1, 4, 2, 3) = 0$$

4 points, 2 loops:



5 points, 2 loops:



2-loop example in detail

$$\begin{aligned}
 \tilde{\Delta} \left(\begin{array}{c} 4 \\ \ell_2 \quad \ell_1 \\ \text{---} \times \text{---} \\ \text{---} \times \text{---} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) &= C \left(\begin{array}{c} 4 \\ \text{---} \times \text{---} \\ \text{---} \times \text{---} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) \Delta \left(\begin{array}{c} 4 \\ \ell_2 \quad \ell_1 \\ \text{---} \times \text{---} \\ \text{---} \times \text{---} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) \\
 &+ C \left(\begin{array}{c} 3 \\ \text{---} \times \text{---} \\ \text{---} \times \text{---} \\ 4 \qquad \qquad 2 \\ 1 \end{array} \right) \Delta \left(\begin{array}{c} 3 \\ \ell_1 \\ \text{---} \times \text{---} \\ \text{---} \times \text{---} \\ 4 \quad \ell_2 \qquad 2 \\ 1 \end{array} \right) \\
 &+ C \left(\begin{array}{c} 4 \\ \text{---} \cup \text{---} \\ \text{---} \cup \text{---} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) \Delta \left(\begin{array}{c} 4 \\ \ell_2 \quad \ell_1 \\ \text{---} \cup \text{---} \\ \text{---} \cup \text{---} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) \\
 &= \left\{ C \left(\begin{array}{c} 4 \\ \text{---} \times \text{---} \\ \text{---} \times \text{---} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) - C \left(\begin{array}{c} 4 \\ \text{---} \cup \text{---} \\ \text{---} \cup \text{---} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) \right\} \Delta \left(\begin{array}{c} 4 \\ \ell_2 \quad \ell_1 \\ \text{---} \times \text{---} \\ \text{---} \times \text{---} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) \\
 &+ \left\{ C \left(\begin{array}{c} 3 \\ \text{---} \times \text{---} \\ \text{---} \times \text{---} \\ 4 \qquad \qquad 2 \\ 1 \end{array} \right) - C \left(\begin{array}{c} 4 \\ \text{---} \cup \text{---} \\ \text{---} \cup \text{---} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) \right\} \Delta \left(\begin{array}{c} 3 \\ \ell_1 \\ \text{---} \times \text{---} \\ \text{---} \times \text{---} \\ 4 \quad \ell_2 \qquad 2 \\ 1 \end{array} \right) \\
 &= C \left(\begin{array}{c} 4 \\ \text{---} \parallel \text{---} \\ \text{---} \parallel \text{---} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) \Delta \left(\begin{array}{c} 4 \\ \ell_2 \quad \ell_1 \\ \text{---} \times \text{---} \\ \text{---} \times \text{---} \\ 3 \qquad \qquad 2 \\ 1 \end{array} \right) + C \left(\begin{array}{c} 3 \\ \text{---} \parallel \text{---} \\ \text{---} \parallel \text{---} \\ 4 \qquad \qquad 2 \\ 1 \end{array} \right) \Delta \left(\begin{array}{c} 3 \\ \ell_1 \\ \text{---} \times \text{---} \\ \text{---} \times \text{---} \\ 4 \quad \ell_2 \qquad 2 \\ 1 \end{array} \right)
 \end{aligned}$$

DDM-based loop decomposition

$$\begin{aligned}
 \mathcal{A}_n^{(L)} &= \sum_{\mathcal{I} \in \text{1PI graphs}} \int \frac{d^{LD} \ell}{(2\pi)^{LD}} \frac{\tilde{\Delta}_{\mathcal{I}}}{S_{\mathcal{I}} \prod_{l \in \mathcal{I}} D_l} \\
 &= \sum_{i \in \text{KK-independent 1PI graphs}} \int \frac{d^{LD} \ell}{(2\pi)^{LD}} \frac{C_i \Delta_i}{S_{\mathcal{I}} \prod_{l \in i} D_l},
 \end{aligned}$$

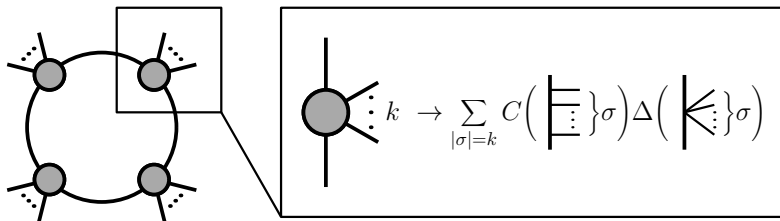
$$\sum_{\sigma \in S_n / D_n} \sigma(2) \begin{array}{c} \cdots \\ \diagup \quad \diagdown \\ \sigma(1) \quad \sigma(n) \end{array} \sigma(n-1) \quad \longrightarrow \quad \sum_{\sigma \in S_{n-2}} \leftarrow 1 \begin{array}{c} \sigma(2) \quad \sigma(3) \quad \cdots \quad \sigma(n-1) \\ | \quad | \quad | \quad \cdots \quad | \\ \hline n \rightarrow \end{array}$$

cyclic-symmetric color-minimal

DDM-based 1-loop decomposition

Del Duca, Dixon, Maltoni (1999)

$$\begin{aligned}
 \mathcal{A}_n^{(1)} &= \sum_{\sigma \in S_n/D_n} \tilde{f}^{b_1 a_{\sigma(1)} b_2} \tilde{f}^{b_2 a_{\sigma(2)} b_3} \dots \tilde{f}^{b_n a_{\sigma(n)} b_1} A^{(1)}(\sigma(1), \sigma(2), \dots, \sigma(n)) \\
 &= \sum_{\sigma \in S_n/D_n} C \left(\begin{array}{c} \dots \\ \sigma(2) \text{---} \bigcirc \text{---} \sigma(n-1) \\ \sigma(1) \quad \sigma(n) \end{array} \right) A^{(1)}(\sigma(1), \sigma(2), \dots, \sigma(n))
 \end{aligned}$$

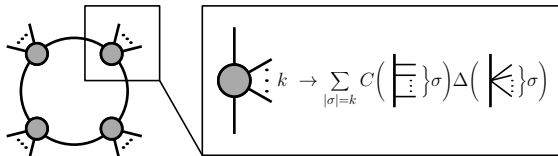


DDM-based 1-loop decomposition

Del Duca, Dixon, Maltoni (1999)

1-loop KK relations by Bern, Kosower (1990)

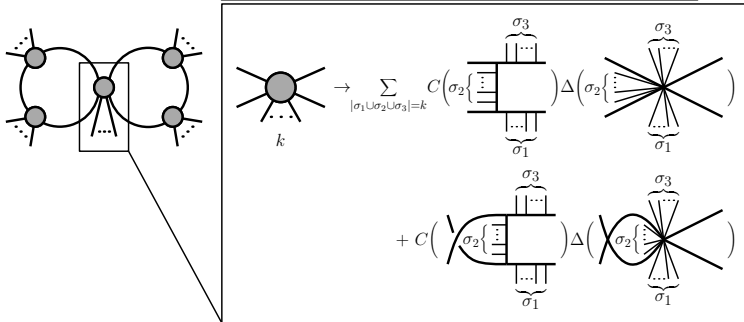
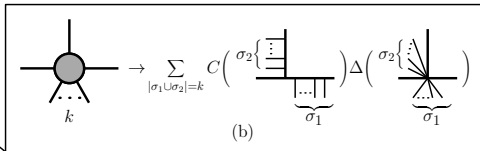
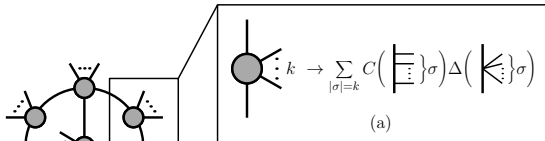
$$\mathcal{A}_n^{(1)} = \sum_{\sigma \in S_n/D_n} C \left(\begin{array}{c} \cdots \\ \sigma(2) \text{ --- } \bigcirc \text{ --- } \sigma(n-1) \\ \sigma(1) \text{ --- } \bigcirc \text{ --- } \sigma(n) \end{array} \right) A^{(1)}(\sigma(1), \sigma(2), \dots, \sigma(n)),$$



$$\begin{aligned} A^{(1)}(1, 2, \dots, n) = I & \left[\sum_{\substack{1 \leq i_1 < i_2 < i_3 \\ < i_4 < i_5 \leq n}} \Delta \left(\begin{array}{c} i_5 \quad i_1-1 \\ \swarrow \quad \searrow \\ i_4 \quad i_2-1 \\ \swarrow \quad \searrow \\ i_3 \quad i_2 \\ \swarrow \quad \searrow \\ i_3-1 \end{array} \right) + \sum_{\substack{1 \leq i_1 < i_2 \\ < i_3 < i_4 \leq n}} \Delta \left(\begin{array}{c} i_1-1 \quad i_1 \\ \swarrow \quad \searrow \\ i_4 \quad i_2-1 \\ \swarrow \quad \searrow \\ i_3 \quad i_3-1 \\ \swarrow \quad \searrow \\ i_4-1 \end{array} \right) \\ + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} \Delta \left(\begin{array}{c} i_3 \quad i_1-1 \\ \swarrow \quad \searrow \\ i_3-1 \quad i_2 \\ \swarrow \quad \searrow \\ i_2 \quad i_2-1 \end{array} \right) + \sum_{1 \leq i_1 < i_2 \leq n} \Delta \left(\begin{array}{c} i_1-1 \quad i_1 \\ \swarrow \quad \searrow \\ i_2 \quad i_2-1 \\ \swarrow \quad \searrow \\ i_2 \quad i_2-1 \end{array} \right) + \sum_{1 \leq i_1 \leq n} \Delta \left(\begin{array}{c} i_1 \\ \swarrow \quad \searrow \\ i_1-1 \end{array} \right) \end{aligned}$$

DDM stretches at 2 loops

Badger, Mogull, AO, O'Connell (2015)



DDM-based 2-loop decomposition

Badger, Mogull, AO, O'Connell (2015)

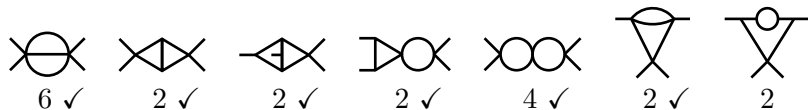
AO, Page (2016)

$$\begin{aligned}
 \mathcal{A}_5^{(2)} = & \sum_{\sigma \in S_5} \sigma \circ I \left[C \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \left\{ \frac{1}{2} \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \frac{1}{2} \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 3 \\ 2 \end{array} \right) + \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 3 \\ 2 \end{array} \right) \right. \\
 & \left. + \frac{1}{2} \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \frac{1}{2} \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \frac{1}{2} \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \frac{1}{2} \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \right\} \\
 & + C \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \left\{ \frac{1}{2} \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \frac{1}{2} \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \right\} \\
 & + \frac{1}{2} C \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \left\{ \frac{1}{2} \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \right\} \\
 & + C \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \left\{ \frac{1}{4} \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \frac{1}{2} \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \frac{1}{2} \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \right\} \\
 & + C \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \left\{ \frac{1}{4} \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \frac{1}{2} \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \right. \\
 & \left. + \frac{1}{2} \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + \Delta \left(\begin{array}{c} 5 \\ 4 \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \right\} + \dots \left. \right]
 \end{aligned}$$

2-loop symmetry factors

AO, Page (2016)

KK-independent basis of color-ordered cuts may be redundant under loop-momentum relabeling, can be further-reducible.



“Stretch” by external legs:

$$\frac{1}{6} \tilde{\Delta} \left(\begin{array}{c} 4 \\ \text{circle with line} \\ 3 \quad 2 \end{array} \right) = C \left(\begin{array}{c} 4 \\ \text{square with lines} \\ 3 \quad 2 \end{array} \right) \Delta \left(\begin{array}{c} 4 \\ \text{circle with line} \\ 3 \quad 2 \end{array} \right) + C \left(\begin{array}{c} 4 \\ \text{square with lines} \\ 3 \quad 2 \end{array} \right) \Delta \left(\begin{array}{c} 4 \\ \text{circle with line} \\ 3 \quad 2 \end{array} \right) \\ + C \left(\begin{array}{c} 4 \\ \text{square with lines} \\ 3 \quad 2 \end{array} \right) \Delta \left(\begin{array}{c} 4 \\ \text{circle with line} \\ 3 \quad 2 \end{array} \right) + \{3 \leftrightarrow 4\}$$

3-loop $\mathcal{N} = 4$ example

Bern, Carrasco, Dixon, Johansson, Kosower, Roiban (2007)

Henn, Mistlberger (2016)

$$\begin{aligned}
 \mathcal{A}_{\mathcal{N}=4}^{(3)} = & \sum_{\sigma \in S_4} \sigma \circ I \left[\frac{1}{8} C \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) \Delta \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) \right. \\
 & + \frac{1}{4} C \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) \Delta \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) + \frac{1}{4} C \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) \Delta \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) \\
 & + \frac{1}{8} C \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) \Delta \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) + \frac{1}{16} C \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) \Delta \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) \\
 & + \frac{1}{2} C \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) \left\{ \Delta \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) + \Delta \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) \right\} \\
 & + \frac{1}{2} C \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) \left\{ \Delta \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) + \Delta \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) \right\} \\
 & + C \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) \left\{ \Delta \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) + \Delta \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) \right\} \\
 & + C \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) \left\{ \frac{1}{2} \Delta \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) + \Delta \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) + \frac{1}{3} \Delta \left(\begin{array}{c} (4) \\ \text{---} \\ (3) \end{array} \begin{array}{c} (1) \\ \text{---} \\ (2) \end{array} \right) \right\} \Big]
 \end{aligned}$$

3-loop $\mathcal{N} = 4$ example's trickiest topology

$$\tilde{\Delta} \left(\begin{array}{c} \ell_3 \quad \ell_1 \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ 2 \\ \text{---} \\ \ell_2 \quad 3 \end{array} \right) = C \left(\begin{array}{c} 4 \quad 1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ 2 \\ \text{---} \\ 3 \end{array} \right) \Delta \left(\begin{array}{c} \ell_3 \quad \ell_1 \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ 2 \\ \text{---} \\ \ell_2 \quad 3 \end{array} \right) \\ + C \left(\begin{array}{c} 4 \quad 1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ 2 \\ \text{---} \\ 3 \end{array} \right) \Delta \left(\begin{array}{c} \ell_3 \quad \ell_1 \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ 2 \\ \text{---} \\ \ell_2 \quad 3 \end{array} \right)$$

$$\mathcal{A}_{\mathcal{N}=4}^{(3)} = \sum_{\sigma \in S_4} \sigma \circ I \left[\frac{1}{8} C \left(\begin{array}{c} 4 \quad 1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ 2 \\ \text{---} \\ 3 \end{array} \right) \Delta \left(\begin{array}{c} \ell_1 \\ \text{---} \\ \ell_2 \quad \ell_3 \\ \text{---} \\ 2 \end{array} \right) + \dots \right. \\ \left. + C \left(\begin{array}{c} 4 \quad 1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ 2 \\ \text{---} \\ 3 \end{array} \right) \left\{ \frac{1}{2} \Delta \left(\begin{array}{c} 4 \quad \ell_3 \quad \ell_1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ 2 \\ \text{---} \\ \ell_2 \quad 3 \end{array} \right) + \Delta \left(\begin{array}{c} 4 \quad \ell_3 \quad 1 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ 2 \\ \text{---} \\ \ell_2 \quad 3 \end{array} \right) + \frac{1}{3} \Delta \left(\begin{array}{c} \ell_3 \quad \ell_1 \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ 2 \\ \text{---} \\ \ell_2 \quad 3 \end{array} \right) \right\} \right]$$

Summary

- ▶ Reviewed tree-level color decompositions
- ▶ Promoted these to loop level within integrand reduction
- ▶ Used loop-level KK relations to reduce decomposition
- ▶ Checked 1-loop consistency with known decomposition

Del Duca, Dixon, Maltoni (1999)

- ▶ 2-loop decomposition used in Badger, Mogull, AO, O'Connell (2015)
- ▶ Most 2-loop symmetry factors are reducible

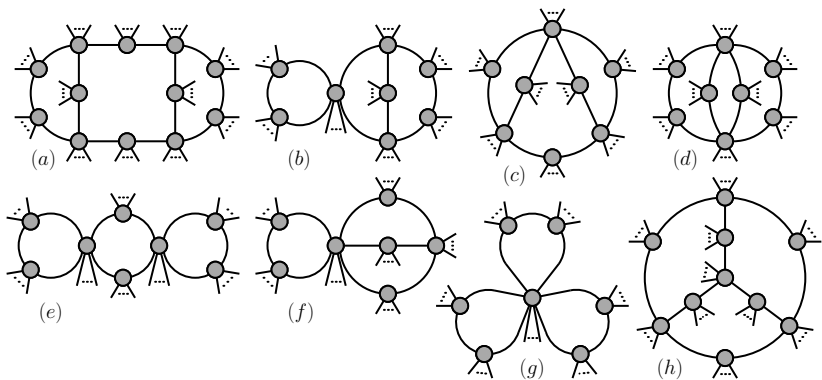
AO, Page (2016)

- ▶ Example of 3-loop decomposition

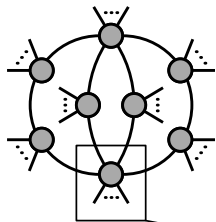
Thank you!

Backup slides

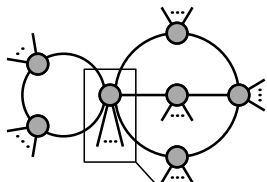
3-loop topologies



3-loop topologies (d) and (f)

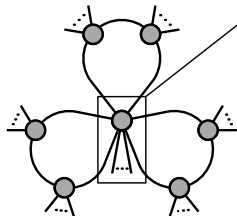


$$\begin{aligned}
 & \text{Diagram } k \rightarrow \sum_{|\sigma_1 \cup \sigma_2 \cup \sigma_3| = k} C \left(\begin{array}{c} \sigma_3 \{ \vdots \} \sigma_1 \\ \ell_2 \quad \underbrace{\quad \quad}_{\sigma_2} \quad \ell_1 \end{array} \right) \Delta \left(\begin{array}{c} \sigma_3 \{ \vdots \} \sigma_1 \\ \ell_2 \quad \underbrace{\quad \quad}_{\sigma_2} \quad \ell_1 \end{array} \right) \\
 & + C \left(\begin{array}{c} \sigma_3 \{ \vdots \} \sigma_1 \\ \underbrace{\quad \quad}_{\sigma_2} \\ \ell_2 \quad \quad \quad \ell_1 \end{array} \right) \Delta \left(\begin{array}{c} \sigma_3 \{ \vdots \} \sigma_1 \\ \underbrace{\quad \quad}_{\sigma_2} \\ \ell_2 \quad \quad \quad \ell_1 \end{array} \right)
 \end{aligned}$$

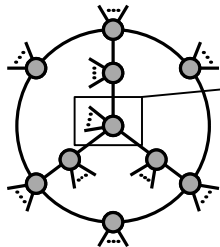


$$\begin{aligned}
 & \text{Diagram } k \rightarrow \sum_{|\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4| = k} C \left(\begin{array}{c} \sigma_1 \\ \vdots \\ \sigma_2 \\ \ell_2 \\ \vdots \\ \sigma_3 \\ \underbrace{\quad \quad}_{\sigma_4} \\ \ell_3 \end{array} \right) \Delta \left(\begin{array}{c} \sigma_1 \\ \vdots \\ \sigma_2 \\ \ell_2 \\ \vdots \\ \sigma_3 \\ \underbrace{\quad \quad}_{\sigma_4} \\ \ell_3 \end{array} \right) \\
 & + C \left(\begin{array}{c} \sigma_1 \\ \vdots \\ \sigma_2 \\ \ell_2 \\ \vdots \\ \sigma_3 \\ \underbrace{\quad \quad}_{\sigma_4} \\ \ell_3 \end{array} \right) \Delta \left(\begin{array}{c} \sigma_1 \\ \vdots \\ \sigma_2 \\ \ell_2 \\ \vdots \\ \sigma_3 \\ \underbrace{\quad \quad}_{\sigma_4} \\ \ell_3 \end{array} \right) + \dots
 \end{aligned}$$

3-loop topologies (g) and (h)



$$\begin{aligned}
 & \text{Diagram } k \rightarrow \sum_{|\sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4 \cup \sigma_5| = k} C \left(\begin{array}{c} \sigma_5 \{ \vdots \} \\ \sigma_4 \{ \vdots \} \\ \vdots \\ \sigma_2 \{ \vdots \} \\ \sigma_1 \{ \vdots \} \\ \underbrace{\quad \quad}_{\sigma_3} \end{array} \right) \Delta \left(\begin{array}{c} \sigma_5 \{ \vdots \} \\ \sigma_4 \{ \vdots \} \\ \vdots \\ \sigma_2 \{ \vdots \} \\ \sigma_1 \{ \vdots \} \\ \underbrace{\quad \quad}_{\sigma_3} \end{array} \right) \\
 & + C \left(\begin{array}{c} \sigma_5 \{ \vdots \} \\ \sigma_4 \{ \vdots \} \\ \vdots \\ \sigma_2 \{ \vdots \} \\ \sigma_1 \{ \vdots \} \\ \underbrace{\quad \quad}_{\sigma_3} \end{array} \right) \Delta \left(\begin{array}{c} \sigma_5 \{ \vdots \} \\ \sigma_4 \{ \vdots \} \\ \vdots \\ \sigma_2 \{ \vdots \} \\ \sigma_1 \{ \vdots \} \\ \underbrace{\quad \quad}_{\sigma_3} \end{array} \right) + \dots
 \end{aligned}$$



$$\text{Diagram } k \rightarrow \sum_{|\sigma_1 \cup \sigma_2| = k} C \left(\begin{array}{c} \sigma_2 \{ \vdots \} \\ \vdots \\ \underbrace{\quad \quad}_{\sigma_1} \end{array} \right) \Delta \left(\begin{array}{c} \sigma_2 \{ \vdots \} \\ \vdots \\ \underbrace{\quad \quad}_{\sigma_1} \end{array} \right)$$