Higgs transverse-momentum resummation at N^3LL

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Debrecen, QCD@LHC 2017

[Monni, Re, Torrielli, Phys. Rev. Lett. 116 (2016), n. 24, 242001]

[Bizon, Monni, Re, Rottoli, Torrielli, 1705.09127]

Higgs p_t precision studies

- Increase in statistics at the LHC allows to study Higgs differential distributions in detail.
- Higgs p_t can be used shed light on potential BSM contributions. For example
  - Gluon-fusion sensitive to various dimension-6 operators in different regions of the p_t spectrum [Grazzini et al., 1612.00283].
  - Very competitive constraints on the charm Yukawa at the high-luminosity LHC using p_t distribution [Bishara et al., 1606.09253].
  - Constraints on the trilinear Higgs coupling in the VH and VBF modes [Bizon et al., 1610.05771].

Fixed-order vs resummed transverse momentum

- In the dominant gluon-fusion mode, fixed-order predictions for $H+1$ jet available at NNLO QCD in the EFT $(m_t \to \infty)$ [Boughezal et al., 1504.07922, 1505.03893], [Caola, Melnikov, Schulze, 1508.02684], [Chen et al., 1607.08817]. Quark-mass effects available at LO, top-bottom interference at NLO [Lindert et al., 1703.03886].

- Higgs p_t accurately predicted $(\sim 5-10\%)$ at fixed order in the hard tails $p_t \sim M$, $M =$ hard scale $\mathcal{O}$(Higgs mass).

- When $p_t \ll M$, soft/collinear QCD radiation generates large logarithms that spoil fixed-order perturbation theory:

$$\frac{d\sigma}{dp_t} \sim \frac{1}{p_t} \alpha_S^n \ln^k(M/p_t), \qquad k \leq 2n-1.$$

  Enhanced logarithmic contributions to be resummed at all orders.

- Logarithmic accuracy usually defined at the level of the logarithm of the cumulative cross section Σ:

$$\Sigma(p_t) = \int_0^{p_t} \frac{d\sigma}{dp'_t} dp'_t \sim e^{\alpha_S^n L^{n+1} + \alpha_S^n L^n + \alpha_S^n L^{n-1} + \alpha_S^n L^{n-2} + \ldots}$$

  for LL, NLL, NNLL, N^{3}LL respectively, with $L = \ln(M/p_t)$.

Conjugate-space vs direct-space resummation

- Different approaches to resummation of transverse observables, usually performed in conjugate spaces where they factorise.
- In most of the cases resummation can be performed in direct space: well developed technology [Banfi, Salam, Zanderighi, 0112156, 0304148, 0407286] for observables V satisfying recursive infrared and collinear (rIRC) safety.
  - Scaling properties of V are the same for any number of soft/collinear emissions.
  - Properties unchanged if one adds an infinitely soft/collinear emission: the more soft/collinear, the less it contributes to the value of the observable.
- Most IRC-safe observables are rIRC-safe, not a severe restriction on the class of resummable observables.
- Logarithmic structure of general rIRC-safe observables known up to NNLL [Banfi et al., 1412.2126, 1607.03111]. For rIRC-unsafe observables, structure unknown beyond LL.
- Higgs p_t is rIRC-safe (and global, rapidity-independent, inclusive). What is the problem then?

Resummation of rIRC-safe observables - sketch

- Contributions to $\Sigma(v) = \int_0^v \frac{d\sigma}{dV} dV$ organised in terms a set of unresolved emissions (softer than some ϵv), giving rise to a Sudakov radiator, and an ensemble of resolved emissions $k_1, ..., k_n$ (harder than ϵv).

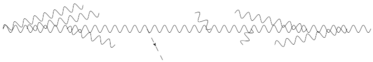
- Analytical manipulations performed on the Sudakov and on the resolved contributions to the observable, $V(k_i)$, in order to only retain contributions up to a given logarithmic accuracy.

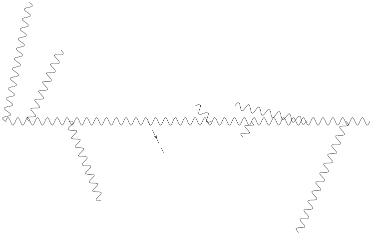
- Usually integrals dominated by $V(k_i) \sim v$. Functions $f(V(k_i))$ expanded around $f(v)$. Subsequent terms in the expansion give subleading $\ln(1/v)$.

- These steps, that work for 'standard' rIRC-safe observables, do not for $v = p_t$: one ends up with a formula that is singular at a finite value of p_t.

- Long-known problem for p_t resummation in momentum space [Frixione, Nason, Ridolfi, 9809367]: at any logarithmic order beyond LL in terms of $\ln(M/p_t)$, resummation in p_t space cannot be simultaneously free of subleading terms and of spurious singularities.

Two competing mechanisms at small p_t





- Each emission i has small k_{ti} (left): $k_{tn} < ... < k_{t2} < k_{t1} \sim p_t \sim 0$. Sudakov limit, sensible $\ln(M/p_t)$ counting, exponential suppression of $\Sigma(p_t)$ at small p_t.

- Large azimuthal cancellations (right): $k_{tn} < ... < k_{t2} < k_{t1} \gg p_t \sim 0$. $p_t \to 0$ away from the Sudakov limit, $\Sigma(p_t) \sim p_t^2$ at small p_t [Parisi, Petronzio, 1979].

- Power-like suppression from the region $k_{ti} \gg p_t$ dominates over the Sudakov mechanism in the $p_t \to 0$ limit.

- Hierarchy in $\ln(M/p_t)$ is not sensible: neglected effects actually dominate the limit. Impossible to recover power behaviour at a given order in $\ln(M/p_t)$.

- Establish a well defined logarithmic counting in momentum space. [Monni, Re, Torrielli, 1604.02191], [Ebert, Tackmann, 1611.08610], [Bizon et al., 1705.09127]

Traditional solution: b space

- For inclusive observables, vectorial nature of the azimuthal cancellations handled via a Fourier transfrom [Parisi, Petronzio, 1979], [Collins, Soper, Sterman, 1985], [Bozzi et al., 0508068], see also [Becher, Neubert, Wilhelm, 1212.2621].

$$\delta^{(2)}(\vec{p_t} - (\vec{k}_{t1} + ... + \vec{k}_{tn})) = \int \frac{d^2\vec{b}}{4\pi^2} e^{-i\vec{b}\cdot\vec{p_t}} \prod_{i=1}^{n} e^{i\vec{b}\cdot\vec{k}_{ti}},$$

$$\frac{d^2\Sigma(p_t)}{d\Phi_B dp_t} = \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1c_2}}{d\Phi_B} \int b\, db\, p_t J_0(p_t b)\, \mathbf{f}^T(b_0/b)\mathbf{C}^{c_1;T}_{N_1}(\alpha_S(b_0/b))H_{\mathrm{CSS}}(M)\mathbf{C}^{c_2}_{N_2}(\alpha_S(b_0/b))\mathbf{f}(b_0/b)$$
$$\times \exp\left\{-\sum_{\ell=1}^{2}\int_{b_0/b}^{M}\frac{dk_t}{k_t}\mathbf{R}'_{\mathrm{CSS},\ell}(k_t)\right\}.$$

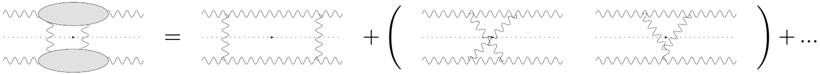
- $\mathbf{C}$ (coefficient functions) and H_{CSS} (form factor) known up to $\mathcal{O}(\alpha_S^2)$ [Catani, Grazzini, 1106.4652], [Gehrmann, Luebbert, Yang, 1403.6451].

$$\sum_{\ell=1}^{2}\int_{b_0/b}^{M}\frac{dk_t}{k_t}\mathbf{R}'_{\mathrm{CSS},\ell}(k_t) = \sum_{\ell=1}^{2}\int_{b_0/b}^{M}\frac{dk_t}{k_t}\Big(A_{\mathrm{CSS},\ell}(\alpha_S(k_t))\ln(M^2/k_t^2) + B_{\mathrm{CSS},\ell}(\alpha_S(k_t))\Big).$$

- A_{CSS} and B_{CSS} anomalous dimensions known up to N^{3}LL (but for the four-loop cusp) [Davies, Stirling, 1984], [de Florian, Grazzini, 0008152], [Becher, Neubert, 1007.4005], [Li, Zhu, 1604.01404].

A solution in momentum space

- Multiple-emission squared amplitude organised into n-particle-correlated blocks. For example 2-particle correlated:



$\mathcal{O}(\alpha_S^2 L^4)$ $\mathcal{O}(\alpha_S^2 L^3)$

- Blocks classified (due to rIRC-safety) according to the logarithmic order at which they contribute. The more correlated, the more logarithmically subleading.

- Introduce a resolution scale ϵk_{t1} (as opposed to ϵp_t!). ϵ is a slicing parameter to be eventually taken $\to 0$.

  - By rIRC safety, blocks with total $k_{ti} < \epsilon k_{t1}$ (unresolved) do not contribute significantly to the observable. Integrated inclusively in d dimensions, they exponentiate and regularise the virtuals giving rise to the Sudakov:

$$e^{-R(\epsilon k_{t1})} = e^{-R(k_{t1}) - \ln(1/\epsilon) R'(k_{t1}) - \dots}$$

  - Blocks with total $k_{ti} > \epsilon k_{t1}$ (resolved) treated exclusively in 4 dimensions and parametrised as derivatives of the Sudakov (wrt $\ln(M/k_{ti})$) $R'(k_{ti})$. Resolved k_{ti} are of the same order as k_{t1}.

- ϵ-dependence in the resolved cancels against the one in the Sudakov, leaving ϵ^p effects.

A solution in momentum space - comments

- Resolved k_{ti} are of the same order of k_{t1} but not necessarily $\sim p_t$: all kinematic configurations are taken into account, without assumptions on the hierarcy between k_{ti} and p_t. In particular the phase-space region $k_{ti} \gg p_t$ is accounted for.
- By including the contributions from the $k_{ti} \gg p_t$ region, the spurious singularities at finite p_t are removed.
- This is also the reason why the b space works.
- Logarithmic counting is defined in terms of $\ln(M/k_{ti})$.
- In the Sudakov limit, where the hierarchy in $\ln(M/p_t)$ makes sense, $k_{ti} \sim p_t \sim 0$ $\Longrightarrow$ same as resummation of $\ln(M/p_t)$. Logarithmic accuracy in $\ln(M/k_{ti})$ translates into the same accuracy in $\ln(M/p_t)$ plus subleading terms.

Momentum-space resummation at N^{3}LL: equivalence with b space

- Result at N^{3}LL is:

$$\frac{d\Sigma(v)}{d\Phi_B} = \int_{\mathcal{C}_1} \frac{dN_1}{2\pi i} \int_{\mathcal{C}_2} \frac{dN_2}{2\pi i} x_1^{-N_1} x_2^{-N_2} \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1 c_2}}{d\Phi_B} \mathbf{f}^T_{N_1}(\mu_0) \hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v) \mathbf{f}_{N_2}(\mu_0)$$

$$\hat{\mathbf{\Sigma}}^{c_1,c_2}_{N_1,N_2}(v) = \left[\mathbf{C}^{c_1;T}_{N_1}(\alpha_s(\mu_0)) H(\mu_R) \mathbf{C}^{c_2}_{N_2}(\alpha_s(\mu_0))\right] \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi}$$
$$\times e^{-\mathbf{R}(\epsilon k_{t1})} \exp\left\{-\sum_{\ell=1}^{2}\left(\int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \frac{\alpha_s(k_t)}{\pi} \mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \int_{\epsilon k_{t1}}^{\mu_0} \frac{dk_t}{k_t} \mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)\right\}$$
$$\sum_{\ell_1=1}^{2}\left(\mathbf{R}'_{\ell_1}(k_{t1}) + \frac{\alpha_s(k_{t1})}{\pi} \mathbf{\Gamma}_{N_{\ell_1}}(\alpha_s(k_{t1})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_1}}(\alpha_s(k_{t1}))\right)$$
$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int_\epsilon^1 \frac{d\zeta_i}{\zeta_i} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \sum_{\ell_i=1}^{2}\left(\mathbf{R}'_{\ell_i}(k_{ti}) + \frac{\alpha_s(k_{ti})}{\pi} \mathbf{\Gamma}_{N_{\ell_i}}(\alpha_s(k_{ti})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_i}}(\alpha_s(k_{ti}))\right)$$
$$\times \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right),$$

- $\zeta_i = k_{ti}/k_{t1}$.
- $\mathbf{\Gamma}_{N_{\ell_i}}$, $\mathbf{\Gamma}^{C}_{N_{\ell_i}}$ = anomalous dimensions of PDFs and coefficient functions.

- Equivalent to b space, up to a resummation-scheme change: using the Θ representation

$$\frac{d^2\Sigma(p_t)}{d\Phi_B dp_t} = \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1 c_2}}{d\Phi_B} \int b\, db\, p_t J_0(p_t b)\, \mathbf{f}^T(b_0/b) \mathbf{C}^{c_1;T}_{N_1}(\alpha_s(b_0/b)) H(M) \mathbf{C}^{c_2}_{N_2}(\alpha_s(b_0/b)) \mathbf{f}(b_0/b)$$
$$\times \exp\left\{-\sum_{\ell=1}^{2} \int_0^M \frac{dk_t}{k_t} \mathbf{R}'_\ell(k_t)\,(1 - J_0(bk_t))\right\},$$

with $\quad (1 - J_0(bk_t)) \simeq \Theta(k_t - \frac{b_0}{b}) + \frac{\zeta_3}{12} \frac{\partial^3}{\partial \ln(Mb/b_0)^3} \Theta(k_t - \frac{b_0}{b}) + \ldots,$

- ζ_3 term starts at N^{3}LL, absorbed in a redefinition of A_4, B_3, and H_2, (or $\mathbf{C}$) wrt CSS.

Momentum-space resummation at $\mathrm{N^3LL}$

- Above formula presented in Mellin space only to diagonalise PDF evolution.
- At any logarithmic order only a finite number of DGLAP-evolution steps necessary: analytic Mellin inversion, dealing only with momentum-space quantities.
- Expand k_{ti} around k_{t1} in the resolved radiation at the desired logarithmic accuracy.

$$\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}}\frac{d\phi_1}{2\pi}\partial_L\left(-e^{-R(k_{t1})}\mathcal{L}_{\mathrm{N^3LL}}(k_{t1})\right)\int d\mathcal{Z}[\{R',k_i\}]\Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1})\right)$$

$$+\int \frac{dk_{t1}}{k_{t1}}\frac{d\phi_1}{2\pi}e^{-R(k_{t1})}\int d\mathcal{Z}[\{R',k_i\}]\int_0^1\frac{d\zeta_s}{\zeta_s}\frac{d\phi_s}{2\pi}\Bigg\{\left(R'(k_{t1})\mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - \partial_L\mathcal{L}_{\mathrm{NNLL}}(k_{t1})\right)$$
$$\times\left(R''(k_{t1})\ln\frac{1}{\zeta_s}+\frac{1}{2}R'''(k_{t1})\ln^2\frac{1}{\zeta_s}\right) - R'(k_{t1})\left(\partial_L\mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - 2\frac{\beta_0}{\pi}\alpha_s^2(k_{t1})\hat{P}^{(0)}\otimes\mathcal{L}_{\mathrm{NLL}}(k_{t1})\ln\frac{1}{\zeta_s}\right)$$
$$+\frac{\alpha_s^2(k_{t1})}{\pi^2}\hat{P}^{(0)}\otimes\hat{P}^{(0)}\otimes\mathcal{L}_{\mathrm{NLL}}(k_{t1})\Bigg\}\Big\{\Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1},k_s)\right) - \Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1})\right)\Big\}$$

$$+\frac{1}{2}\int \frac{dk_{t1}}{k_{t1}}\frac{d\phi_1}{2\pi}e^{-R(k_{t1})}\int d\mathcal{Z}[\{R',k_i\}]\int_0^1\frac{d\zeta_{s1}}{\zeta_{s1}}\frac{d\phi_{s1}}{2\pi}\int_0^1\frac{d\zeta_{s2}}{\zeta_{s2}}\frac{d\phi_{s2}}{2\pi}R'(k_{t1})$$
$$\times\Bigg\{\mathcal{L}_{\mathrm{NLL}}(k_{t1})\left(R''(k_{t1})\right)^2\ln\frac{1}{\zeta_{s1}}\ln\frac{1}{\zeta_{s2}} - \partial_L\mathcal{L}_{\mathrm{NLL}}(k_{t1})R''(k_{t1})\left(\ln\frac{1}{\zeta_{s1}}+\ln\frac{1}{\zeta_{s2}}\right)$$
$$+\frac{\alpha_s^2(k_{t1})}{\pi^2}\hat{P}^{(0)}\otimes\hat{P}^{(0)}\otimes\mathcal{L}_{\mathrm{NLL}}(k_{t1})\Bigg\}$$
$$\times\Big\{\Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1},k_{s1},k_{s2})\right) - \Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1},k_{s1})\right) -$$
$$\Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1},k_{s2})\right) + \Theta\left(v - V(\{\tilde{p}\},k_1,\ldots,k_{n+1})\right)\Big\} + \mathcal{O}\left(\alpha_s^n\ln^{2n-6}\frac{1}{v}\right),$$

- H and C absorbed in $\mathcal{L}$.
- Reproduces correct p_t^2 scaling at small p_t (see backup).
- Evaluated numerically by means of fast Monte Carlo code: `RadISH`. $\int d\mathcal{Z}[\{R',k_i\}]$ generated as a parton shower.

Advantages with respect to b-space solution?

- If it were only for p_t, there wouldn't be a clear advantage with respect to b space
  - ... but possibly for the easier interpretation of the dominant dynamics at $p_t \to 0$
  - ... and for potentially more efficient numerical implementations
  - ... and for the possible connections with parton-shower formalisms
- However a solution in momentum space is much less observable-dependent.
  - What is learnt for p_t can be immediately exported to all other observables of the same class (global, rapidity-independent, inclusive). Extension to more general rIRC-safe observables is conceptually known.
  - For example, ϵk_{t1} is a correct resolution scale for all observables with the same LL as p_t. One can write a generator that computes all of them at the same time (p_t, ϕ^* in DY, $p_t(j_1)$, E_T, ...).
  - It gives access to joint resummations at high logarithmic accuracies.

Matching to fixed order

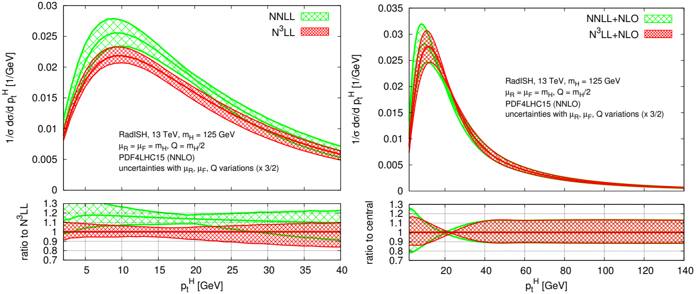
- Resummation matched to fixed order with a multiplicative scheme $(\Sigma(p_t, \Phi_B) \equiv d\Sigma(p_t)/d\Phi_B)$:

$$\Sigma_{\text{MAT}}(p_t, \Phi_B) = (\Sigma_{\text{RES}}(p_t, \Phi_B))^Z \, \frac{\Sigma_{\text{FO}}(p_t, \Phi_B)}{(\Sigma_{\text{EXP}}(p_t, \Phi_B))^Z},$$

$$\Sigma_{\text{FO}}(p_t, \Phi_B) = \sigma_{pp\to H}^{\text{N}^k\text{LO}}(\Phi_B) - \int_{p_t} dp_t' \, \frac{d\sigma_{pp\to Hj}^{\text{N}^{k-1}\text{LO}}(\Phi_B)}{dp_t'},$$

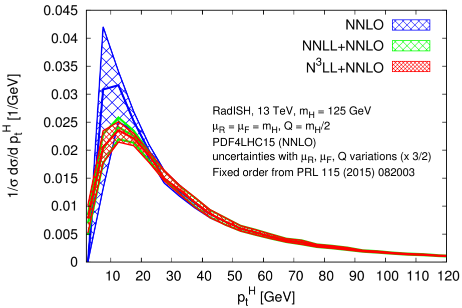
- $k = 2, 3$ for fixed-order p_t spectrum at NLO or NNLO.
- $Z \to 1$ at small p_t and $Z \to 0$ at high p_t: resummation turned off asymptotically and total cross section recovered.
- Σ_{EXP} = expansion of Σ_{RES} up to the relevant order in α_{S}.
- Σ_{EXP} determined as linear combination (with analytic coefficients) of master integrals evaluated numerically with high precision.
- At NNLO ($k = 3$), the multiplicative scheme recovers constant terms of $\mathcal{O}(\alpha_{\text{S}}^3)$.

Phenomenological results (EFT)



- Left: pure resummation at N^{3}LL and NNLL.
  - Pure N^{3}LL correction amounts to 10-15%, in part due to inclusion of constant $\mathcal{O}(\alpha_S^2)$ coefficient functions and form factor, absent at NNLL.
  - Reduction in theoretical uncertainty (μ_R, μ_F, Q) compared to NNLL.
- Right: NLO matching (i.e. using $\sigma^{\mathrm{NNLO}}_{pp\to H}$ and $\sigma^{\mathrm{NLO}}_{pp\to Hj}$).
  - N^{3}LL+NLO correction is $\mathcal{O}(10\%)$ around the peak, and somewhat larger at smaller p_t.
  - Perturbative uncertainty halved below 10 GeV, unchanged elsewhere.

Phenomenological results (EFT)



- NNLO matching (i.e. using $\sigma^{\mathrm{N^3LO}}_{pp\to H}$ and $\sigma^{\mathrm{NNLO}}_{pp\to Hj}$).
- $\mathrm{N^3LO}$ $pp \to H$ cross section from [Anastasiou et al., 1503.06056]. NNLO $pp \to Hj$ cross section from [Boughezal et al., 1504.07922]

- $\mathrm{N^3LL}$+NNLO corrections are a few percent around the peak, and get more sizeable $\mathcal{O}(10\%)$ below 10 GeV.
- $\mathrm{N^3LL}$+NNLO display only moderate reduction in uncertainty with respect to NNLL+NNLO. Need for very stable NNLO distributions below 10 GeV to appreciate reduction / make a quantitative statement.
- Quark-mass corrections necessary to improve further.

Conclusions

- New formalism for p_t resummation up to N^{3}LL in momentum space.
- Presented for p_t but valid for all inclusive, rapidity-independent, global rIRC-safe observables with (or without) azimuthal cancellations away from the Sudakov limit.
- Extension to more general rIRC-safe observables possible and under study.
- Formalism allows an efficient implementation in a computer code. `RadISH` can process any colour singlet with arbitrary cuts at Born level. To be publicly released soon.
- Higgs-p_t phenomenology (EFT)
    - N^{3}LL+NLO corrects NNLL+NLO by $\mathcal{O}(10\%)$ around and below the peak. Uncertainty nearly halved below 10 GeV.
    - N^{3}LL+NNLO corrects NNLL+NNLO by few percent at the peak, and $\mathcal{O}(10\%)$ below. Moderate reduction in theo. uncertainty, now below $\sim 10\%$ in the whole spectrum.

Thank you for your attention

Backup: reproducing the $\Sigma(p_t) \sim p_t^2$ scaling at $p_t \to 0$

- Computation at NLL (for DY and $n_f = 4$) gives exactly the original Parisi-Petronzio result [Parisi, Petronzio, 1979]:

$$\frac{d^2\Sigma(p_t)}{dp_t d\Phi_B} = 4\,\frac{d\sigma_B}{d\Phi_B}\,p_t \int_{\Lambda_{\mathrm{QCD}}}^{M} \frac{dk_{t1}}{k_{t1}^3} e^{-R(k_{t1})} \simeq 2\frac{d\sigma_B}{d\Phi_B} p_t \left(\frac{\Lambda_{\mathrm{QCD}}^2}{M^2}\right)^{\frac{16}{25}\ln\frac{41}{16}} .$$

- As now higher logarithmic terms (up to N^{3}LL) are under control, one can systematically improve the *perturbative* prediction of the coefficient in front of p_t (non-perturbative effects – of the same order – not considered in this analysis).

- Each new subleading-logarithmic order induces a relative $\mathcal{O}(\alpha_{\mathrm{S}})$ correction with respect to the previous order: scaling $L \sim 1/\alpha_{\mathrm{S}}$.

Backup: NLL result and the finiteness in four dimensions

$$
\begin{aligned}
\frac{d\Sigma(p_t)}{d\Phi_B} &= \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \partial_L \left(-e^{-R'(k_{t1})} \mathcal{L}_{\mathrm{NLL}}(k_{t1})\right) \times \\
&\quad \times \underbrace{\epsilon^{R'(k_{t1})} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_{i=2}^{n+1} \int_{\epsilon k_{t1}}^{k_{t1}} \frac{dk_{ti}}{k_{ti}} \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'(k_{t1}) \right) \Theta(p_t - |\vec{k}_{t1} + \ldots + \vec{k}_{t(n+1)}|)}_{\equiv \int d\mathcal{Z}[\{R', k_i\}] \Theta(p_t - |\vec{k}_{t1} + \ldots + \vec{k}_{t(n+1)}|)} .
\end{aligned}
$$

- $L = \ln(M/k_{t1})$; luminosity $\mathcal{L}_{\mathrm{NLL}}(k_{t1}) = \sum_{c_1,c_2} \frac{d|M_B|^2_{c_1 c_2}}{d\Phi_B} f_{c_1}(x_1, k_{t1}) f_{c_2}(x_2, k_{t1})$.
- $\int d\mathcal{Z}[\{R', k_i\}]\Theta$ finite as $\epsilon \to 0$:

$$
\epsilon^{R'(k_{t1})} = 1 - R'(k_{t1}) \ln(1/\epsilon) + \ldots = 1 - \int_{\epsilon k_{t1}}^{k_{t1}} R'(k_{t1}) + \ldots,
$$

$$
\begin{aligned}
\int d\mathcal{Z}[\{R', k_i\}]\Theta &= \left[1 - \int_{\epsilon k_{t1}}^{k_{t1}} R'(k_{t1}) + \ldots\right] \left[\Theta(p_t - |\vec{k}_{t1}|) + \int_{\epsilon k_{t1}}^{k_{t1}} R'(k_{t1}) \Theta(p_t - |\vec{k}_{t1} + \vec{k}_{t2}|) + \ldots\right] \\
&= \Theta(p_t - |\vec{k}_{t1}|) + \underbrace{\int_0^{k_{t1}}}_{\epsilon \to 0} R'(k_{t1}) \underbrace{\left[\Theta(p_t - |\vec{k}_{t1} + \vec{k}_{t2}|) - \Theta(p_t - |\vec{k}_{t1}|)\right]}_{\text{finite: real-virtual cancellation}} + \ldots
\end{aligned}
$$

- Evaluated with Monte Carlo techniques: $\int d\mathcal{Z}[\{R', k_i\}]$ is generated as a parton shower over secondary emissions.

Backup: generating secondary radiation as a parton shower

- Secondary radiation:

$$
\begin{aligned}
d\mathcal{Z}[\{R', k_i\}] &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_{i=2}^{n+1} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \int_{\epsilon k_{t1}}^{k_{t1}} \frac{dk_{ti}}{k_{ti}} R'(k_{t1}) \right) \epsilon^{R'(k_{t1})} \\
&= \sum_{n=0}^{\infty} \left(\prod_{i=2}^{n+1} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \int_{\epsilon k_{t1}}^{k_{t(i-1)}} \frac{dk_{ti}}{k_{ti}} R'(k_{t1}) \right) \epsilon^{R'(k_{t1})}, \\
\epsilon^{R'(k_{t1})} &= e^{-R'(k_{t1}) \ln 1/\epsilon} = \prod_{i=2}^{n+2} e^{-R'(k_{t1}) \ln k_{t(i-1)}/k_{ti}},
\end{aligned}
$$

  with $k_{t(n+2)} = \epsilon k_{t1}$.

- Each secondary emissions has differential probability

$$
dw_i = \frac{d\phi_i}{2\pi} \frac{dk_{ti}}{k_{ti}} R'(k_{t1}) e^{-R'(k_{t1}) \ln k_{t(i-1)}/k_{ti}} = \frac{d\phi_i}{2\pi} d\left(e^{-R'(k_{t1}) \ln k_{t(i-1)}/k_{ti}}\right).
$$

- $k_{t(i-1)} \geq k_{ti}$. Scale k_{ti} extracted by solving $e^{-R'(k_{t1}) \ln k_{t(i-1)}/k_{ti}} = r$, with r random number extracted uniformly in $[0, 1]$. Shower ordered in k_{ti}.

- Extract ϕ_i randomly in $[0, 2\pi]$.

Backup: checks

- b-space resummation reproduced analytically.
- Correct small-p_t scaling reproduced analytically.
- Numerical checks down to very low p_t against b-space codes at the resummed level (`HqT` [Bozzi et al., 0302104, 0508068], [de Florian et al., 1109.2109 , `CuTe` [Becher et al., 1109.6027, 1212.2621]).
- Expansion of the momentum-space formula up to $\mathcal{O}(\alpha_S^3)$ checked against b space.
- Expansion checked against `MCFM` [Campbell, Ellis, 9905386], [Campbell et al., 1105.0020, 1503.06182] up to $\mathcal{O}(\alpha_S^2)$.