Calculation of 3-loop operator matrix elements with two masses

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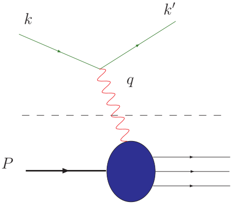
Contents

Introduction

Unpolarized Deep–Inelastic Scattering (DIS):



$\longrightarrow L_{\mu\nu}$ $\qquad Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q}$ Bjorken–x

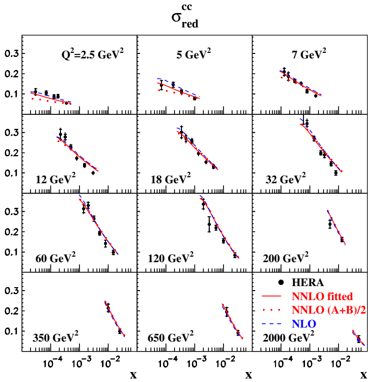
$\longrightarrow W_{\mu\nu}$ $\qquad \frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$

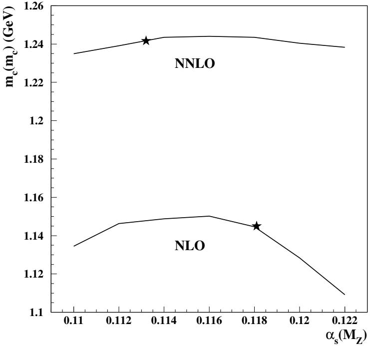
$$W_{\mu\nu}(q, P, s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P, s \rangle =$$

$$\frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x, Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x, Q^2) .$$

Structure Functions: $F_{2,L}$
contain light and heavy quark contributions.

Deep–Inelastic Scattering (DIS):





NNLO:

S. Alekhin, J. Blümlein, K. Daum, K. Lipka, Phys.Lett. B720 (2013) 172 [1212.2355]

$$m_c(m_c) = 1.24 \pm 0.03(exp) \; {}^{+0.03}_{-0.02} \; (scale) \; {}^{+0.00}_{-0.07} \; (thy),$$

$$\alpha_s(M_Z^2) = 0.1132 \pm 0.011$$

Yet approximate NNLO treatment [Kawamura et al. [1205.5227].

$\alpha_s(M_Z^2)$ from NNLO DIS(+) analyses [from ABM13]

	$\alpha_s(M_Z^2)$	
BBG	$0.1134\ ^{+0.0019}_{-0.0021}$	valence analysis, NNLO
GRS	0.112	valence analysis, NNLO
ABKM	0.1135 ± 0.0014	HQ: FFNS $N_f = 3$
JR	0.1128 ± 0.0010	dynamical approach
JR	0.1140 ± 0.0006	including NLO-jets
MSTW	0.1171 ± 0.0014	
MSTW	$0.1155 - 0.1175$	(2013)
ABM11_J	$0.1134 - 0.1149 \pm 0.0012$	Tevatron jets (NLO) incl.
ABM13	0.1133 ± 0.0011	
ABM13	0.1132 ± 0.0011	(without jets)
CTEQ	0.1159..0.1162	
CTEQ	0.1140	(without jets)
NN21	$0.1174 \pm 0.0006 \pm 0.0001$	
Gehrmann et al.	$0.1131\ ^{+\ 0.0028}_{-\ 0.0022}$	e^+e^- thrust
Abbate et al.	0.1140 ± 0.0015	e^+e^- thrust
BBG	$0.1141\ ^{+0.0020}_{-0.0022}$	valence analysis, $\mathrm{N^3LO}$

$$\Delta_{\mathrm{TH}}\alpha_s = \alpha_s(\mathrm{N^3LO}) - \alpha_s(\mathrm{NNLO}) + \Delta_{\mathrm{HQ}} = +0.0009 \pm 0.0006_{\mathrm{HQ}}$$

NNLO accuracy is needed to analyze the world data. $\Longrightarrow$ NNLO HQ corrections needed.

Goals

- Complete the NNLO heavy flavor Wilson coefficients for twist-2 in the dynamical safe region $Q^2 > 20 GeV^2$ (no higher twist) for $F_2(x, Q^2)$
- Measure m_c and α_s as precisely as possible
- Provide precise CC heavy flavor corrections
- Consequences for LHC:
  - NNLO VFNS will be provided
  - better constraint on sea quarks and the gluon
  - precise m_c and α_s on input

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x, Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x, \mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).

$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz\, \delta(x - yz) f(y) g(z)\ .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx\, x^{N-1} f(x)\ .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)+H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)\ .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=\sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right)=\langle j \mid O_i \mid j\rangle\ .$$

→ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO [Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

By now we have computed 6 out of 7 OMEs at $O(\alpha_s^3)$ in the single mass case, namely,

$A_{qq}^{(3),\mathrm{PS}}$, $A_{qg}^{(3)}$ [Ablinger, Blümlein, Klein, Schneider, Wissbrock, arXiv:1008.3347]

$A_{qq}^{(3),\mathrm{NS,\,TR}}$ [Ablinger, Behring, Blümlein, ADF, von Manteuffel, Schneider, *et. al.*, arXiv:1406.4654]

$A_{gq}^{(3)}$ [Ablinger, Blümlein, ADF, von Manteuffel, Schneider, *et. al.*, arXiv:1402.0359]

$A_{Qq}^{(3),\mathrm{PS}}$ [Ablinger, Behring, Blümlein, ADF, von Manteuffel, Schneider, arXiv:1409.1135]

$A_{gg}^{(3)}$ to be published soon

and the corresponding Wilson coefficients:

$$L_{q,2}^{\mathrm{PS}}, \quad L_{g,2}^{\mathrm{S}}, \quad L_{q,2}^{\mathrm{NS}}, \quad \text{and} \quad H_{q,2}^{\mathrm{PS}}$$

We have also partial results for $A_{Qg}^{(3)}$ (terms $\propto N_F T_F^2$ and Ladder diagrams).

The logarithmic contributions to all OMEs and WCs were published recently

[Behring, Blümlein, ADF, Bierembaum, Klein, Wißbrock, arXiv:1403.6356]

Starting at 3-loop order, we have to consider the simultaneous contributions of quarks of different mass

[Ablinger, Blümlein, ADF, Hasselhuhn, Schneider, Wißbrock *et. al.*, arXiv:1705.07030],

since m_b is not much larger than m_c

$$\frac{m_c^2}{m_b^2} \sim 0.1$$

The heavy flavor contribution with two masses is given by

$$\begin{aligned}
&\frac{1}{x} F_{(2,L)}{}^{\text{heavy}}(x, N_F+2, Q^2, m_1^2, m_2^2) = \\
&\sum_{k=1}^{N_F} e_k^2 \Bigg\{ L_{q,(2,L)}^{\rm NS}\left(x, N_F+2, \frac{Q^2}{\mu^2}, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \otimes \left[f_k(x,\mu^2,N_F) + f_{\overline{k}}(x,\mu^2,N_F)\right] \\
&\qquad + \frac{1}{N_F} L_{q,(2,L)}^{\rm PS}\left(x, N_F+2, \frac{Q^2}{\mu^2}, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,N_F) \\
&\qquad + \frac{1}{N_F} L_{g,(2,L)}^{\rm S}\left(x, N_F+2, \frac{Q^2}{\mu^2}, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \otimes G(x,\mu^2,N_F) \Bigg\} \\
&\quad + \tilde{\tilde{H}}_{q,(2,L)}^{\rm PS}\left(x, N_F+2, \frac{Q^2}{\mu^2}, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,N_F) \\
&\quad + \tilde{\tilde{H}}_{g,(2,L)}^{\rm S}\left(x, N_F+2, \frac{Q^2}{\mu^2}, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \otimes G(x,\mu^2,N_F)
\end{aligned}$$

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
L_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+2)\ \delta_2 + A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big]\\
L_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= a_s^2 A_{gg,Q}^{(1)}(N_F+2) N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + a_s^3\Big[\ A_{qg,Q}^{(3)}(N_F+2)\ \delta_2\\
&\quad + A_{gg,Q}^{(1)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+2) + A_{gg,Q}^{(2)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2)\\
&\quad + A_{Qg}^{(1)}(N_F+2)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ ,\\
L_{q,(2,L)}^{\mathsf{NS}}(N_F+2) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2)\ \delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big]\\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+2)\ \delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2) C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big]\\
\tilde{\tilde{H}}_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2 a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\ \delta_2 + \ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big] + a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+2)\ \delta_2\\
&\quad + \sum_{i=1}^{2} e_{Q_i}^2\Big[\tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+2) + A_{gq,Q}^{(2)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2)\\
&\quad + A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2)\Big]\Big]\ ,\\
\tilde{\tilde{H}}_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2\Big[a_s\Big[\ A_{Qg}^{(1)}(N_F+2)\ \delta_2 + \ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2)\Big] + a_s^2\Big[\ A_{Qg}^{(2)}(N_F+2)\ \delta_2\\
&\quad + A_{Qg}^{(1)}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) + \ A_{gg,Q}^{(1)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2)\\
&\quad + \tilde{C}_{g,(2,L)}^{(2)}(N_F+2)\Big]\Big]\\
&\quad + a_s^3\Big[\ A_{Qg}^{(3)}(N_F+2)\ \delta_2 + \ \sum_{i=1}^{2} e_{Q_i}^2\Big[A_{Qg}^{(2)}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2)\\
&\quad + A_{gg,Q}^{(2)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + \ A_{Qg}^{(1)}(N_F+2)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+2)\\
&\quad + \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big\} + \ A_{gg,Q}^{(1)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+2) + \ \tilde{C}_{g,(2,L)}^{(3)}(N_F+2)\Big]\Big]
\end{aligned}
$$

The Wilson Coefficients at large Q^2

$$L_{q,(2,L)}^{\mathsf{PS}}(N_F+2) = a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+2)\ \delta_2 + A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big]$$

$$\begin{aligned}
L_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= a_s^2 A_{gg,Q}^{(1)}(N_F+2)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + a_s^3\Big[\ A_{qg,Q}^{(3)}(N_F+2)\ \delta_2 \\
&\quad + A_{gg,Q}^{(1)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+2) + A_{gg,Q}^{(2)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&\quad + A_{Qg}^{(1)}(N_F+2)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ ,
\end{aligned}$$

$$\begin{aligned}
L_{q,(2,L)}^{\mathsf{NS}}(N_F+2) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2)\ \delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] \\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+2)\ \delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big]
\end{aligned}$$

$$\begin{aligned}
\tilde{\tilde{H}}_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2 a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\ \delta_2 +\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big] + a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+2)\ \delta_2 \\
&\quad + \sum_{i=1}^{2} e_{Q_i}^2\Big[\tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+2) + A_{gq,Q}^{(2)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&\quad + A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2)\Big]\Big]\ ,
\end{aligned}$$

$$\begin{aligned}
\tilde{\tilde{H}}_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2\Big[a_s\Big[\ A_{Qg}^{(1)}(N_F+2)\ \delta_2 +\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2)\Big] + a_s^2\Big[\ A_{Qg}^{(2)}(N_F+2)\ \delta_2 \\
&\quad + A_{Qg}^{(1)}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) +\ A_{gg,Q}^{(1)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&\quad + \tilde{C}_{g,(2,L)}^{(2)}(N_F+2)\Big]\Big] \\
&\quad + a_s^3\Big[\ A_{Qg}^{(3)}(N_F+2)\ \delta_2 +\ \sum_{i=1}^{2} e_{Q_i}^2\Big[A_{Qg}^{(2)}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) \\
&\quad + A_{gg,Q}^{(2)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) +\ A_{Qg}^{(1)}(N_F+2)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+2) \\
&\quad + \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big\} +\ A_{gg,Q}^{(1)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+2) +\ \tilde{C}_{g,(2,L)}^{(3)}(N_F+2)\Big]\Big]
\end{aligned}$$

The Wilson Coefficients at large Q^2

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L_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+2)\ \delta_2 + A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big] \\
L_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= a_s^2 A_{gg,Q}^{(1)}(N_F+2)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + a_s^3\Big[\ A_{qg,Q}^{(3)}(N_F+2)\ \delta_2 \\
&\quad + A_{gg,Q}^{(1)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+2) + A_{gg,Q}^{(2)}(N_F+2)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&\quad + A_{Qg}^{(1)}(N_F+2)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ , \\
L_{q,(2,L)}^{\mathsf{NS}}(N_F+2) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2)\ \delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] \\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+2)\ \delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big] \\
\tilde{\tilde{H}}_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2 a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\ \delta_2 +\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big] + a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+2)\ \delta_2 \\
&\quad + \sum_{i=1}^{2} e_{Q_i}^2\Big[\tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+2) + A_{gq,Q}^{(2)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&\quad + A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2)\Big]\Big]\ , \\
\tilde{\tilde{H}}_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2\Big[a_s\Big[\ A_{Qg}^{(1)}(N_F+2)\ \delta_2 +\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2)\Big] + a_s^2\Big[\ A_{Qg}^{(2)}(N_F+2)\ \delta_2 \\
&\quad + A_{Qg}^{(1)}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) +\ A_{gg,Q}^{(1)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&\quad + \tilde{C}_{g,(2,L)}^{(2)}(N_F+2)\Big]\Big] \\
&\quad + a_s^3\Big[\ A_{Qg}^{(3)}(N_F+2)\ \delta_2 +\ \sum_{i=1}^{2} e_{Q_i}^2\Big[A_{Qg}^{(2)}(N_F+2)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) \\
&\quad + A_{gg,Q}^{(2)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+2) +\ A_{Qg}^{(1)}(N_F+2)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+2) \\
&\quad + \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big\} +\ A_{gg,Q}^{(1)}(N_F+2)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+2) +\ \tilde{C}_{g,(2,L)}^{(3)}(N_F+2)\Big]\Big]
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L_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= a_s^3\Big[\; A_{qq,Q}^{(3),\mathsf{PS}}(N_F+2)\;\delta_2 + A_{gq,Q}^{(2)}(N_F)\;\; N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big] \\
L_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= a_s^2 A_{gg,Q}^{(1)}(N_F+2) N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + a_s^3\Big[\; A_{qg,Q}^{(3)}(N_F+2)\;\delta_2 \\
&\quad + A_{gg,Q}^{(1)}(N_F+2)\;\; N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+2) + A_{gg,Q}^{(2)}(N_F+2)\;\; N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&\quad + A_{Qg}^{(1)}(N_F+2)\;\; N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\;, \\
L_{q,(2,L)}^{\mathsf{NS}}(N_F+2) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2)\;\delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] \\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+2)\;\delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+2) C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big] \\
\tilde{\tilde{H}}_{q,(2,L)}^{\mathsf{PS}}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2 a_s^2\Big[\; A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\;\delta_2 + \;\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big] + a_s^3\Big[\; A_{Qq}^{(3),\mathsf{PS}}(N_F+2)\;\delta_2 \\
&\quad + \;\sum_{i=1}^{2} e_{Q_i}^2\Big[\tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+2) + A_{gq,Q}^{(2)}(N_F+2)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&\quad + A_{Qq}^{(2),\mathsf{PS}}(N_F+2)\; C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2)\Big]\Big]\;, \\
\tilde{\tilde{H}}_{g,(2,L)}^{\mathsf{S}}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2\Big[a_s\Big[\; A_{Qg}^{(1)}(N_F+2)\;\delta_2 + \;\tilde{C}_{g,(2,L)}^{(1)}(N_F+2)\Big] + a_s^2\Big[\; A_{Qg}^{(2)}(N_F+2)\;\delta_2 \\
&\quad + A_{Qg}^{(1)}(N_F+2)\; C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) + \; A_{gg,Q}^{(1)}(N_F+2)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) \\
&\quad + \tilde{C}_{g,(2,L)}^{(2)}(N_F+2)\Big]\Big] \\
&\quad + a_s^3\Big[\; A_{Qg}^{(3)}(N_F+2)\;\delta_2 + \;\sum_{i=1}^{2} e_{Q_i}^2\Big[A_{Qg}^{(2)}(N_F+2)\; C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+2) \\
&\quad + A_{gg,Q}^{(2)}(N_F+2)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+2) + \; A_{Qg}^{(1)}(N_F+2)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+2) \\
&\quad + \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+2)\Big\} + \; A_{gg,Q}^{(1)}(N_F+2)\;\tilde{C}_{g,(2,L)}^{(2)}(N_F+2) + \;\tilde{C}_{g,(2,L)}^{(3)}(N_F+2)\Big]\Big]
\end{aligned}
$$

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
L^{\mathsf{PS}}_{q,(2,L)}(N_F+2) &= a_s^3\Big[\ A^{(3),\mathsf{PS}}_{qq,Q}(N_F+2)\ \delta_2 + A^{(2)}_{gq,Q}(N_F)\ \ N_F\tilde{C}^{(1)}_{g,(2,L)}(N_F+2) + N_F\hat{\tilde{C}}^{(3),\mathsf{PS}}_{q,(2,L)}(N_F)\Big] \\
L^{\mathsf{S}}_{g,(2,L)}(N_F+2) &= a_s^2 A^{(1)}_{gg,Q}(N_F+2)N_F\tilde{C}^{(1)}_{g,(2,L)}(N_F+2) + a_s^3\Big[\ A^{(3)}_{qg,Q}(N_F+2)\ \delta_2 \\
&\quad + A^{(1)}_{gg,Q}(N_F+2)\ \ N_F\tilde{C}^{(2)}_{g,(2,L)}(N_F+2) + A^{(2)}_{gg,Q}(N_F+2)\ \ N_F\tilde{C}^{(1)}_{g,(2,L)}(N_F+2) \\
&\quad + A^{(1)}_{Qg}(N_F+2)\ \ N_F\tilde{C}^{(2),\mathsf{PS}}_{q,(2,L)}(N_F+2) + N_F\hat{\tilde{C}}^{(3)}_{g,(2,L)}(N_F)\Big]\ , \\
L^{\mathsf{NS}}_{q,(2,L)}(N_F+2) &= a_s^2\Big[A^{(2),\mathsf{NS}}_{qq,Q}(N_F+2)\ \delta_2 + \hat{C}^{(2),\mathsf{NS}}_{q,(2,L)}(N_F)\Big] \\
&+ a_s^3\Big[A^{(3),\mathsf{NS}}_{qq,Q}(N_F+2)\ \delta_2 + A^{(2),\mathsf{NS}}_{qq,Q}(N_F+2)C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+2) + \hat{C}^{(3),\mathsf{NS}}_{q,(2,L)}(N_F)\Big] \\
\tilde{\tilde{H}}^{\mathsf{PS}}_{q,(2,L)}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2 a_s^2\Big[\ A^{(2),\mathsf{PS}}_{Qq}(N_F+2)\ \delta_2 +\ \tilde{C}^{(2),\mathsf{PS}}_{q,(2,L)}(N_F+2)\Big] + a_s^3\Big[\ A^{(3),\mathsf{PS}}_{Qq}(N_F+2)\ \delta_2 \\
&\quad + \sum_{i=1}^{2} e_{Q_i}^2\Big[\tilde{C}^{(3),\mathsf{PS}}_{q,(2,L)}(N_F+2) + A^{(2)}_{gq,Q}(N_F+2)\ \tilde{C}^{(1)}_{g,(2,L)}(N_F+2) \\
&\quad + A^{(2),\mathsf{PS}}_{Qq}(N_F+2)\ C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+2)\Big]\Big]\ , \\
\tilde{\tilde{H}}^{\mathsf{S}}_{g,(2,L)}(N_F+2) &= \sum_{i=1}^{2} e_{Q_i}^2\Big[a_s\Big[\ A^{(1)}_{Qg}(N_F+2)\ \delta_2 +\ \tilde{C}^{(1)}_{g,(2,L)}(N_F+2)\Big] + a_s^2\Big[\ A^{(2)}_{Qg}(N_F+2)\ \delta_2 \\
&\quad + A^{(1)}_{Qg}(N_F+2)\ C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+2) +\ A^{(1)}_{gg,Q}(N_F+2)\ \tilde{C}^{(1)}_{g,(2,L)}(N_F+2) \\
&\quad + \tilde{C}^{(2)}_{g,(2,L)}(N_F+2)\Big]\Big] \\
&\quad + a_s^3\Big[\ A^{(3)}_{Qg}(N_F+2)\ \delta_2 +\ \sum_{i=1}^{2} e_{Q_i}^2\Big[A^{(2)}_{Qg}(N_F+2)\ C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+2) \\
&\quad + A^{(2)}_{gg,Q}(N_F+2)\ \tilde{C}^{(1)}_{g,(2,L)}(N_F+2) +\ A^{(1)}_{Qg}(N_F+2)\Big\{C^{(2),\mathsf{NS}}_{q,(2,L)}(N_F+2) \\
&\quad + \tilde{C}^{(2),\mathsf{PS}}_{q,(2,L)}(N_F+2)\Big\} +\ A^{(1)}_{gg,Q}(N_F+2)\ \tilde{C}^{(2)}_{g,(2,L)}(N_F+2) +\ \tilde{C}^{(3)}_{g,(2,L)}(N_F+2)\Big]\Big]
\end{aligned}
$$

Variable Flavor Number Scheme

$$f_k(n_f+2,\mu^2) + f_{\overline{k}}(n_f+2,\mu^2) = A_{qq,Q}^{\rm NS}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right) \otimes \left[f_k(n_f,\mu^2) + f_{\overline{k}}(n_f,\mu^2)\right]$$

$$+\frac{1}{n_f} A_{qq,Q}^{\rm PS}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right) \otimes \Sigma(n_f,\mu^2)$$

$$+\frac{1}{n_f} A_{qg,Q}^{\rm S}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right) \otimes G(n_f,\mu^2)$$

$$f_{Q+\bar{Q}}(n_f+2,\mu^2) = A_{Qq}^{\rm PS}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right) \otimes \Sigma(n_f,\mu^2) + A_{Qg}^{\rm S}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right) \otimes G(n_f,\mu^2)\,.$$

$$G(n_f+2,\mu^2) = A_{gq,Q}^{\rm S}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right) \otimes \Sigma(n_f,\mu^2) + A_{gg,Q}^{S}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right) \otimes G(n_f,\mu^2)\,.$$

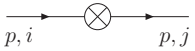
$$\Sigma(n_f+2,\mu^2) = \sum_{k=1}^{n_f+2}\left[f_k(n_f+2,\mu^2) + f_{\overline{k}}(n_f+2,\mu^2)\right]$$

$$= \left[A_{qq,Q}^{\rm NS}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right) + A_{qq,Q}^{\rm PS}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right) + A_{Qq}^{\rm PS}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right)\right]$$

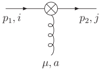
$$\otimes\Sigma(n_f,\mu^2)$$

$$+\left[A_{qg,Q}^{\rm S}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right) + A_{Qg}^{\rm S}\left(n_f+2,\frac{\mu^2}{m_1^2},\frac{\mu^2}{m_2^2}\right)\right] \otimes G(n_f,\mu^2)$$

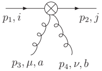
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



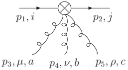
$$\delta^{ij} \not\Delta \gamma_\pm (\Delta \cdot p)^{N-1}, \quad N \geq 1$$



$$g t^a_{ji} \Delta^\mu \not\Delta \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}, \quad N \geq 2$$



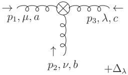
$$g^2 \Delta^\mu \Delta^\nu \not\Delta \gamma_\pm \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2}$$
$$\left[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \right],$$
$$N \geq 3$$



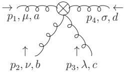
$$g^3 \Delta_\mu \Delta_\nu \Delta_\rho \not\Delta \gamma_\pm \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta.p_2)^j (\Delta.p_1)^{N-m-2}$$
$$\Big[\quad (t^a t^b t^c)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1}$$
$$+ (t^a t^c t^b)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1}$$
$$+ (t^b t^a t^c)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1}$$
$$+ (t^b t^c t^a)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1}$$
$$+ (t^c t^a t^b)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1}$$
$$+ (t^c t^b t^a)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1} \Big],$$
$$N \geq 4$$

$\gamma_+ = 1, \quad \gamma_- = \gamma_5 .$

$$\frac{1+(-1)^N}{2} \delta^{ab} (\Delta \cdot p)^{N-2}$$
$$\left[g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu \right], \quad N \geq 2$$



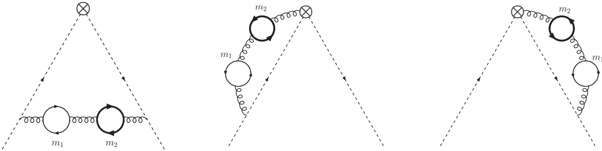
$$-ig \frac{1+(-1)^N}{2} f^{abc} \Bigg($$
$$\left[(\Delta_\nu g_{\lambda\mu} - \Delta_\lambda g_{\mu\nu}) \Delta \cdot p_1 + \Delta_\mu (p_{1,\nu} \Delta_\lambda - p_{1,\lambda} \Delta_\nu) \right] (\Delta \cdot p_1)^{N-2}$$
$$+ \Delta_\lambda \left[\Delta \cdot p_1 p_{2,\mu} \Delta_\nu + \Delta \cdot p_2 p_{1,\nu} \Delta_\mu - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_\mu \Delta_\nu \right]$$
$$\times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j}$$
$$+ \left\{ {p_1 \to p_2 \to p_3 \to p_1 \atop \mu \to \nu \to \lambda \to \mu} \right\} + \left\{ {p_1 \to p_3 \to p_2 \to p_1 \atop \mu \to \lambda \to \nu \to \mu} \right\} \Bigg), \quad N \geq 2$$



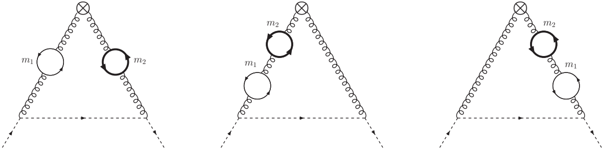
$$g^2 \frac{1+(-1)^N}{2} \Bigg(f^{abe} f^{cde} O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4)$$
$$+ f^{ace} f^{bde} O_{\mu\lambda\nu\sigma}(p_1, p_3, p_2, p_4) + f^{ade} f^{bce} O_{\mu\sigma\nu\lambda}(p_1, p_4, p_2, p_3) \Bigg),$$

$$O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) = \Delta_\nu \Delta_\lambda \Big\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2}$$
$$+ [p_{4,\mu} \Delta_\sigma - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i}$$
$$- [p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i}$$
$$+ [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_\mu \Delta_\sigma - \Delta \cdot p_4 p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 p_{4,\mu} \Delta_\sigma]$$
$$\times \sum_{i=0}^{N-4} \sum_{j=0}^{i} (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \Big\}$$
$$- \left\{ {p_1 \leftrightarrow p_2 \atop \mu \leftrightarrow \nu} \right\} - \left\{ {p_3 \leftrightarrow p_4 \atop \lambda \leftrightarrow \sigma} \right\} + \left\{ {p_1 \leftrightarrow p_2, \; p_3 \leftrightarrow p_4 \atop \mu \leftrightarrow \nu, \; \lambda \leftrightarrow \sigma} \right\}, \quad N \geq 2$$

Diagrams for $A_{qq}^{(3),\mathrm{NS}}$



Diagrams for $A_{gq}^{(3)}$

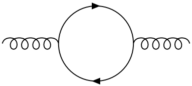


The diagrams are generated using QGRAF [Nogueira 1993 J. Comput. Phys].

	$A_{qq,Q}^{(3),\mathrm{NS}}$	$A_{gq,Q}^{(3)}$	$A_{Qq}^{(3),\mathrm{PS}}$	$A_{gg,Q}^{(3)}$	$A_{Qg}^{(3)}$
No. diagrams	6	6	16	72	256

The NS and gq contributions at general values of N

One massive fermion loop insertion is effectively rendered massless via a Mellin-Barnes representation:



$$= a_s T_F \frac{4}{\pi} (4\pi)^{-\varepsilon/2} \left(k_\mu k_\nu - k^2 g_{\mu\nu}\right) \times \int_{-i\infty}^{+i\infty} d\sigma \left(\frac{m^2}{\mu^2}\right)^{\sigma} \left(-k^2\right)^{\varepsilon/2-\sigma} \times \frac{\Gamma(\sigma-\varepsilon/2)\Gamma^2(2-\sigma+\varepsilon/2)\Gamma(-\sigma)}{\Gamma(4-2\sigma+\varepsilon)}$$

The Introduction of Feynman parameters then leads to an expression for the integrals of the form

$$I \propto C(\varepsilon, N) \int_{-i\infty}^{+i\infty} d\xi\, \eta^{\xi} \Gamma\begin{bmatrix} g_1(\varepsilon)+\xi, g_2(\varepsilon)+\xi, g_3(\varepsilon)+\xi, g_4(\varepsilon)-\xi, g_5(\varepsilon)-\xi \\ g_6(\varepsilon)+\xi, g_7(\varepsilon)-\xi \end{bmatrix}$$

where $\eta = m_1^2/m_2^2$ and the g_j are linear functions in ε.

After closing the contour and collecting the residues a linear combination of generalized hypergeometric ${}_4F_3$–functions is obtained

$$I = \sum_j C_j\,(\varepsilon, N)\,{}_4F_3\left[\begin{matrix} a_1(\varepsilon), a_2(\varepsilon), a_3(\varepsilon), a_4(\varepsilon) \\ b_1(\varepsilon), b_2(\varepsilon), b_3(\varepsilon) \end{matrix}, \eta\right].$$

For $A_{qq}^{(3),\mathrm{NS}}$ and $A_{gq}^{(3)}$ the arguments of the hypergeometric ${}_PF_Q$-function are completely independent of the Mellin variable N
$\rightarrow$ the N and $\eta = m_1/m_2$ dependendence factorize!

The ε expansion can be done using `HypExp 2`.
The results are given in terms of the following (poly)logarithmic functions:

$$\{\ln(\eta),\ \ln(1 \pm \eta),\ \ln\left(1 \pm \sqrt{\eta}\right),\ \mathrm{Li}_2\left(\pm\sqrt{\eta}\right),\ \mathrm{Li}_2\left(\pm\eta\right),\ \mathrm{Li}_3\left(\pm\sqrt{\eta}\right)\}$$

The pre-factor $C_j\,(\varepsilon, N)$ may contain a sum stemming from the operator insertion on the vertex. This sum is evaluated in terms of harmonic sums using the summation package `Sigma` (see C. Schneider's talk).

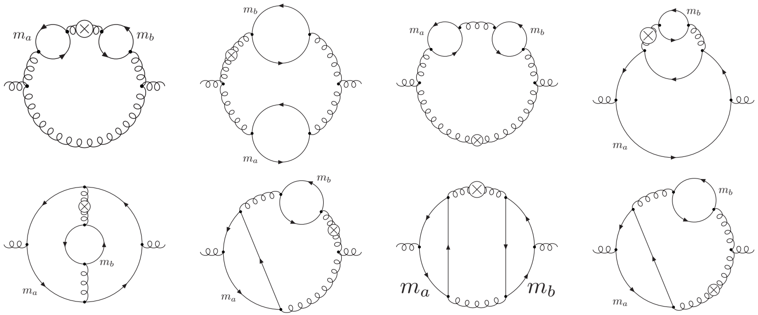
The flavor non-singlet contribution

$$
\begin{aligned}
\tilde{a}_{qq,Q}^{(3),\mathrm{NS}} = \; & C_F T_F^2 \Biggl\{ \left(\frac{4}{9} S_1 - \frac{3N^2+3N+2}{9N(N+1)} \right) \Biggl[-24 \left(L_1{}^3 + L_2{}^3 + (L_1 L_2 + 2\zeta_2 + 5)(L_1 + L_2) \right) \\
& + \frac{\eta+1}{\eta^{3/2}} \left(5\eta^2 + 22\eta + 5\right) \left(-\frac{1}{4} \ln^2(\eta) \ln\left(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}} \right) + 2 \ln(\eta) \mathrm{Li}_2\left(\sqrt{\eta}\right) - 4 \mathrm{Li}_3\left(\sqrt{\eta}\right) \right) \\
& + \frac{\left(\sqrt{\eta}+1\right)^2}{2\eta^{3/2}} \left(-10\eta^{3/2} + 5\eta^2 + 42\eta - 10\sqrt{\eta} + 5 \right) \left[\mathrm{Li}_3(\eta) - \ln(\eta) \mathrm{Li}_2(\eta) \right] + \frac{64}{3}\zeta_3 \\
& + \frac{8}{3} \ln^3(\eta) - 16 \ln^2(\eta) \ln(1-\eta) + 10 \frac{\eta^2 - 1}{\eta} \ln(\eta) \Biggr] + \frac{16\left(405\eta^2 - 3238\eta + 405\right)}{729\eta} S_1 \\
& + \frac{4}{3} \left(\frac{3N^4 + 6N^3 + 47N^2 + 20N - 12}{3N^2(N+1)^2} - \frac{40}{3} S_1 + 8 S_2 \right) \left[\frac{4}{3}\zeta_2 + (L_1 + L_2)^2 \right] \\
& + \frac{8}{9} \left(\frac{130N^4 + 84N^3 - 62N^2 - 16N + 24}{3N^3(N+1)^3} - \frac{52}{3} S_1 + \frac{80}{3} S_2 - 16 S_3 \right) (L_1 + L_2) \\
& + \left[-\frac{R_1}{18N^2(N+1)^2\eta} + \frac{2\left(5\eta^2 + 2\eta + 5\right)}{9\eta} S_1 + \frac{32}{9} S_2 \right] \ln^2(\eta) - \frac{4R_2}{729N^4(N+1)^4\eta} \\
& + \frac{3712}{81} S_2 - \frac{1280}{81} S_3 + \frac{256}{27} S_4 \Biggr\} .
\end{aligned}
$$

The A_{gq} contribution

$$
\begin{aligned}
\tilde{a}_{gq,Q}^{(3)} \quad = \quad & C_F T_F^2 \Bigg\{ p_{gq}^{(0)} \Bigg[16 \Bigg(L_1{}^3 + L_2{}^3 + \Bigg(L_1 L_2 + 2\zeta_2 + \frac{26}{3} \Bigg) (L_1 + L_2) \Bigg) \\
& - \frac{4}{3\eta^{3/2}} \Big(\left(\sqrt{\eta}+1\right)^2 R_8 \mathrm{Li}_3(-\sqrt{\eta}) - \left(\sqrt{\eta}-1\right)^2 R_9 \mathrm{Li}_3(\sqrt{\eta}) \Big) - \frac{16}{9} \ln^3(\eta) \\
& + \Bigg(\frac{2\left(\sqrt{\eta}+1\right)^2}{3\eta^{3/2}} R_8 \mathrm{Li}_2(-\sqrt{\eta}) - \frac{2\left(\sqrt{\eta}-1\right)^2}{3\eta^{3/2}} R_9 \mathrm{Li}_2(\sqrt{\eta}) - \frac{20}{3\eta} \left(\eta^2 - 1\right) \Bigg) \ln(\eta) \\
& + \Bigg(\frac{\left(\sqrt{\eta}+1\right)^2}{6\eta^{3/2}} R_8 \ln(1+\sqrt{\eta}) - \frac{\left(\sqrt{\eta}-1\right)^2}{6\eta^{3/2}} R_9 \ln(1-\sqrt{\eta}) - \frac{16}{3} S_1 \Bigg) \ln^2(\eta) \\
& - \frac{64}{27} S_1^3 - \frac{128}{27} S_3 - \frac{64}{3} \left(\zeta_2 + \frac{1}{3} S_2 \right) S_1 - \frac{128}{9} \zeta_3 \Bigg] - \frac{R_{10} \ln^2(\eta)}{3\eta(N-1)N(N+1)^2} \\
& + 16 \left[-\frac{1}{(N+1)^2} + p_{gq}^{(0)} \left(\frac{8}{3} - S_1 \right) \right] \left((L_1 + L_2)^2 - \frac{4}{3} (L_1 + L_2) S_1 \right) \\
& + \left[\frac{32}{3} p_{gq}^{(0)} \left(S_2 - S_1^2 \right) - \frac{64(8N+5)}{9(N+1)^3} \right] (L_1 + L_2) - \frac{64 R_{11} S_1}{27(N-1)N(N+1)^3} \\
& + \frac{64\left(8N^3 + 13N^2 + 27N + 16\right)}{27(N-1)N(N+1)^2} \left(S_1^2 + S_2 + 3\zeta_2 \right) - \frac{8 R_{12}}{243\eta(N-1)N(N+1)^4} \Bigg\}
\end{aligned}
$$

Scalar $A_{gg,Q}$ diagrams with $m_1 \neq m_2$



The strategy:

- Introduce Feynman parameters and do the momentum integration for one of the closed fermion lines $\rightarrow$ effective propagator.
- Detach mass using the Mellin-Barnes representation

$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\xi \frac{B^{\xi}}{A^{\lambda+\xi}} \Gamma(\lambda+\xi)\Gamma(-\xi)$$

- Perform the remaining momentum integrals and the Feynman parameter integrals (except the one where both ξ and N appear)

$$\rightarrow \quad C(N,m_1,m_2,\varepsilon)\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\xi \int_0^1 dX\, \eta^\xi X^{\xi+N+\alpha\varepsilon+\beta}(1-X)^{-\xi+\gamma\varepsilon+\delta} \times \Gamma\begin{bmatrix} a_1+b_1\varepsilon+c_1\xi,\ldots,a_i+b_i\varepsilon+c_i\xi \\ d_1+e_1\varepsilon+f_1\xi,\ldots,d_j+e_j\varepsilon+f_j\xi \end{bmatrix}$$

$a_k,\ d_k,\ \beta,\ \delta \in \mathbb{Z}$, $b_k,\ e_k,\ \alpha,\ \gamma \in \mathbb{Z}/2$, $c_k \in \{-1,1\}$ and $f_k \in \{-2,-1,1,2\}$, with $\sum_{k=1}^{i} c_k = \sum_{k=1}^{j} f_k$

- Split integration range and remap to $[0,1]$ using

$$\int_{-i\infty}^{+i\infty} d\xi \int_0^1 dX f(\xi,X)\left(\frac{\eta X}{1-X}\right)^\xi = \left(\int_0^{\frac{1}{1+\eta}} dX + \int_{\frac{1}{1+\eta}}^1 dX\right) f(\xi,X)\left(\frac{\eta X}{1-X}\right)^\xi$$

$$= \int_{-i\infty}^{+i\infty} d\xi \int_0^1 dT \left[\frac{\eta}{(\eta+T)^2} f\left(\xi,\frac{T}{\eta+T}\right) T^\xi + \frac{1}{(1+\eta T)^2} f\left(\xi,\frac{1}{1+\eta T}\right) T^{-\xi}\right]$$

- Regulate poles and expand in ε.
- Take residues and sum using `Sigma` (see C. Schneider's talk) and `HarmonicSums` (see J. Ablinger's talk). Results in terms of GHPLs, i.e., iterated integrals over the alphabet

$$\left\{ \frac{1}{\tau}, \frac{1}{\tau + T}, \frac{1}{1 + T\tau^2} \right\}$$

- Rewrite GHPLs so that T appears only in the argument.
- Absorb rational, N-dependent factors into the integrals using

$$\begin{aligned} N \int_0^1 dx g(x)^N f(x) &= g(x)^{N+1} \frac{f(x)}{g'(x)} \Big|_0^1 \\ &\quad - \int_0^1 dx\, (g(x))^N \frac{d}{dx} \left[\frac{f(x) g(x)}{g'(x)} \right] \\ \frac{1}{(N+a)} \int_0^1 dx g(x)^N f(x) &= \frac{1}{(N+a)} g(x)^{N+a} \left(\int_0^x dy \frac{f(y)}{g(y)^a} \right) \Big|_{x=0}^1 \\ &\quad - \int_0^1 dx g(x)^{N+a-1} \frac{dg(x)}{dx} \left(\int_0^x dy \frac{f(y)}{g(y)^a} \right) \end{aligned}$$

- Rewrite the remaining integral doing the change of variable

$$g(x,\eta) \to x'$$

- Final result in z-space in terms of generalized iterated integrals

$$G\left(\{f_1(\tau), f_2(\tau), \cdots, f_n(\tau)\}, z\right) = \int_0^z d\tau_1\, f_1(\tau_1) G\left(\{f_2(\tau), \cdots, f_n(\tau)\}, \tau_1\right)$$

with

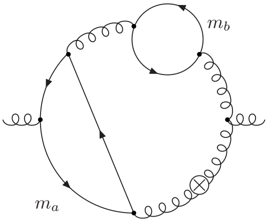
$$G\left(\Big\{\underbrace{\frac{1}{\tau}, \frac{1}{\tau}, \cdots, \frac{1}{\tau}}_{\text{n times}}\Big\}, z\right) = \frac{1}{n!}\mathrm{H}_0(z)^n \equiv \frac{1}{n!}\ln^n(z)\ .$$

and in N-space in terms of generalized harmonic sums

$$S_{b,\vec{a}}(c,\vec{d};N) = \sum_{k=1}^{N} \frac{c^k}{k^b} S_{\vec{a}}(\vec{d};k), c, d_i \in \mathbb{R}\backslash\{0\}; \quad b, a_i \in \mathbb{N}\backslash\{0\},$$

and other generalizations including inverse binomial sums.

Example:



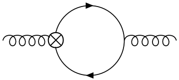
$$
\begin{aligned}
D_{8a}^{(+)}(z) \quad = \quad & \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{-3+\varepsilon}\left\{-\frac{1}{\varepsilon}\frac{1}{90(1-z)}-\frac{1}{450(1-z)}+\frac{1}{180(1-z)}\mathrm{H}_1\left(z\right)\right. \\
& +\frac{25+(63\eta-100)(1-z)}{3360\eta(1-z)^{3/2}}\sqrt{z}\left[\left(\eta-1-\frac{1+\eta}{2}\ln(\eta)\right)G\left(\left\{\sqrt{1-\tau}\sqrt{\tau}\right\},z\right)\right. \\
& +\frac{(1-\eta)^2}{8}\left[-\ln\left(\eta\right)G\left(\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau}\right\},z\right)+G\left(\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{\tau}\right\},z\right)\right. \\
& \left.+G\left(\left\{\frac{\sqrt{1-\tau}\sqrt{\tau}}{-\eta-\tau+\eta\tau},\frac{1}{1-\tau}\right\},z\right)\right]+\frac{1+\eta}{2}\left[G\left(\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau}\right\},z\right)\right. \\
& \left.\left.\left.+G\left(\left\{\sqrt{1-\tau}\sqrt{\tau},\frac{1}{\tau}\right\},z\right)\right]\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
D_{8a}(N) \quad = \quad & \left(m_1^2\right)^{\varepsilon/2}\left(m_2^2\right)^{-3+\varepsilon}\left[\frac{1+(-1)^N}{2}\right]\Biggl\{-\frac{N+2}{45\varepsilon^2(N+1)} \\
& +\frac{1}{\varepsilon}\left[\frac{(N+2)S_1(N)}{90(N+1)}-\frac{8N^3+(4-25\eta)N^2-(25\eta+24)N+20}{1800N(N+1)^2}\right] \\
& -\frac{\left(7N\left(N^2+3N+2\right)-3\eta^3\right)S_1^2(N)}{2520N(N+1)^2}+\frac{2^{-2N-8}\binom{2N}{N}P_{45}}{105\sqrt{\eta}(N+1)^2}\left[\mathrm{H}_{-1,0,0}\left(\sqrt{\eta}\right)+\mathrm{H}_{1,0,0}\left(\sqrt{\eta}\right)\right] \\
& +\frac{2^{-2N}\binom{2N}{N}P_{45}}{53760(\eta-1)\eta(N+1)^2}\sum_{i_1=1}^{N}\frac{2^{2i_1}\left(\frac{\eta}{-1+\eta}\right)^{i_1}\left[S_2\left(\frac{-1+\eta}{\eta},i_1\right)-S_{1,1}\left(\frac{-1+\eta}{\eta},1,i_1\right)\right]}{\binom{2i_1}{i_1}} \\
& +\ln^2(\eta)\Biggl[\frac{\eta^3}{840N(N+1)^2}-\frac{(\eta-1)^{-N-1}\eta^N}{53760N(N+1)^2}P_{44}-\frac{2^{-2N-10}\binom{2N}{N}P_{45}}{105(\eta-1)(N+1)^2} \\
& -\frac{2^{-2N-10}\binom{2N}{N}P_{45}}{105(\eta-1)\eta(N+1)^2}\sum_{i_1=1}^{N}\frac{2^{2i_1}(-1+\eta)^{-i_1}\eta^{i_1}}{\binom{2i_1}{i_1}}\Biggr]+\frac{P_{48}}{9072000\eta N^2(N+1)^3} \\
& -\frac{P_{47}S_1(N)}{403200\eta(N+1)^2}-\frac{\left(3\eta^3+7N\left(N^2+3N+2\right)\right)S_2(N)}{2520N(N+1)^2}+\ln(\eta)\Biggl[-\frac{S_1(N)\,\eta^3}{420N(N+1)^2} \\
& -\frac{2^{-2N-8}\binom{2N}{N}P_{45}}{105\eta(N+1)^2}+\frac{2^{-2N-9}\binom{2N}{N}P_{45}}{105(\eta-1)\eta(N+1)^2}\sum_{i_1=1}^{N}\frac{2^{2i_1}(-1+\eta)^{-i_1}\eta^{i_1}S_1\left(\frac{-1+\eta}{\eta},i_1\right)}{\binom{2i_1}{i_1}} \\
& +\frac{(\eta-1)^{-N-1}\eta^N P_{44}}{26880N(N+1)^2}S_1\left(\frac{\eta-1}{\eta},N\right)+\frac{P_{46}}{80640(N+1)^2\eta}\Biggr]-\frac{2^{-2N-7}\binom{2N}{N}P_{45}}{105\eta(N+1)^2} \\
& +\frac{(\eta-1)^{-N-1}\eta^N P_{44}}{26880N(N+1)^2}\left[S_2\left(\frac{\eta-1}{\eta},N\right)-S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)\right]-\frac{(N+2)\zeta_2}{120(N+1)}\Biggr\}
\end{aligned}
$$

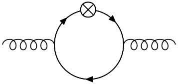
The PS contribution at general values of N

We use the same trick we used before for A_{qq}^{NS} and A_{gq} to decouple the mass coming from the fermion loop without operator insertion.

For the fermion loop with operator insertion we use



$$= 16\delta_{ab} T_F g_s^2 \frac{(\Delta.k)^{N-2}}{(4\pi)^{D/2}} \Gamma(2-D/2) \times \int_0^1 dx\; x^N (1-x) \frac{(\Delta.k)\Delta_\mu k_\nu - k^2 \Delta_\mu \Delta_\nu}{(m^2 - x(1-x)k^2)^{2-D/2}}$$



$$\begin{aligned} = {} & 4\delta_{ab} T_F g_s^2 \frac{(\Delta.k)^{N-2}}{(4\pi)^{D/2}} \int_0^1 dx\; x^{N-2}(1-x) \Big[\\ & - \Big(x(1-x)(g_{\mu\nu} k^2 - 2k_\mu k_\nu) + 2m^2 g_{\mu\nu} \Big) \frac{x^2 \Gamma(3-D/2)(\Delta.k)^2}{(m^2 - x(1-x)k^2)^{3-D/2}} \\ & + \Gamma(2-D/2)(2Nx + 1 - N) \frac{x(k_\mu \Delta_\nu + k_\nu \Delta_\mu)(\Delta.k)}{(m^2 - x(1-x)k^2)^{2-D/2}} \\ & + \Gamma(2-D/2)((N-1)(1-2x) - Dx) \frac{x g_{\mu\nu}(\Delta.k)^2}{(m^2 - x(1-x)k^2)^{2-D/2}} \\ & - \Gamma(1-D/2) \frac{N-1}{1-x} (N(1-x) - 1) \frac{\Delta_\mu \Delta_\nu}{(m^2 - x(1-x)k^2)^{1-D/2}} \Big] \end{aligned}$$

The diagrams end up being expressed as a linear combination of integrals of the form

$$I_1 = C_1(N, m_1, m_2, \varepsilon)\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\xi \int_0^1 dx\, \eta^{\xi} x^{\xi+N+\alpha\varepsilon+\beta}(1-x)^{\xi+\gamma\varepsilon+\delta} \times \Gamma\begin{bmatrix} a_1+b_1\varepsilon+c_1\xi, \dots, a_i+b_i\varepsilon+c_i\xi \\ d_1+e_1\varepsilon+f_1\xi, \dots, d_j+e_j\varepsilon+f_j\xi \end{bmatrix}$$

or

$$I_2 = C_2(N, m_1, m_2, \varepsilon)\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\xi \int_0^1 dx\, \eta^{\xi} x^{-\xi+N+\alpha'\varepsilon+\beta'}(1-x)^{-\xi+\gamma'\varepsilon+\delta'} \times \Gamma\begin{bmatrix} a_1'+b_1'\varepsilon+c_1'\xi, \dots, a_i'+b_i'\varepsilon+c_i'\xi \\ d_1'+e_1'\varepsilon+f_1'\xi, \dots, d_j'+e_j'\varepsilon+f_j'\xi \end{bmatrix}$$

Notice the difference with $A_{gg}^{(3)}$, where the relevant variable was

$$\frac{\eta x}{1-x}.$$

Now the relevant variables are

$$\eta x(1-x) \;\text{ for }\; I_1, \quad \text{and} \quad \frac{\eta}{x(1-x)} \;\text{ for }\; I_2$$

For the calculation of I_2 we need to split the integration range, since

$$\begin{aligned} \frac{\eta}{x(1-x)} &< 1, \quad \text{for} \quad x \in (\eta_-, \eta_+) \\ \frac{\eta}{x(1-x)} &> 1, \quad \text{for} \quad x \in (0, \eta_-) \quad \text{or} \quad x \in (\eta_+, 1) \end{aligned}$$

where

$$\eta_\pm = \frac{1}{2}\left(1 \pm \sqrt{1-\eta}\right)$$

We can, however, perform the contour integrals for both, I_1 and I_2 by taking residues and summing them with `Sigma` and `HarmonicSums`. The results are expressed in terms of GHPLs.

Unlike the case of $A_{gg}^{(3)}$, the arguments of the GHPLs depend on both x and η, so it's not so easy to absorb the rational factors of N coming from $C_1(N, m_1, m_2, \varepsilon)$ and $C_2(N, m_1, m_2, \varepsilon)$ and rewrite the result in terms of GHPL's of higher weight.

The result for the constant term of the pure singlet operator matrix element is

$$
\begin{aligned}
a_{Qq}^{(3),\mathrm{PS}} \quad = \quad & \int_0^1 dx\, x^{N-1} \Big\{ K(\eta,x) + \left(\theta(\eta_- - x) + \theta(x - \eta_+)\right) x\, g_0(\eta,x) \\
& + \theta(\eta_+ - x)\theta(x - \eta_-) \left[x\, f_0(\eta,x) - \int_{\eta_-}^{x} dy \left(f_1(\eta,y) + \frac{y}{x} f_2(\eta,y) + \frac{x}{y} f_3(\eta,y) \right) \right] \\
& + \theta(\eta_- - x) \int_x^{\eta_-} dy \left(g_1(\eta,y) + \frac{y}{x} g_2(\eta,y) + \frac{x}{y} g_3(\eta,y) \right) \\
& - \theta(x - \eta_+) \int_{\eta_+}^{x} dy \left(g_1(\eta,y) + \frac{y}{x} g_2(\eta,y) + \frac{x}{y} g_3(\eta,y) \right) \\
& + x\, h_0(\eta,x) + \int_x^1 dy \left(h_1(\eta,y) + \frac{y}{x} h_2(\eta,y) + \frac{x}{y} h_3(\eta,y) \right) \\
& + \int_{\eta_-}^{\eta_+} dy \left(\eta_+^N f_1(\eta,y) + \eta_+^{N-1} \frac{y}{x} f_2(\eta,y) + \eta_+^{N+1} \frac{x}{y} f_3(\eta,y) \right) \\
& + \int_{\eta_+}^1 dy \left(g_1(\eta,y) + \frac{y}{x} g_2(\eta,y) + \frac{x}{y} g_3(\eta,y) \right) \Big\}
\end{aligned}
$$

The integrals $\int_{\eta_-}^{x} dy$, $\int_{\eta_+}^{x} dy$, $\int_x^1 dy$, $\int_{\eta_-}^{\eta_+} dy$ and $\int_{\eta_+}^{1} dy$ arise from the absorption of N dependant factors.

These are two of the functions:

$$
\begin{aligned}
f_2(\eta, y) \quad = \quad & -\frac{64P_1\left(\eta + 4y^2 - 4y\right)^{3/2}}{9\eta^{3/2}(1-y)y^2} G\left(\left\{\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, \frac{\eta}{y(1-y)}\right) \\
& + G\left(\left\{\sqrt{4-\tau}\sqrt{\tau}\right\}, \frac{\eta}{y(1-y)}\right)\left\{\frac{128}{3}(1-y) G\left(\left\{\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, \frac{\eta}{y(1-y)}\right)\right. \\
& \left. - \frac{32P_1\left(\eta + 4y^2 - 4y\right)^{3/2}}{9\eta^{3/2}(1-y)y^2}\left[1 - 2\ln\left(\frac{\eta}{y(1-y)}\right)\right]\right\} + \frac{1280}{9}(1-y)\ln^2\left(\frac{\eta}{y(1-y)}\right) \\
& - \frac{128}{3}(1-y) G\left(\left\{\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, \frac{\eta}{y(1-y)}\right) - \frac{256}{9}(1-y)\ln^3\left(\frac{\eta}{(1-y)y}\right) \\
& + \frac{32}{3}(1-y)\left[1 - 2\ln\left(\frac{\eta}{y(1-y)}\right)\right] G\left(\left\{\sqrt{4-\tau}\sqrt{\tau}\right\}, \frac{\eta}{y(1-y)}\right)^2 + \frac{4P_2}{9(1-y)^3 y^4} \\
& - \left(\frac{16P_3}{9(1-y)^3 y^4} + \frac{512}{3}(1-y)\zeta_2\right)\ln\left(\frac{\eta}{y(1-y)}\right) + \frac{2560}{9}(1-y)\zeta_2 - \frac{1024}{3}(1-y)\zeta_3
\end{aligned}
$$

$$
\begin{aligned}
h_3(\eta, y) \quad = \quad & (1-y)\left\{\frac{512}{9}\left(1 - 4\eta y(1-y)\right)^{3/2} G\left(\left\{\frac{1}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}\right\}, \eta y(1-y)\right)\right. \\
& + \frac{1024}{3} G\left(\left\{\frac{\sqrt{1-4\tau}}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}\right\}, \eta y(1-y)\right) + \frac{512}{3} G\left(\left\{\frac{\sqrt{1-4\tau}}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}, \frac{1}{\tau}\right\}, \eta y(1-y)\right) \\
& + \left(\frac{512}{3}\zeta_2 - \frac{1024}{9}\left(4\eta y^2 - 4\eta y - 1\right)^2\right)\ln(\eta y(1-y)) + \frac{4096}{9}\eta y(1-y) + \frac{256}{9}\ln^3(\eta y(1-y)) \\
& + \frac{1280}{9}\ln^2(\eta y(1-y)) + \frac{512}{9}\zeta_2\left(1 - 4\eta y(1-y)\right)^{3/2} - \frac{512}{9}\zeta_2 \\
& \left. + \left(\frac{512}{3}\zeta_2 - \frac{2}{9}\left(1 - 4\eta y(1-y)\right)^{3/2}\left[2 + \ln(\eta y(1-y))\right]\right) G\left(\left\{\frac{\sqrt{1-4\tau}}{\tau}\right\}, \eta y(1-y)\right)\right\}
\end{aligned}
$$

In total, eleven new GHPL's appear:

$$\left\{ G\left(\left\{\sqrt{4-\tau}\sqrt{\tau}\right\}, \frac{\eta}{(1-x)x}\right), G\left(\left\{\frac{\sqrt{1-4\tau}}{\tau}\right\}, \frac{(1-x)x}{\eta}\right), G\left(\left\{\frac{\sqrt{1-4\tau}}{\tau}\right\}, \eta x(1-x)\right),\right.$$

$$G\left(\left\{\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, \frac{\eta}{(1-x)x}\right), G\left(\left\{\frac{1}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}\right\}, \frac{(1-x)x}{\eta}\right),$$

$$G\left(\left\{\frac{1}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}\right\}, \eta x(1-x)\right), G\left(\left\{\frac{\sqrt{1-4\tau}}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}\right\}, \frac{(1-x)x}{\eta}\right),$$

$$G\left(\left\{\frac{\sqrt{1-4\tau}}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}\right\}, \eta x(1-x)\right), G\left(\left\{\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, \frac{\eta}{(1-x)x}\right),$$

$$\left. G\left(\left\{\frac{\sqrt{1-4\tau}}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}, \frac{1}{\tau}\right\}, \frac{(1-x)x}{\eta}\right), G\left(\left\{\frac{\sqrt{1-4\tau}}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}, \frac{1}{\tau}\right\}, \eta x(1-x)\right) \right\}$$

All of the generalized harmonic polylogarithms appearing in the functions f_i, g_i and h_i ($i = 0 \ldots 3$) can be written in terms of logarithms and polylogarithms of complicated arguments. E.g.

$$
\begin{aligned}
&G\left(\left\{\frac{\sqrt{1-4\tau}}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}, \frac{1}{\tau}\right\}, \eta x(1-x)\right) = \\
&\frac{20}{3}\ln^3(2) + \ln\left(1+\sqrt{1-4\eta x(1-x)}\right)\Bigg[-10\ln^2(2) - (8\ln(2)+4)\sqrt{1-4\eta x(1-x)} - 4\ln(2) - 4\mathrm{Li}_2\left(\frac{1}{2}-\frac{1}{2}\sqrt{1-4\eta x(1-x)}\right) \\
&-4\mathrm{Li}_2\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-4\eta x(1-x)}\right) - 4\left(2\eta x^2 - 2\eta x - 3\right)\Bigg] + 4\left(\ln(2) + 1 + \sqrt{1-4\eta x(1-x)}\right)\mathrm{Li}_2\left(\frac{1}{2}-\frac{1}{2}\sqrt{1-4\eta x(1-x)}\right) \\
&+\ln\left(1-\sqrt{1-4\eta x(1-x)}\right)\Bigg[-10\ln^2(2) + (4-4\ln(2))\left(\sqrt{1-4\eta x(1-x)} + \ln(\eta x(1-x))\right) - 4\left(2\eta x^2 - 2\eta x - 1\right) \\
&+\left(8\ln(2) + 4\sqrt{1-4\eta x(1-x)}\right)\ln\left(1+\sqrt{1-4\eta x(1-x)}\right) + 4\ln(2) - \ln^2(\eta x(1-x)) - 4\ln^2\left(1+\sqrt{1-4\eta x(1-x)}\right) - 4\zeta_2\Bigg] \\
&+\left(6\ln^2(2) - 8\right)\sqrt{1-4\eta x(1-x)} + \left(2\ln^2(2) - 4\ln(2) + 2\zeta_2 - 8\right)\ln(\eta x(1-x)) + (4\ln(2)+4)\mathrm{Li}_2\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-4\eta x(1-x)}\right) \\
&+16\ln(2)\left(\eta x^2 - \eta x - 1\right) + \left(\ln(2) - \sqrt{1-4\eta x(1-x)} - 1\right)\ln^2(\eta x(1-x)) + (4\ln(2)-4)\zeta_2 + 24\eta x^2 - 24\eta x + 8 \\
&+\left(6\ln(2) + 2\ln(\eta x(1-x)) - 2\right)\ln^2\left(1-\sqrt{1-4\eta x(1-x)}\right) + \left(6\ln(2) + 2\sqrt{1-4\eta x(1-x)} + 2\right)\ln^2\left(1+\sqrt{1-4\eta x(1-x)}\right) \\
&+2\mathrm{Li}_3\left(-\frac{1+\sqrt{1-4\eta x(1-x)}}{1-\sqrt{1-4\eta x(1-x)}}\right) + 4\mathrm{Li}_3\left(\frac{1}{2}-\frac{1}{2}\sqrt{1-4\eta x(1-x)}\right) + 4\mathrm{Li}_3\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-4\eta x(1-x)}\right) + \frac{1}{2}\ln^3(\eta x(1-x)) \\
&-2\ln^3\left(1-\sqrt{1-4\eta x(1-x)}\right) - \frac{2}{3}\ln^3\left(1+\sqrt{1-4\eta x(1-x)}\right) - 4\zeta_3
\end{aligned}
$$

The integrals can therefore be performed numerically without problems.

Summary

- In the calculation of 3-loop heavy flavor corrections to DIS Wilson coefficients we need to consider the contribution from diagrams with two different masses since $m_c^2/m_b^2 \sim 0.1$.
- We have computed the 3-loop 2-mass contributions to $A_{qq}^{(3),\mathrm{NS}}$ and $A_{gq}^{(3)}$ for general values of the Mellin variable N in analytic form.
- We have calculated scalar diagrams for $A_{gg}^{(3)}$, witnessing the appearance of generalized harmonic sums and generalized harmonic polylogarithms.
- The operator matrix element $A_{Qq}^{(3),\mathrm{PS}}$ has been computed. The result is given in terms of 11 GHPLs with the alphabet

$$\left\{\frac{1}{\tau},\ \frac{\sqrt{1-4\tau}}{\tau},\ \sqrt{4-\tau}\sqrt{\tau}\right\}$$

- Different new Computer-algebra and mathematical technologies have been and continue to be developed.