Chiral Anomalies

I. Adler-Bell-Jackiw Anomaly:

- Triangle diagram:

Consider QED action

$$S_f[A,\psi,\bar{\psi}]$$

$$\|$$

$$S[A,\psi,\bar{\psi}] = -\frac{1}{4}\int d^4x F_{\mu\nu}F_{\mu\nu} - \int d^4x \bar{\psi}[\gamma_\mu(\partial_\mu - ieA_\mu) + m]\psi$$

Path integral

$$\int [dA]\,[d\psi d\bar{\psi}] e^{iS[A,\psi,\bar{\psi}]} = \int [dA] \exp\left[-\frac{i}{4}\int d^4x F_{\mu\nu}F_{\mu\nu} + i\Gamma[A]\right]$$

$$e^{i\Gamma[A]} = \int [d\psi d\bar{\psi}] e^{iS_f[A,\psi,\bar{\psi}]} = \det[\gamma_\mu(\partial_\mu - ieA_\mu) + m]$$

$\Gamma[A] =$ sum of diagrams with one fermion loop decorated by external photon vertices

- Triangle diagram:(cont.)

Classical symmetry in the chiral limit $m = 0$:

$U_V(1)$: $\psi \to e^{-i\alpha}\psi \qquad \bar{\psi} \to \bar{\psi}e^{i\alpha}$

$U_A(1)$: $\psi \to e^{-i\alpha\gamma_5}\psi \qquad \bar{\psi} \to \bar{\psi}e^{-i\alpha\gamma_5}$

$$S[A,\psi,\bar{\psi}] \to S[A,\psi,\bar{\psi}]$$

Currents: $J_\mu = i\bar{\psi}\gamma_\mu\psi \qquad J_{5\mu} = i\bar{\psi}\gamma_\mu\gamma_5\psi$

$\partial_\mu J_\mu = 0 \implies$ electric charge conservation

$\partial_\mu J_{5\mu} = 0 \implies$ axial charge conservation

Quantum mechanical:

UV divergence demands regulators.

$U_V(1)$ has to be preserved because of gauge invariance.

$U_A(1)$ is explicitly broken by a gauge invariant regulator;

is not recovered when the regulator mass $M \to \infty \implies$ anomalous!

$$\partial_\mu J_\mu = 0 \quad \text{but} \quad \partial_\mu J_{5\mu} = \text{anomaly} \neq 0$$

- Triangle diagram:(cont.)

Beyond chiral limit: $m \neq 0$:

Classical: $\partial_\mu J_\mu = 0$ $\partial_\mu J_{5\mu} = 2im\bar{\psi}\gamma_5\psi$

Quantum: $\partial_\mu J_\mu = 0$ $\partial_\mu J_{5\mu} = 2im\bar{\psi}\gamma_5\psi$ + anomaly

Axial current in an external EM field

$$< J_{5\mu}(x) >^A \equiv \frac{\int [d\psi d\bar{\psi}] e^{iS_1[A,\psi,\bar{\psi}]} J_{5\mu}(x)}{\int [d\psi d\bar{\psi}] e^{iS_1[A,\psi,\bar{\psi}]}}$$

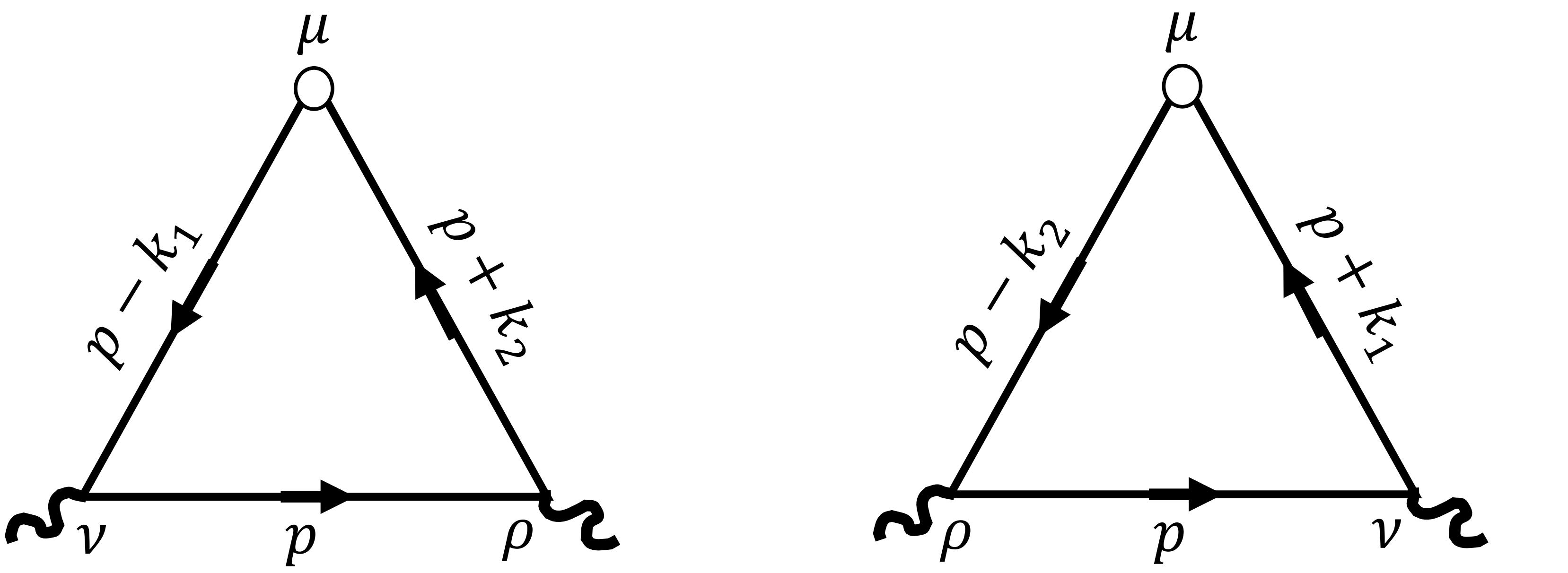
$$A_\mu(k) \equiv \int d^4x \, e^{-ik\cdot x} A_\mu(x)$$

$$= \frac{1}{2}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} e^{i(k_1+k_2)\cdot x} \Delta_{\mu\nu\rho}(k_1,k_2) A_\nu(k_1) A_\rho(k_2) + O(A^3)$$

$$\partial_\mu < J_{5\mu}(x) >^A =$$

$$\frac{i}{2}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} e^{i(k_1+k_2)\cdot x} (k_1+k_2)_\mu \Delta_{\mu\nu\rho}(k_1,k_2) A_\nu(k_1) A_\rho(k_2) + O(A^3)$$

- <u>Triangle diagram:(cont.)</u>



$$\Delta_{\mu\nu\rho}(k_1,k_2|m) = -e^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\gamma_5\gamma_\mu \left(\frac{1}{\not{p}+\not{k}_2-m}\gamma_\rho \frac{1}{\not{p}-m}\gamma_\nu \frac{1}{\not{p}-\not{k}_1-m} + \frac{1}{\not{p}+\not{k}_1-m}\gamma_\nu \frac{1}{\not{p}-m}\gamma_\rho \frac{1}{\not{p}-\not{k}_2-m} \right)$$

- <u>Triangle diagram:(cont.)</u>

$$(k_1+k_2)_\mu \Delta_{\mu\nu\rho}(k_1,k_2|m) = -ie^2\int \frac{d^4p}{(2\pi)^4}\mathrm{tr}\gamma_5(\not{k}_1+\not{k}_2)\times$$

$$\left(\frac{1}{\not{p}+\not{k}_2-m}\gamma_\rho\frac{1}{\not{p}-m}\gamma_\nu\frac{1}{\not{p}-\not{k}_1-m}+\frac{1}{\not{p}+\not{k}_1-m}\gamma_\nu\frac{1}{\not{p}-m}\gamma_\rho\frac{1}{\not{p}-\not{k}_2-m}\right)$$

Using the identity

$$\frac{1}{\not{p}+\not{q}-m}\gamma_5\not{q}\frac{1}{\not{p}-m} = -\gamma_5\frac{1}{\not{p}-m}-\frac{1}{\not{p}+\not{q}-m}\gamma_5$$
$$-2m\frac{1}{\not{p}+\not{q}-m}\gamma_5\frac{1}{\not{p}-m}$$

We find

$$(k_1+k_2)_\mu \Delta_{\mu\nu\rho}(k_1,k_2|m) = I_{\nu\rho}(m)+2im\Delta_{\nu\rho}(m)$$

- Triangle diagram:(cont.)

$$I_{\nu\rho}(m) = ie^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\gamma_5 \left(-\gamma_\nu \frac{1}{\not{p} + \not{k}_2 - m} \gamma_\rho \frac{1}{\not{p} - m} + \gamma_\nu \frac{1}{\not{p} - m} \gamma_\rho \frac{1}{\not{p} - \not{k}_2 - m} \right.$$

$$\left. -\gamma_\rho \frac{1}{\not{p} - m} \gamma_\nu \frac{1}{\not{p} - \not{k}_1 - m} + \gamma_\rho \frac{1}{\not{p} + \not{k}_1 - m} \gamma_\nu \frac{1}{\not{p} - m} \right)$$

The 1st (3nd) term differ from 2nd (4th) term by a shift of integration momentum,
but each integral is linearly divergent $\Longrightarrow I_{\nu\rho}(m) \neq 0.$

$$\Delta_{\nu\rho}(m) = e^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\gamma_5 \left(\frac{1}{\not{p} + \not{k}_2 - m} \gamma_\rho \frac{1}{\not{p} - m} \gamma_\nu \frac{1}{\not{p} - \not{k}_1 - m} \right.$$

$$\left. + \frac{1}{\not{p} + \not{k}_1 - m} \gamma_\nu \frac{1}{\not{p} - m} \gamma_\rho \frac{1}{\not{p} - \not{k}_2 - m} \right)$$

Convergent integral

- Triangle diagram:(cont.)

Pauli-Villars regularization: Preserve the vector current conservation

$$\Delta^R_{\mu\nu\rho}(k_1,k_2|m) \equiv \lim_{M\to\infty}\left[\Delta_{\mu\nu\rho}(k_1,k_2|m) - \Delta_{\mu\nu\rho}(k_1,k_2|M)\right]$$
$$= \lim_{M\to\infty}\left[I_{\nu\rho}(m) - I_{\nu\rho}(M)\right] + 2i\lim_{M\to\infty}\left[m\Delta_{\nu\rho}(m) - M\Delta_{\nu\rho}(M)\right]$$

Shift integration momentum becomes legitimate!

$$\lim_{M\to\infty}\left[I_{\nu\rho}(m) - I_{\nu\rho}(M)\right] = \lim_{M\to\infty}\int\frac{d^4p}{(2\pi)^4}\left[\mathcal{J}_{\nu\rho}(p|m) - \mathcal{J}_{\nu\rho}(p|M)\right] = 0$$

$$(k_1+k_2)_\mu\Delta^R_{\mu\nu\rho}(k_1,k_2|m) = 2im\Delta_{\nu\rho}(m) - 2i\lim_{M\to\infty}M\Delta_{\nu\rho}(M)$$

Naïve Ward identity

Anomalous Ward identity

$$(k_1+k_2)_\mu\Delta^R_{\mu\nu\rho}(k_1,k_2|m) = 2im\Delta_{\nu\rho}(m) + \frac{e^2}{2\pi^2}\epsilon_{\nu\rho\alpha\beta}k_{1\alpha}k_{2\beta}$$

Anomaly!

- Triangle diagram:(cont.)

$$-2iM\Delta_{\nu\rho}(M) = 2ie^2M\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\gamma_5\left\{\frac{(\not{p}+\not{k}_2+M)\gamma_\rho(\not{p}+M)\gamma_\nu(\not{p}-\not{k}_1+M)}{[(p+k_2)^2+M^2](p^2+M^2)[(p-k_1)^2+M^2]}\right.$$

$$\left.+(k_1\leftrightarrow k_2,\nu\leftrightarrow\rho)\right\}$$

$$= 4ie^2M\int_0^1 dx\int_0^{1-x}dy\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\gamma_5\left\{\frac{(\not{p}+\not{k}_2+M)\gamma_\rho(\not{p}+M)\gamma_\nu(\not{p}-\not{k}_1+M)}{[(p+k_2)^2y+(p-k_1)^2x+p^2(1-x-y)+M^2]^3}\right.$$

$$\left.+(k_1\leftrightarrow k_2,\nu\leftrightarrow\rho)\right\}$$

$$= 4ie^2M^2\int_0^1 dx\int_0^{1-x}dy\int\frac{d^4l}{(2\pi)^4}\frac{N_{\rho\nu}(l,k_1,k_2)+N_{\nu\rho}(l,k_2,k_1)}{[l^2+M^2+k_1^2x+k_2^2y-(k_1x-k_2y)^2]^3}$$

$$N_{\rho\nu}(l,k_1,k_2)\equiv\mathrm{tr}\gamma_5\gamma_\rho(\not{l}+\not{k}_1x-\not{k}_2y)\gamma_\nu[\not{l}-\not{k}_1(1-x)-\not{k}_2y]$$

Working out the trace

$$-2iM\Delta_{\nu\rho}(M) = \frac{4ie^2M^2}{\pi^2}\epsilon_{\nu\rho\alpha\beta}k_{1\alpha}k_{2\beta}\int_0^1 dx\int_0^{1-x}dy\frac{1}{M^2+k_1^2x+k_2^2y-(k_1x-k_2y)^2}$$

$$\longrightarrow\frac{e^2}{2\pi^2}\epsilon_{\nu\rho\alpha\beta}k_{1\alpha}k_{2\beta}\qquad\text{as } M\to\infty$$

- Triangle diagram:(cont.)

Coordinate space:

$$\partial_\mu J_{5\mu} = 2im\bar{\psi}\gamma_5\psi + \frac{ie^2}{16\pi^2}\epsilon_{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}$$

Generalization to QCD+QED

$$\partial_\mu J_{5\mu} = 2im\bar{\psi}\gamma_5\psi + i\frac{N_f g^2}{32\pi^2}\epsilon_{\mu\nu\rho\lambda}F^l_{\mu\nu}F^l_{\rho\lambda} + \frac{i\eta e^2}{16\pi^2}\epsilon_{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$F^l_{\mu\nu} = \partial_\mu A^l_\nu - \partial_\nu A^l_\mu + gf^{lmn}A^m_\mu A^n_\nu$

color-flavor factor

$$\eta = N_c \sum_f q_f^2$$

UV divergence $\Longrightarrow$ Anomaly

$\Longrightarrow$ Anomaly is **independent of temperature and chemical potential.**

- <u>Other approches:</u>

Point-splitting: *Schwinger*

$$J_{5\mu}(x) = \lim_{\delta\to 0} J_{5\mu}(x,\delta)$$

$$J_{5\mu}(x,\delta) \equiv iU(x_+,x_-)\bar{\psi}(x_+)\gamma_\mu\gamma_5\psi(x_-)$$

$$U(x_+,x_-) = \exp\left[ie\int_{x_-}^{x_+} d\xi_\rho\, A_\rho(\xi)\right] \quad x_\pm = x \pm \frac{\delta}{2}$$

Maintain gauge invariance

$$< J_{5\mu}(x,\delta) >^A = -iU(x_+,x_-)\mathrm{tr}\gamma_\mu\gamma_5 S_A(x_-,x_+)$$

where $S_A(x_-,x_+)$ = Dirac propagator in an external EM field

$$-\gamma_\mu(\partial_\mu - ieA_\mu)S_A(x_-,x_+) = i\delta^4(x-y) \qquad m=0$$

$$S_A(x_-,x_+) = -i < x\left|\frac{1}{\gamma_\mu(\partial_\mu - ieA_\mu)}\right|y >$$

- <u>*Other approaches (cont.)*</u>**:**

$$\partial_\mu U(x_+, x_-) = ie\delta_\rho \partial_\mu A_\rho(x) + O(\delta^2)$$

$$-\partial_\mu S_A(x_-, x_+) = \left[ie\delta_\rho \partial_\rho A_\mu(x) + O(\delta^3)\right]$$

$$\partial_\mu < J_{5\mu}(x,\delta) >^A = eF_{\mu\rho}(x)\delta_\rho \mathrm{tr}\gamma_\mu\gamma_5 S_A(x_-, x_+)$$

$$= -e^2 F_{\mu\rho}(x)\delta_\rho \int d^4y\, \mathrm{tr}\gamma_\mu\gamma_5 S_F(x_- - y)\gamma_\nu A_\nu(y) S_F(y - x_+)$$

$$= -\frac{e^2}{16\pi^2} e^2 F_{\mu\rho}(x)\delta_\rho \epsilon_{\mu\alpha\nu\beta} \int d^4y \left[\frac{\partial}{\partial x_{-\alpha}} \frac{1}{(x_1 - y)^2}\right] \frac{1}{(y - x_+)^2} A_\beta(y)$$

$$= \frac{ie^2}{4\pi^2\delta^2} F_{\mu\rho}F_{\beta\nu}\epsilon_{\mu\alpha\nu\beta}\delta_\rho\delta_\alpha = \frac{ie^2}{16\pi^2}\epsilon_{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}$$

where Schouten identity

$$\epsilon_{\mu\alpha\nu\beta}\delta_\rho + \epsilon_{\rho\mu\alpha\nu}\delta_\beta + \epsilon_{\beta\rho\mu\alpha}\delta_\nu + \epsilon_{\nu\beta\rho\mu}\delta_\alpha + \epsilon_{\alpha\nu\beta\rho}\delta_\mu = 0$$

has been employed.

- *Other approaches (cont.)***:**

Change of the path integral measure: *Fujikawa*

$$S_1[A,\psi,\bar{\psi}] = i\int d^4x\bar{\psi}\not{D}\psi \qquad \not{D} = -i\gamma_\mu(\partial_\mu - ieA_\mu)$$

$$Z[A] = \int [d\psi d\bar{\psi}]e^{iS_1[A,\psi,\bar{\psi}]}$$

$$\psi(x) \to e^{-i\alpha(x)\gamma_5}\psi(x) \qquad \bar{\psi}(x) \to \bar{\psi}(x)e^{-i\alpha(x)\gamma_5}$$

$$\frac{\delta\psi_\alpha(x)}{\delta\psi_\beta(x')} = \delta_{\alpha\beta}e^{-i\alpha(x)\gamma_5}\delta(x-x') \equiv (\mathcal{U}^A)_{\alpha\beta}(x,x')$$

$$\frac{\delta\bar{\psi}_\alpha(x)}{\delta\bar{\psi}_\beta(x')} = \delta_{\alpha\beta}e^{-i\alpha(x)\gamma_5}\delta(x-x') \equiv (\mathcal{U}^A)_{\alpha\beta}(x,x')$$

- *Other approaches (cont.):*

$$S_1[A,\psi,\bar{\psi}] \to S_1[A,\psi,\bar{\psi}] - \int d^4x\alpha\, \partial_\mu J_{5\mu}$$

$$[d\psi d\bar{\psi}] \to [d\psi d\bar{\psi}](\det\mathcal{U}^A)^2 \equiv [d\psi d\bar{\psi}]e^{i\int d^4x\alpha d}$$

$$d = i\delta^4(0)\mathrm{tr}\gamma_5 = \infty \times 0$$ Need regularization!

$$d \equiv i \lim_{\Lambda\to\infty} \mathrm{Tr}\gamma_5 f(\Lambda^{-2}\not{D}^2) = \frac{ie^2}{16\pi^2}\epsilon_{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}$$

$$Z[A] \to \int [d\psi d\bar{\psi}]\, e^{iS_1[A,\psi,\bar{\psi}]+\int d^4x\alpha[d-\partial_\mu J_{5\mu}]} = Z[A]$$

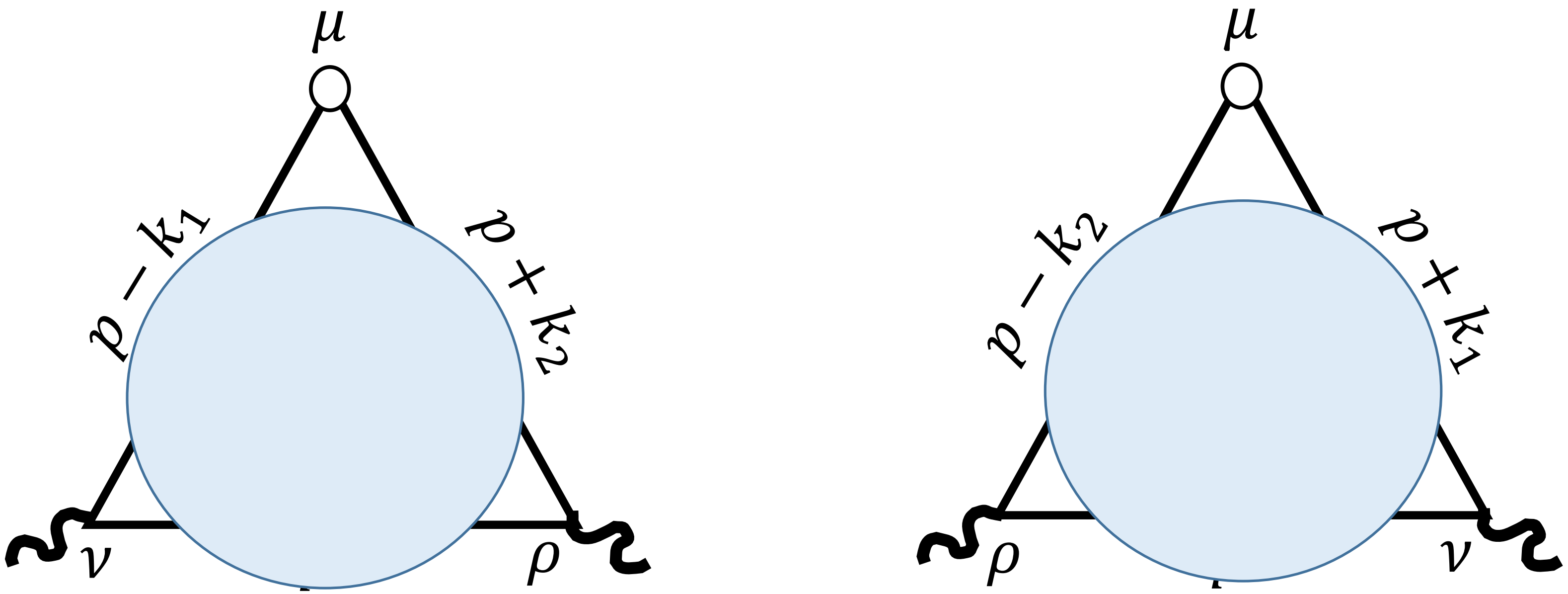
$$\Longrightarrow \qquad \partial_\mu < J_{5\mu}(x) >^A = \frac{ie^2}{16\pi^2}\epsilon_{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}$$

In contrast: $\psi \to e^{-i\alpha}\psi \qquad \bar{\psi} \to \bar{\psi}e^{i\alpha}$ for $U_V(1)$

$[d\psi d\bar{\psi}] \to [d\psi d\bar{\psi}] \quad \Longrightarrow$ No anomaly

- Non-renormalization

Multi-loop diagrams can be regularized by Pauli-Villars scheme applied to internal Bose lines without offsetting gauge invariance. Therefore



$$(k_1+k_2)_\mu \Delta^R_{\mu\nu\rho}(k_1,k_2|m) = 2i\Delta_{\nu\rho}(m) + \frac{e^2}{2\pi^2}\epsilon_{\nu\rho\alpha\beta}k_{1\alpha}k_{2\beta}$$

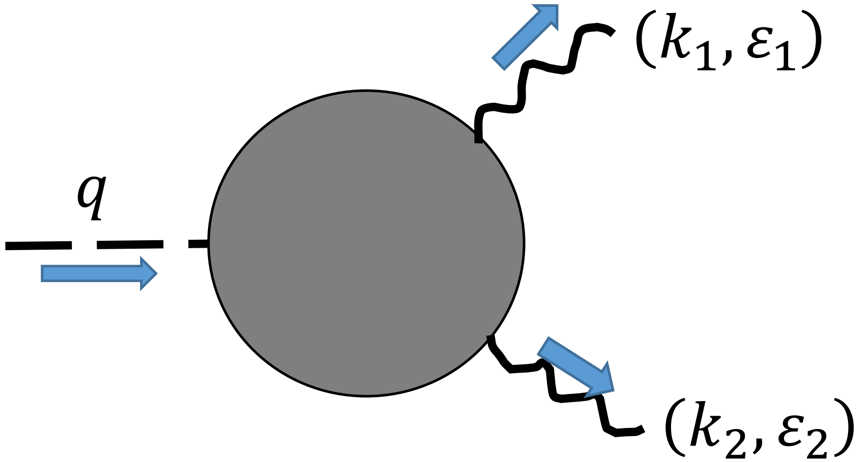
from one-loop only

II. Applications:

- $\underline{\pi^0 \longrightarrow 2\gamma}$

  Decay amplitude

$$\mathcal{T} = \lim_{q^2 \to -m_\pi^2} \mathcal{F}(q^2)$$



$$\mathcal{F}(q^2) = (q^2 + m_\pi^2) e^2 \varepsilon_{1\mu} \varepsilon_{2\nu} \int d^4x d^4y < 0 \left| T J_\mu(x) J_\nu(y) \pi(0) \right| 0 > e^{i(k_1 \cdot x + k_2 \cdot y)}$$

Naïve PCAC relation $\partial_\mu J_{5\mu} = m_\pi^2 f_\pi \pi$

$$\mathcal{F}(q^2) = \frac{q^2 + m_\pi^2}{f_\pi m_\pi^2} \varepsilon_{1\mu} \varepsilon_{2\nu} q_\rho T_{\mu\nu\rho}$$

$$T_{\mu\nu\rho} \equiv -ie^2 \int d^4x d^4y < 0 \left| T J_\mu(x) J_\nu(y) J_{5\rho}(0) \right| 0 > e^{i(k_1 \cdot x + k_2 \cdot y)}$$

- <u>$\pi^0 \longrightarrow 2\gamma$ (cont.):</u>

Electric current conservation + Bose symmetry

$$q_\rho T_{\mu\nu\rho} = \epsilon_{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta} \frac{q^2 F(q^2)}{q^2 + m_\pi^2}$$

where we assumed that π^0 is the only low-lying pole

$$\mathcal{F}(q^2) = \frac{q^2 + m_\pi^2}{f_\pi m_\pi^2} \varepsilon_{1\mu} \varepsilon_{2\nu} q_\rho T_{\mu\nu\rho} = \frac{1}{f_\pi m_\pi^2} \varepsilon_{1\mu} \varepsilon_{2\nu} \epsilon_{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta} q^2 F(q^2)$$

$$\Longrightarrow \qquad \mathcal{F}(0) = 0$$

Low π^0 mass $\quad \mathcal{F}(m_\pi^2) \cong \mathcal{F}(0) = 0 \quad$ disagree with experiments.

Modified PCAC relation

$$\partial_\mu J_{5\mu} = m_\pi^2 f_\pi \pi + \frac{ie^2}{32\pi^2} \epsilon_{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}$$

$\Longrightarrow$ Decay width $\cong$ 7.63 eV

Experimental value $= (7.31 \pm 1.5)$eV

- <u>Chiral magnetic effect:</u>

QCD+QED Lagrangian with ordinary and chiral chemical potentials

$$\mathcal{L}[A,\psi,\bar{\psi}] = -\frac{1}{4}F_{\mu\nu}^{l}F_{\mu\nu}^{l} - \frac{1}{4}F_{\mu\nu}F_{\mu\nu} - \bar{\psi}\gamma_\mu\left(\partial_\mu - igA_\mu^l T^l - ieA_\mu\right)\psi$$
$$+\mu\psi^\dagger\psi + \mu_5\psi^\dagger\gamma_5\psi + J_\mu A_\mu$$

+gauge fixing terms and counter terms

Both μ and μ_5 can be functions of space and time.

Generating functional of connected Green functions of photons

$$Z[J] = \int [dA^l][dA][d\psi d\bar{\psi}]\, e^{i\int d^4x \mathcal{L}[A,\psi,\bar{\psi}]}$$

$\int dt$ may follow a closed time path to handle the non-equilibrium case.

$$\mathcal{A}_\mu(x) = -i\frac{\delta \ln Z}{\delta J_\mu(x)}$$

- *Chiral magnetic effect (cont.):*

Quantum effective action:

$$\Gamma[\mathcal{A}] = -i \ln Z[J] - \int d^4x \, J_\mu \mathcal{A}_\mu$$

$$J_\mu(x) = \frac{\delta\Gamma}{\delta\mathcal{A}_\mu(x)} = \int \frac{d^4q}{(2\pi)^4} e^{iq\cdot x} J_\mu(q) = J_\mu^{(0)} + J_\mu^{(1)} + \cdots$$

$$J_\mu^{(0)} = O(\mathcal{A}), \qquad J_\mu^{(1)} = O(\mu_5\mathcal{A}), \qquad \ldots$$

$$J_\mu^{(1)}(q) = -i \int \frac{d^4k}{(2\pi)^4} \Delta_{4\mu\nu}(-q, q-k|0)\mu_5(k)\mathcal{A}_\nu(q-k)$$

$$\mathcal{A}_\mu(q) = \int d^4x \, e^{-iq\cdot x} \mathcal{A}_\mu(x) \qquad \mu_5(k) = \int d^4x \, e^{-iq\cdot x} \mu_5(x)$$

- *Chiral magnetic effect (cont.):*

Consider $\mathcal{A}_\mu(x) = \left(\vec{\mathcal{A}}(\vec{r}), 0\right), \;\; \mu_5(x) = \mu_5 e^{-i\omega t}$

$$\mu_5(k) = (2\pi)^4 \mu_5 \delta^3\left(\vec{k}\right)\delta(k_0 - \omega)$$

$$J_i(q) = -i\mu_5 \Delta_{4ij}(-q, q-k|0)\mathcal{A}_j(q-k)$$

$$k = \left(\vec{0}, i\omega\right) \quad q = (\vec{q}, i\omega)$$

Anomalous Ward identity

$$i\omega\Delta_{4ij}(-q, q-k|0) = -\frac{e^2}{2\pi^2}\epsilon_{ij\alpha\beta} q_\alpha (q-k)_\beta = i\frac{e^2}{2\pi^2}\omega\epsilon_{ijk} q_k$$

$$\Delta_{4ij}(-q, q-k|0) = \frac{e^2}{2\pi^2}\epsilon_{ijk} q_k$$

In the limit $\omega \to 0$ $\qquad J_i(q) = -i\frac{e^2\mu_5}{2\pi^2}\epsilon_{ijk} q_k \mathcal{A}_j(q)$

$$\vec{J}(\vec{r}) = \frac{e^2\mu_5}{2\pi^2}\vec{B}(\vec{r}) \qquad \vec{B} = \vec{\nabla}\times\vec{\mathcal{A}}$$

III. Miscellaneous Topics:

- Other chiral anomalies:

Non-Abelian gauge anomalies:

$$S_1[A,\psi,\bar{\psi}] = i\int d^4x(\bar{\psi}_L \not{D}_L \psi_L + \bar{\psi}_R \not{D}_R \psi_R)$$

$$\not{D}_L = \not{\partial} - i\not{A}^l T_L^l \qquad \not{D}_R = \not{\partial} - i\not{A}^l T_R^l$$

$$J_\mu = i\left(\bar{\psi}_L \gamma_\mu T_L \psi_L + \bar{\psi}_R \gamma_\mu T_R \psi_R\right)$$

$(T_L, T_R) =$ generators of gauge transformation of left and right fermions

$$D_\mu J_\mu^a \equiv \partial_\mu J_\mu^a + f^{abc} A_\mu^b J_\mu^c$$

Classical $\quad D_\mu J_\mu^a = 0$

Quantum $\quad D_\mu J_\mu^a = \kappa \epsilon_{\mu\nu\rho\lambda} d^{abc} F_{\mu\nu}^b F_{\rho\lambda}^c$

$$d^{abc} = \frac{1}{2}\left(\mathrm{tr} T_R^a \{T_R^b, T_R^c\} - \mathrm{tr} T_L^a \{T_L^b, T_L^c\}\right)$$

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- *Other chiral anomalies (cont.):*

Chiral anomaly in curved space:

$$D^\mu J_\mu^a = \kappa \epsilon^{\mu\nu\rho\lambda}\left(d^{abc} F_{\mu\nu}^b F_{\rho\lambda}^c + \kappa' b^a R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\lambda}\right)$$

$$D^\mu J_\mu^a \equiv \frac{1}{\sqrt{-g}} \partial_\mu\left(\sqrt{-g} J_\mu^a\right) + f^{abc} A_\mu^b J_\mu^c$$

$$R^\alpha{}_{\beta\mu\nu} = \text{Riemann tensor}$$

$$d^{abc} = \frac{1}{2}\left(\mathrm{tr} T_R^a \{T_R^b, T_R^c\} - \mathrm{tr} T_L^a \{T_L^b, T_L^c\}\right)$$

$$b^a = \mathrm{tr} T_R^a - \mathrm{tr} T_L^a$$

Gauge-gravity mixed anomaly

Pure gravitational anomaly:

$$D^\mu T_{\mu\nu} \neq 0$$

Exists only in $d = 4k + 2$ $(2,6,8,10, \dots)$ dimensions

- Covariant and consistent anomalies:

One-loop effective action:

$$\Gamma[A] = -i \ln \int [d\psi d\bar{\psi}] e^{iS_1[A,\psi,\bar{\psi}]}$$

An infinitesimal gauge transformation:

$$\delta A_\mu = D_\mu \theta$$

$$\delta\Gamma[A] = 2\mathrm{tr} \int d^4x\, D_\mu \theta \frac{\delta \Gamma}{\delta A_\mu(x)} \propto \mathrm{tr} \int d^4x\, \theta D_\mu J_\mu \propto tr \int d^4x \theta \mathcal{A}$$

Anomaly

Introduce the generator in functional space

$$\mathcal{G}(x) \equiv -D_\mu \frac{\delta}{\delta A_\mu}$$

Lie algebra:

$$[\mathcal{G}^a(x), \mathcal{G}^b(y)] = i f^{abc} \mathcal{G}^c(x) \delta^4(x-y)$$

- *Covariant and consistent anomalies (cont):*

Wess-Zumino consistent condition:

$$\mathcal{G}^a(x)\mathcal{A}^b(y) - \mathcal{G}^b(y)\mathcal{A}^a(x) = if^{abc}\mathcal{G}^c(x)\mathcal{A}^c(x)\delta^4(x-y)$$

For non-Abelian chiral gauge theory

---- Covariant anomaly is not consistent;

---- Consistent anomaly is not covariant;

---- They share the same group theoretic factor.

- Anomaly cancellation:

Benign anomaly: Anomaly of global symmetry

$\Longrightarrow$ Interesting physics, e.g., CME

Bad anomaly: Anomaly of gauge symmetry

$\Longrightarrow$ Jeopardize unitary and renormalizability

$\Longrightarrow$ **Has to be cancelled!**

Example 1: Electroweak theory, gauge group $= SU(2) \times U(1)$

$$d^{abc} = \frac{1}{2}\left(\mathrm{tr}T_R^a\{T_R^b, T_R^c\} - \mathrm{tr}T_L^a\{T_L^b, T_L^c\}\right) = 0$$

$$b^a = \mathrm{tr}T_R^a - \mathrm{tr}T_L^a = 0$$

$\Longrightarrow$ **Anomaly free!**

Example 2: Supersymmetric Yang-Mills in $d = 10$

Anomalies to be cancelled: **gauge, mixed and gravity**

Anomaly free gauge group: $SO(32)$ and $\boldsymbol{E_8} \times \boldsymbol{E_8}$

⇩

Superstring

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