Full-color decompositions for loop amplitudes in Yang-Mills

based on work with Ben Page
arXiv:1612.04366 [hep-ph]

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Invitation

- HEP experiment wants cross-sections
- QFT analytics gives S-matrix (scattering amplitudes ♡)
- Amplitudes — one of many ingredients in HEP calculations
- On-shell techniques: simple tree amplitudes

  Berends, Kleiss, De Causmaecker, Gastmans, Wu, Parke, Taylor, Xu, Zhang, Chang etc.
- Unitarity methods: trees recycled into loops

  Bern, Dixon, Dunbar, Kosower (1994)
  Britto, Cachazo, Feng (2004)

This talk is about

managing adjoint color at loop level

- systematically
- including nonplanar information
- consistently with unitarity methods

Badger, Mogull, AO, O'Connell (2015)
AO, Page (2016)

This talk: example-based
1612.04366: general treatment
Ongoing work: extend to fundamental representation

also Johansson, AO (2015)

Reduction to traces

Any Feynman diagram $\Rightarrow \sum N_c^{\#}\,\mathrm{Tr}(T^{a_1}\ldots T^{a_\#})\ldots\mathrm{Tr}(T^{a_\#}\ldots T^{a_\#})$

review by Dixon (1996)

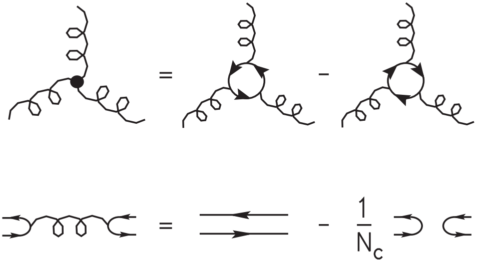


Figure 1: Diagrammatic equations for simplifying $SU(N_c)$ color algebra. Curly lines ("gluon propagators") represent adjoint indices, oriented solid lines ("quark propagators") represent fundamental indices, and "quark-gluon vertices" represent the generator matrices $(T^a)_i^{\ \bar{\jmath}}$.

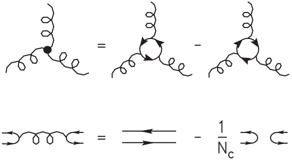
$$\tilde{f}^{abc} = \mathrm{Tr}(T^aT^bT^c) - \mathrm{Tr}(T^cT^bT^a)$$

$$T^a_{i\bar{\jmath}}\,T^a_{k\bar{l}} = \delta_{i\bar{l}}\,\delta_{k\bar{\jmath}} - \frac{1}{N_c}\delta_{i\bar{\jmath}}\,\delta_{k\bar{l}}$$

Reduction to leading traces

Any Feynman diagram $\Rightarrow \sum N_c^{\#}\,\mathrm{Tr}(T^{a_1}\ldots T^{a_\#})\ldots\mathrm{Tr}(T^{a_\#}\ldots T^{a_\#})$

review by Dixon (1996)



Maximal power of $N_c = \mathrm{Tr}\,1$ in planar diagrams:

$$\mathcal{A}_n^{(L)} = \Big\{ N_c^L \sum_{\sigma\in S_n/Z_n} \mathrm{Tr}\big(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}\ldots T^{a_{\sigma(n)}}\big) A(\sigma(1),\sigma(2),\ldots,\sigma(n)) + O(N_c^{L-1}) \Big\}$$

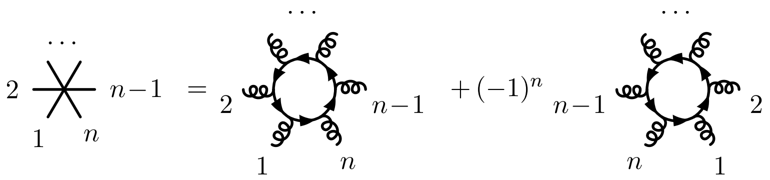
Trace-based tree decomposition

At tree level, leading color = full color:

$$\mathcal{A}_n^{(0)} = \sum_{\sigma \in S_n/D_n} T(\sigma(1), \sigma(2), \ldots, \sigma(n)) A(\sigma(1), \sigma(2), \ldots, \sigma(n))$$

sum is over $(n-1)!/2$ noncyclic
reflection-inequivalent permutations

$$T(1, 2, \ldots, n) \equiv \mathrm{Tr}\big(T^{a_1} T^{a_2} \ldots T^{a_n}\big) + (-1)^n \,\mathrm{Tr}\big(T^{a_n} \ldots T^{a_2} T^{a_1}\big)$$



Kleiss-Kuijf relations

Color ordering $\Rightarrow$ $(n-1)!$ gluonic color-ordered amplitudes:

$$\mathcal{A}_n^{(0)} = \sum_{\sigma \in S_{n-1}(\{2,\ldots,n\})} \mathrm{Tr}\big(T^{a_1} T^{a_{\sigma(2)}} \ldots T^{a_{\sigma(n)}}\big) A(1, \sigma(2), \ldots, \sigma(n))$$

KK relations: Kleiss, Kuijf (1988)

$$A(1, \beta, 2, \alpha) = (-1)^{|\beta|} \sum_{\sigma \in \alpha \sqcup\!\sqcup \beta^T} A(1, 2, \sigma)$$

$\Rightarrow$ KK basis of $(n-2)!$ color-ordered amplitudes:

$$\big\{A(1, 2, \sigma) \;\big|\; \sigma \in S_{n-2}(\{3, \ldots, n\})\big\}$$

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$\Rightarrow$ DDM decomposition: Del Duca, Dixon, Maltoni (1999)

$$\mathcal{A}_n^{(0)} = \sum_{\sigma \in S_{n-2}(\{3,\dots,n\})} \tilde{f}^{\,a_2 a_{\sigma(3)} b_1} \tilde{f}^{\,b_1 a_{\sigma(4)} b_2} \dots \tilde{f}^{\,b_{n-3} a_{\sigma(n)} a_1}$$

$$\times A(1, 2, \sigma(3), \dots, \sigma(n))$$

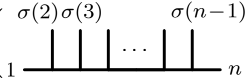
DDM tree decomposition

Del Duca, Dixon, Maltoni (1999)

$$\mathcal{A}_n^{(0)} = \sum_{\sigma\in S_{n-2}} \tilde{f}^{\,a_1 a_{\sigma(2)} b_1} \tilde{f}^{\,b_1 a_{\sigma(3)} b_2} \ldots \tilde{f}^{\,b_{n-4} a_{\sigma(n-2)} b_{n-3}} \tilde{f}^{\,b_{n-3} a_{\sigma(n-1)} a_n}$$

$$\times A(1, \sigma(2), \ldots, \sigma(n-1), n)$$

$$= \sum_{\sigma\in S_{n-2}} C\left(\quad \right) A(1, \sigma(2), \ldots, \sigma(n-1), n)$$



DDM tree decomposition

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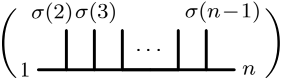
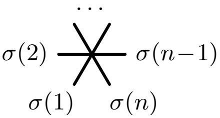
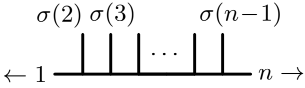


“DDM stretch”

$$\sum_{\sigma\in S_n/D_n}$$



$$\longrightarrow \sum_{\sigma\in S_{n-2}}$$

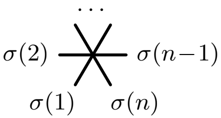


$(n-1)!/2$

$(n-2)!$

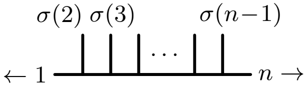
Done with trees ✓

$\sum_{\sigma \in S_n/D_n}$



$\longrightarrow$

$\sum_{\sigma \in S_{n-2}}$



cyclic-symmetric

color-minimal

From trees to cut loops

Generic colored cut:

$$\mathcal{C}\text{ut}_{\mathcal{I}} \equiv \prod_{j=1}^{v} \mathcal{A}^{(0)}(\tau_j), \qquad\qquad \mathcal{I} = \{\tau_1, \ldots, \tau_v\}$$

$$= \sum_{\substack{\sigma_1 \in S_{|\tau_1|}/D_{|\tau_1|} \\ \cdots \\ \sigma_v \in S_{|\tau_v|}/D_{|\tau_v|}}} \prod_{j=1}^{v} T(\sigma_j(\tau_j)) A(\sigma_j(\tau_j))$$

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Generic color-ordered cut:

$$\mathrm{Cut}_i \equiv \prod_{j=1}^{v} A(\sigma_j), \qquad i = \{\sigma_1, \ldots, \sigma_v\}$$

Its natural trace-based color factor:

$$T_i \equiv \prod_{j=1}^{v} T(\sigma_j), \qquad i = \{\sigma_1, \ldots, \sigma_v\}$$

From trees to loops

Bern, Dixon, Dunbar, Kosower (1994)
Britto, Cachazo, Feng (2004)
Ossola, Papadopoulos, Pittau (2006)
Mastrolia, Mirabella, Ossola, Peraro; Badger, Frellesvig, Zhang (2012)

Unitarity method (integrand reduction version):
bijection between each cut and its **irreducible numerator**:

$$\mathcal{C}\mathrm{ut}_{\mathcal{I}} = \prod_{j=1}^{v} \mathcal{A}^{(0)}(\tau_j) \qquad \Leftrightarrow \qquad \tilde{\Delta}_{\mathcal{I}}$$

$$\mathrm{Cut}_i = \prod_{j=1}^{v} A(\sigma_j) \qquad \Leftrightarrow \qquad \Delta_i$$

such that

$$\mathcal{A}^{(L)}\big|_{\mathcal{I}} = \mathcal{C}\mathrm{ut}_{\mathcal{I}} \quad \Leftrightarrow \quad \int \frac{\mathrm{d}^{LD}\ell}{(2\pi)^{LD}} \frac{\Delta_i}{\prod_{k\in i} D_k} \in \mathcal{A}^{(L)}.$$

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Puzzle: once Δ_i are computed, how to combine them into $\mathcal{A}^{(L)}$?

Trace-based loop decomposition

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Solution (trivial):

$$\mathcal{A}_n^{(L)} = \sum_{\mathcal{I}\,\in\,1\text{PI graphs}} \int \frac{\mathrm{d}^{LD}\ell}{(2\pi)^{LD}} \frac{\tilde{\Delta}_{\mathcal{I}}}{S_{\mathcal{I}} \prod_{l\in\mathcal{I}}(-iD_l)},$$

$$\tilde{\Delta}_{\mathcal{I}} = \sum_{i=\left\{\substack{\sigma_1\in S_{|\tau_1|}/D_{|\tau_1|}\\ \cdots \\ \sigma_v\in S_{|\tau_v|}/D_{|\tau_v|}}\right\}} T_i\,\Delta_i, \qquad T_i = \prod_{j=1}^{v} T(\sigma_j)$$

subtleties discussed in AO, Page (2016)

Loop-level KK relations

Badger, Mogull, AO, O'Connell (2015)

AO, Page (2016)

Question: do irred. numerators Δ_i satisfy extra relations?
Answer: **yes**, they inherit KK relations from cuts.

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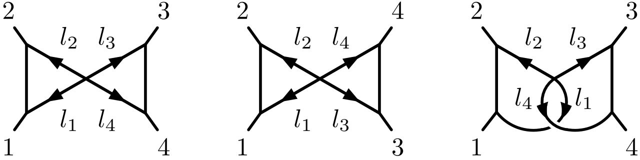
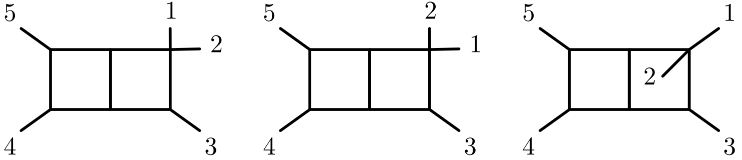
Answer: **yes**, they inherit KK relations from cuts.

$$A(1,2,3,4) + A(1,2,4,3) + A(1,4,2,3) = 0$$

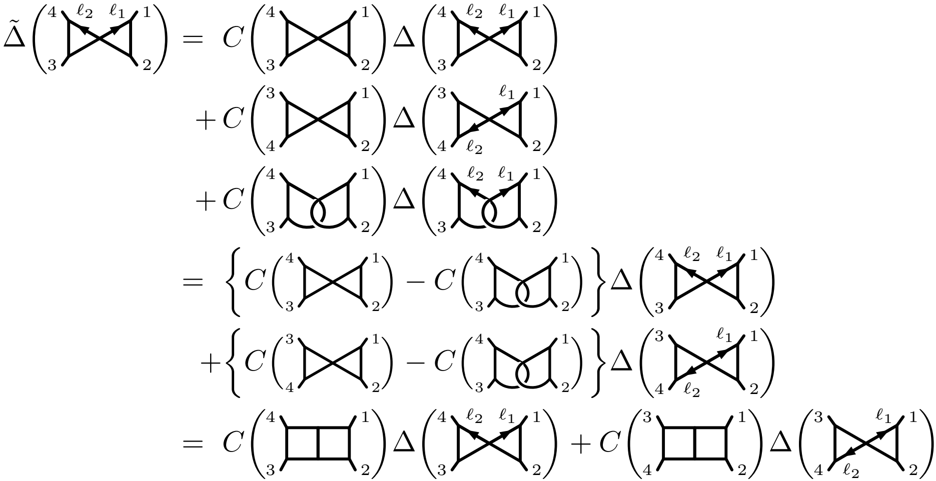
4 points, 2 loops:



5 points, 2 loops:

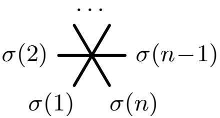


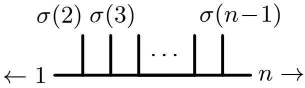
2-loop example in detail



DDM-based loop decomposition

$$\mathcal{A}_n^{(L)} = \sum_{\mathcal{I}\,\in\,\text{1PI graphs}} \int \frac{\mathrm{d}^{LD}\ell}{(2\pi)^{LD}} \frac{\tilde{\Delta}_{\mathcal{I}}}{S_{\mathcal{I}} \prod_{l\in\mathcal{I}} D_l}$$
$$= \sum_{i\,\in\,\text{KK-independent 1PI graphs}} \int \frac{\mathrm{d}^{LD}\ell}{(2\pi)^{LD}} \frac{C_i\,\Delta_i}{S_{\mathcal{I}} \prod_{l\in i} D_l},$$

$$\sum_{\sigma\in S_n/D_n}$$


$\longrightarrow$
$$\sum_{\sigma\in S_{n-2}}$$


cyclic-symmetric

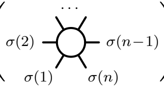
color-minimal

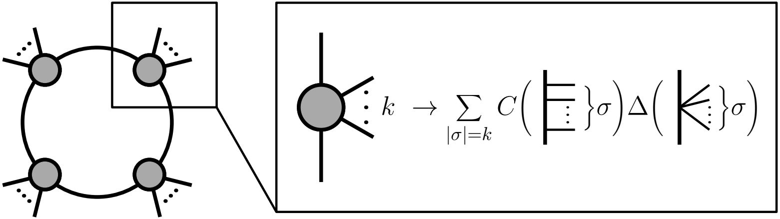
DDM-based 1-loop decomposition

Del Duca, Dixon, Maltoni (1999)

$$\mathcal{A}_n^{(1)} = \sum_{\sigma \in S_n/D_n} \tilde{f}^{\,b_1 a_{\sigma(1)} b_2}\, \tilde{f}^{\,b_2 a_{\sigma(2)} b_3} \ldots \tilde{f}^{\,b_n a_{\sigma(n)} b_1}\, A^{(1)}(\sigma(1), \sigma(2), \ldots, \sigma(n))$$

$$= \sum_{\sigma \in S_n/D_n} C\left(\cdots \right) A^{(1)}(\sigma(1), \sigma(2), \ldots, \sigma(n))$$

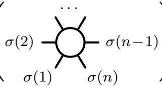




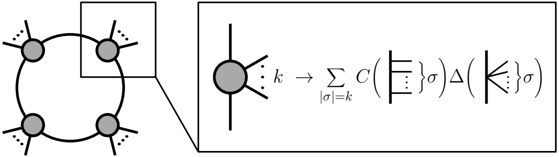
DDM-based 1-loop decomposition

Del Duca, Dixon, Maltoni (1999)
1-loop KK relations by Bern, Kosower (1990)

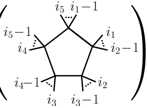
$$\mathcal{A}_n^{(1)} = \sum_{\sigma\in S_n/D_n} C\left(\vphantom{\sum}\right.$$



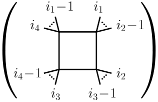
$$\left.\vphantom{\sum}\right) A^{(1)}(\sigma(1),\sigma(2),\ldots,\sigma(n)),$$



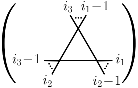
$$A^{(1)}(1,2,\ldots,n) = I\Bigg[\sum_{\substack{1\le i_1<i_2<i_3\\<i_4<i_5\le n}} \Delta\Big($$



$$\Big) + \sum_{\substack{1\le i_1<i_2\\<i_3<i_4\le n}} \Delta\Big($$



$$\Big) + \sum_{1\le i_1<i_2<i_3\le n} \Delta\Big($$



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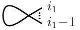
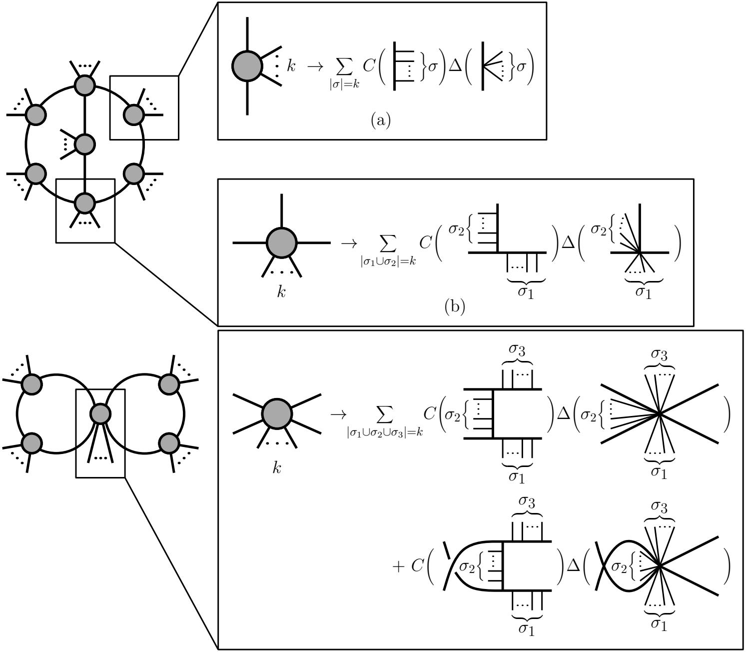


$$\Big)\Bigg]$$

DDM stretches at 2 loops

Badger, Mogull, AO, O'Connell (2015)

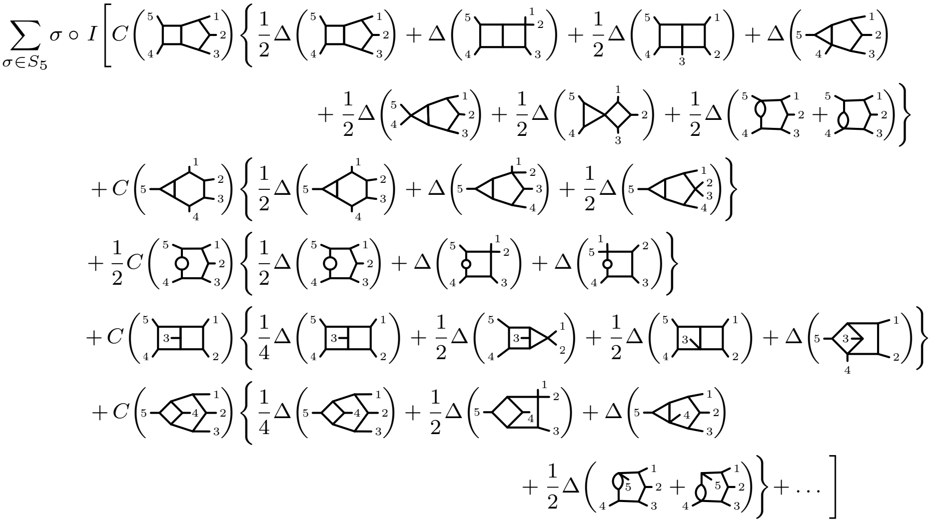


DDM-based 2-loop decomposition

Badger, Mogull, AO, O'Connell (2015)

AO, Page (2016)

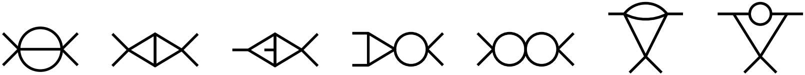
$\mathcal{A}_5^{(2)} =$



2-loop symmetry factors

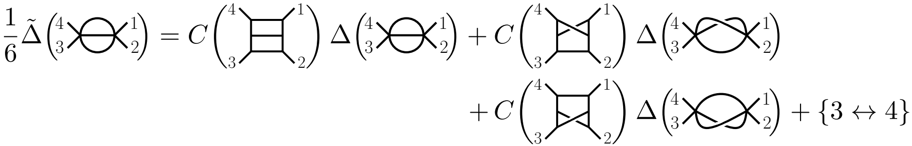
AO, Page (2016)

KK-independent basis of color-ordered cuts may be redundant under loop-momentum relabeling, can be further-reducible.



6 ✓ 2 ✓ 2 ✓ 2 ✓ 4 ✓ 2 ✓ 2

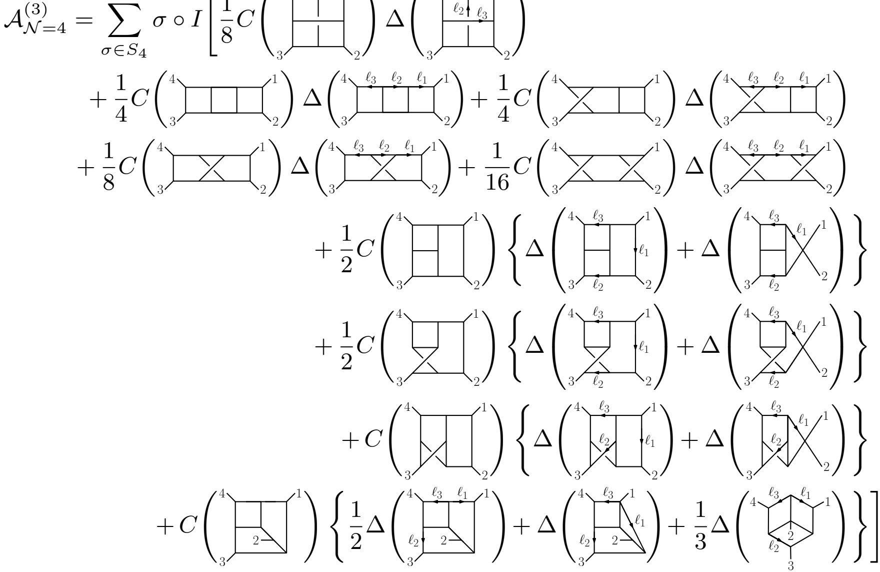
“Stretch” by external legs:



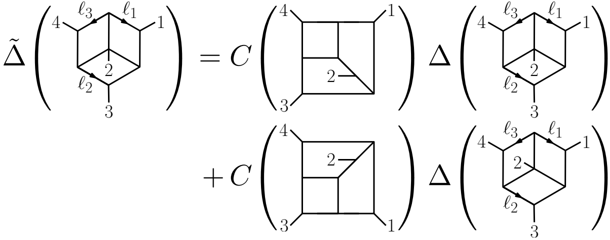
3-loop $\mathcal{N} = 4$ example

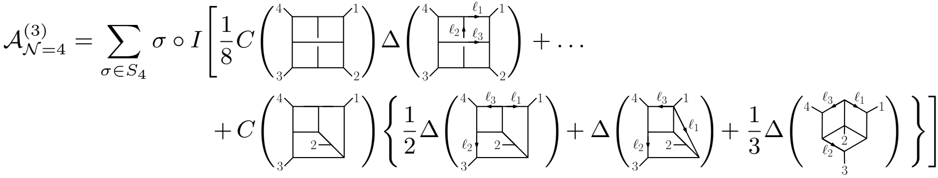
Bern, Carrasco, Dixon, Johansson, Kosower, Roiban (2007)

Henn, Mistlberger (2016)



3-loop $\mathcal{N} = 4$ example's trickiest topology





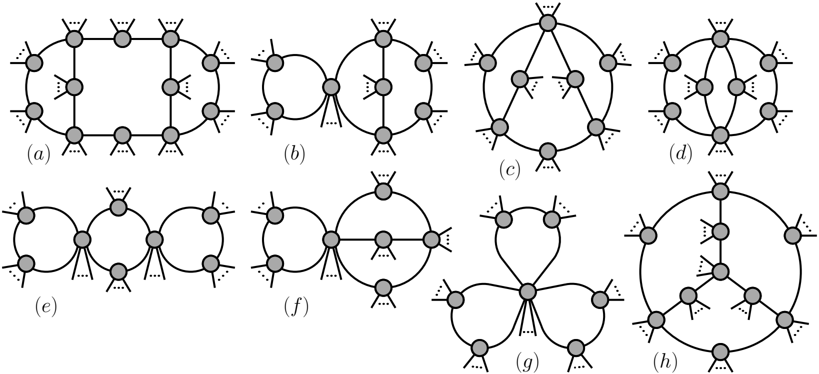
Summary

- Reviewed tree-level color decompositions
- Promoted these to loop level within integrand reduction
- Used loop-level KK relations to reduce decomposition
- Checked 1-loop consistency with known decomposition
  Del Duca, Dixon, Maltoni (1999)
- 2-loop decomposition used in Badger, Mogull, AO, O'Connell (2015)
- Most 2-loop symmetry factors are reducible
  AO, Page (2016)
- Example of 3-loop decomposition

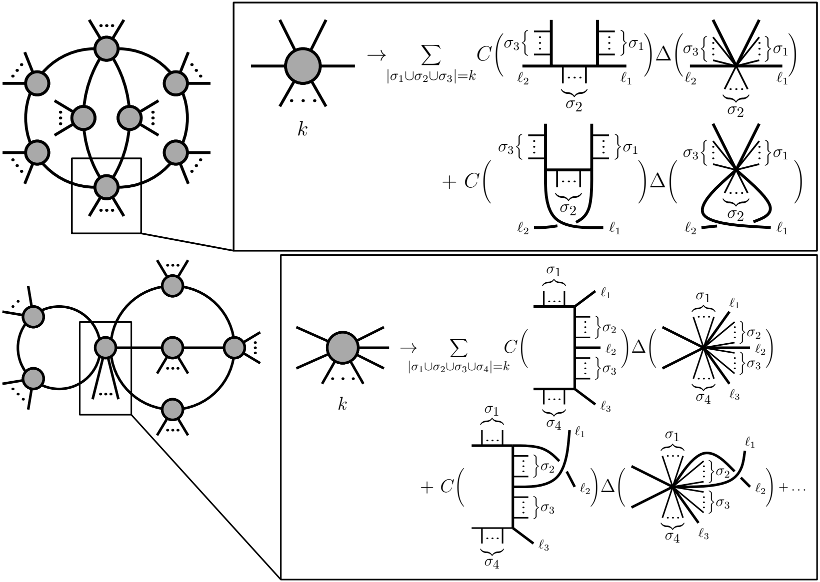
Thank you!

Backup slides

3-loop topologies



3-loop topologies (d) and (f)



3-loop topologies (g) and (h)