Chiral Kinetic Theory by Wigner Function

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Outline

- Wigner function in quantum mechanics
- Wigner function in field theory: complex scalar and Dirac field
- Wigner function of chiral fermions in external electromagnetic field
- Chiral kinetic equation from 4D to 3D

Single-particle distribution function in classical theory

- Single particle distribution function in phase space $f(t,\mathbf{x},\mathbf{p})$

$$f(t,\mathbf{x},\mathbf{p})d^3xd^3p$$

particle number in volume element d^3xd^3p

- Macroscopic particle current

$$j^\mu(t,\mathbf{x}) = \int d^3p \frac{p^\mu}{E_p} f(t,\mathbf{x},\mathbf{p})$$

- Macroscopic energy-momentum tensor current

$E_p = \sqrt{m^2+\mathbf{p}^2}$

$$T^{\mu\nu}(t,\mathbf{x}) = \int d^3p \frac{p^\mu p^\nu}{E_p} f(t,\mathbf{x},\mathbf{p})$$

Heisenberg uncertainty principle

- Position and momentum of a particle cannot be determined simultaneously

$$\Delta x \cdot \Delta p \geqslant \frac{\hbar}{2}$$

- Quasi-classical approximation

$$l_{\mathrm{mfp}} \gg \lambda_{\mathrm{deBroglie}}$$

- Boltzmann equation

$$\begin{aligned} \frac{d}{dt} f(t, \mathbf{x}, \mathbf{p}) &= \left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{E_p} \cdot \nabla + \mathbf{F} \cdot \nabla_{\mathbf{p}} \right) f(t, \mathbf{x}, \mathbf{p}) = \mathcal{C}[f] \\ \mathcal{C}[f] &= \int_{124} d\tilde{\Gamma}_{1,2\to p,4} (f_1 f_2 F_p F_4 - F_1 F_2 f_p f_4) \end{aligned}$$

CM position and relative momentum

- Particle 1 with $\hat{\mathbf{x}}_1, \hat{\mathbf{p}}_1$, particle 2 with $\hat{\mathbf{x}}_2, \hat{\mathbf{p}}_2$

$$[\hat{x}_a^i, \hat{p}_b^j] = i\hbar\delta_{ab}\delta_{ij}, \quad [\hat{x}_a^i, \hat{x}_b^j] = 0, \quad [\hat{p}_a^i, \hat{p}_b^j] = 0, \quad (a, b = 1, 2)$$

- Position and momentum of the center of mass for Particles 1 and 2

$$\hat{\mathbf{X}} = \frac{1}{2}(\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2), \quad \hat{\mathbf{P}} = \hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2, \quad [\hat{X}^i, \hat{P}^j] = i\hbar\delta_{ij}$$

- Relative position and relative momentum of Particles 1 and 2

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2, \quad \hat{\mathbf{p}} = \frac{1}{2}(\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_2), \quad [\hat{x}^i, \hat{p}^j] = i\hbar\delta_{ij}$$

- CM position and relative momentum are commutable

$$[\hat{X}^i, \hat{p}^j] = 0$$

Definition of Wigner function

- Schroedinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(t,\mathbf{x}) = H\Psi(t,\mathbf{x}), \quad H = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})$$

- Definition of Wigner function through wave function

$$W(t,\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi\hbar)^3}\int d^3y\ e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}}\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)$$

- Wigner fucntion $W(t,\mathbf{x},\mathbf{p})$ in phase space is well defined

Property of Wigner function

- Properties of Wigner function

$$W^*(t,\mathbf{x},\mathbf{p}) = W(t,\mathbf{x},\mathbf{p})$$

$$\begin{aligned}
\rho &= \Psi^*(t,\mathbf{x})\Psi(t,\mathbf{x}) = \int d^3p W(t,\mathbf{x},\mathbf{p}) \\
\mathbf{j} &= \frac{i\hbar}{2m}\left(\Psi(t,\mathbf{x})\nabla\Psi^*(t,\mathbf{x}) - \Psi^*(t,\mathbf{x})\nabla\Psi(t,\mathbf{x})\right) \\
&= \int d^3p \frac{\mathbf{p}}{m} W(t,\mathbf{x},\mathbf{p})
\end{aligned}$$

Property of Wigner function

- Proof of the current

$$\int d^3p \frac{\mathbf{p}}{m} W(t,\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi\hbar)^3}\int d^3p \int d^3y \left(\frac{-i\hbar}{m}\right)\left(\nabla_{\mathbf{y}} e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}}\right) \times \Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)$$

$\overleftarrow{\nabla}_{\mathbf{y}} \to \overrightarrow{\nabla}_{\mathbf{y}}$

drop complete derivative in $\mathbf{y}$

$$= i\frac{\hbar}{m}\int d^3y \frac{1}{(2\pi\hbar)^3}\int d^3p e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}} \times \nabla_{\mathbf{y}}\left[\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right]$$

$\nabla_{\mathbf{y}} \to \pm\frac{1}{2}\nabla_{\mathbf{x}}$

$$= i\frac{\hbar}{2m}\int d^3y \delta^{(3)}(\mathbf{y}) \times \left[\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\nabla_{\mathbf{x}}\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right) - \Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\nabla_{\mathbf{x}}\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right]$$

EOM for Wigner function

- Equation of motion for the Wigner function

$$
\begin{aligned}
\frac{\partial}{\partial t} W(t, \mathbf{x}, \mathbf{p}) &= -\frac{1}{m}\mathbf{p} \cdot \nabla_{\mathbf{x}} W(t, \mathbf{x}, \mathbf{p}) + [\nabla_{\mathbf{x}} V(\mathbf{x})] \cdot \nabla_{\mathbf{p}} W(t, \mathbf{x}, \mathbf{p}) \\
&\quad + \sum_{n=1} \left(\frac{\hbar}{2}\right)^{2n} \frac{(-1)^n}{(2n+1)!} \\
&\quad \times (\nabla_{\mathbf{p}} \cdot \nabla_{\mathbf{x}})^{2n+1} V(\mathbf{x}) W(t, \mathbf{x}, \mathbf{p})
\end{aligned}
$$

- Proof.

$$
\begin{aligned}
\frac{\partial}{\partial t} W(t, \mathbf{x}, \mathbf{p}) &= \frac{1}{(2\pi\hbar)^3} \int d^3y \, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}} \left[\Psi\left(t, \mathbf{x} - \frac{\mathbf{y}}{2}\right) \partial_t \Psi^*\left(t, \mathbf{x} + \frac{\mathbf{y}}{2}\right) \right. \\
&\quad \left. + \Psi^*\left(t, \mathbf{x} + \frac{\mathbf{y}}{2}\right) \partial_t \Psi\left(t, \mathbf{x} - \frac{\mathbf{y}}{2}\right) \right]
\end{aligned}
$$

EOM for Wigner function

- Proof.

$$\begin{aligned}
\frac{\partial}{\partial t}W &= \frac{1}{(2\pi\hbar)^3}\int d^3y\, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}} \\
& \left[\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\left(-\frac{i\hbar}{2m}\nabla_{\mathbf{x}}^2+\frac{i}{\hbar}V(\mathbf{x}+\frac{\mathbf{y}}{2})\right)\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right. \\
& \left.+\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\left(\frac{i\hbar}{2m}\nabla_{\mathbf{x}}^2-\frac{i}{\hbar}V(\mathbf{x}-\frac{\mathbf{y}}{2})\right)\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right] \\
&= I(\nabla_{\mathbf{x}}^2)+I(V) \\
\partial_t\Psi^* &= \frac{i}{\hbar}H\Psi^* \\
\partial_t\Psi &= -\frac{i}{\hbar}H\Psi
\end{aligned}$$

EOM for Wigner function

$$
\begin{aligned}
I(\nabla_{\mathbf{x}}^2) &= \frac{1}{(2\pi\hbar)^3}\left(-\frac{i\hbar}{2m}\right)\nabla_{\mathbf{x}}\cdot\int d^3y\, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}} \\
&\quad \left[\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\nabla_{\mathbf{x}}\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right. \\
&\quad \left.-\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\nabla_{\mathbf{x}}\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right] \\
&= \frac{1}{(2\pi\hbar)^3}\left(-\frac{i\hbar}{m}\right)\nabla_{\mathbf{x}}\cdot\int d^3y\, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}} \\
&\quad \times\nabla_{\mathbf{y}}\left[\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right] \\
&= -\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{x}}W(t,\mathbf{x},\mathbf{p})
\end{aligned}
$$

EOM for Wigner function

$$
\begin{aligned}
I(V) &= \frac{1}{(2\pi\hbar)^3}\frac{i}{\hbar}\int d^3y\, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}}\left[V\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)-V\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right] \\
&\quad \times\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right) \\
&= \frac{1}{(2\pi\hbar)^3}\frac{i}{\hbar}\int d^3y\, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}}\sum_{n=0}\frac{2}{(2n+1)!} \\
&\quad \times\left(\frac{1}{2}\mathbf{y}\cdot\nabla_{\mathbf{x}}\right)^{2n+1}V(\mathbf{x})\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right) \\
&= \sum_{n=0}\left(\frac{\hbar}{2}\right)^{2n}\frac{(-1)^n}{(2n+1)!}\left(\nabla_{\mathbf{p}}\cdot\nabla_{\mathbf{x}}\right)^{2n+1}V(\mathbf{x})W(t,\mathbf{x},\mathbf{p}) \\
&= [\nabla_{\mathbf{x}}V(\mathbf{x})]\cdot\nabla_{\mathbf{p}}W(t,\mathbf{x},\mathbf{p})+O(\hbar^2)
\end{aligned}
$$

Liouville equation for Wigner function

- Wigner function satisfies the Liouville equation at classical limit $\hbar = 0$,

$$\frac{\partial}{\partial t}W(t,\mathbf{x},\mathbf{p}) + \frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{x}}W(t,\mathbf{x},\mathbf{p}) - [\nabla_{\mathbf{x}}V(\mathbf{x})]\cdot\nabla_{\mathbf{p}}W(t,\mathbf{x},\mathbf{p}) = 0$$

velocity force

- Quantum effect

$$-\sum_{n=1}\left(\frac{\hbar}{2}\right)^{2n}\frac{(-1)^n}{(2n+1)!}\left(\nabla_{\mathbf{p}}\cdot\nabla_{\mathbf{x}}\right)^{2n+1}V(\mathbf{x})W(t,\mathbf{x},\mathbf{p})$$

Wigner function: complex scalar field

- Lagrangian and Euler-Lagrange equation

$$\begin{aligned} \mathscr{L} &= (\partial^\mu \phi^\dagger)(\partial_\mu \phi) - m^2 \phi^\dagger \phi \\ & (\partial^2 + m^2)\phi = (\partial^2 + m^2)\phi^\dagger = 0 \end{aligned}$$

- Current and energy-moment tensor

$$\begin{aligned} j^\mu &= i\phi^\dagger(\overrightarrow{\partial^\mu} - \overleftarrow{\partial^\mu})\phi \\ T^{\mu\nu} &= \frac{i^2}{2}\phi^\dagger(\overrightarrow{\partial^\mu} - \overleftarrow{\partial^\mu})(\overrightarrow{\partial^\mu} - \overleftarrow{\partial^\mu})\phi \end{aligned}$$

Wigner function: complex scalar field

- Definition and EOM

$$W(x,p) = 2\int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \left\langle \phi^\dagger\left(x+\frac{y}{2}\right)\phi\left(x-\frac{y}{2}\right)\right\rangle$$

quasi-on-shell $\left(p^2 - m^2 - \frac{1}{4}\hbar^2\partial_x^2\right) W(x,p) = 0$

Vlasov eq. $p\cdot\partial_x W(x,p) = 0$

- Current and energy-moment tensor

$$j^\mu = \int d^4p\, p^\mu W(x,p)$$

$$T^{\mu\nu} = \int d^4p\, p^\mu p^\nu W(x,p)$$

Wigner function: complex scalar field

- Decomposition of Wigner function

$$\mathscr{L} = W^{(+)}(x,p) + W^{(-)}(x,p) + W^{(0)}(x,p)$$

- where

time-like p positive energy	$W^{(+)}(x,p)$	$\sim \Theta(p_0)\Theta(p^2)$
time-like p negative energy	$W^{(-)}(x,p)$	$\sim \Theta(-p_0)\Theta(p^2)$
space-like p	$W^{(0)}(x,p)$	$\sim \Theta(-p^2)$

Wigner function: complex scalar field

- If p is space like, $p^2 < 0$, we would have

$$\begin{aligned}
\frac{1}{4}\hbar^2\partial_x^2 W(x,p) &= (p^2 - m^2)W(x,p) \\
\left(\frac{\hbar}{2m}\right)^2 \left|\partial_x^2 W(x,p)\right| &= \left|1 - \frac{p^2}{m^2}\right| |W(x,p)| > |W(x,p)|
\end{aligned}$$

- which means that Wigner function has large fluctuation in the scale of Compton wave length. This arises from interference of wavepackets of postive and negative frequency, a quantum effect. We won't consider this case, instead we assume

$$\left(\frac{\hbar}{2m}\right)^2 \left|\partial_x^2 W(x,p)\right| \ll |W(x,p)|$$

Wigner function: complex scalar field

- So Wigner function is on mass-shell

$$
\begin{aligned}
(p^2 - m^2) W(x,p) &= 0 \\
W(x,p) &\sim \delta(p^2 - m^2)
\end{aligned}
$$

- We have

$$
\begin{aligned}
W(x,p) &= 2\delta(p^2 - m^2)\left[\Theta(p_0) f(x,p) + \Theta(-p_0)\bar{f}(x,-p)\right] \\
j^\mu &= \int d^4p p^\mu W(x,p) = \int d^3p \frac{p^\mu}{E_p}\left[f(x,p) - \bar{f}(x,p)\right] \\
T^{\mu\nu} &= \int d^4p\, p^\mu p^\nu W(x,p) = \int d^3p \frac{p^\mu p^\nu}{E_p}\left[f(x,p) + \bar{f}(x,p)\right]
\end{aligned}
$$

Wigner function: complex scalar field

- Use second quantization to compute $W(x, p)$

$$
\begin{aligned}
\phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_{\mathbf{p}} e^{-ip\cdot x} + b_{\mathbf{p}}^{\dagger} e^{ip\cdot x}) \\
&= \phi^{(+)}(x) + \phi^{(-)}(x)
\end{aligned}
$$

- We have

$$
\begin{aligned}
W^{(+)}(x, p) &= 2 \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \left\langle \phi^{(+)\dagger}\left(x + \frac{y}{2}\right) \phi^{(+)}\left(x - \frac{y}{2}\right) \right\rangle \\
W^{(-)}(x, p) &= 2 \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \left\langle \phi^{(-)\dagger}\left(x + \frac{y}{2}\right) \phi^{(-)}\left(x - \frac{y}{2}\right) \right\rangle
\end{aligned}
$$

Wigner function: complex scalar field

- Let's compute $W^{(+)}(x,p)$ and $W^{(-)}(x,p)$

$$W^{(+)}(x,p) = 2\int \frac{d^4y}{(2\pi)^4}\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{1}{\sqrt{2E_{p1}}}\frac{1}{\sqrt{2E_{p2}}} \times \exp\left[-ip\cdot y + ip_1\cdot(x+\frac{y}{2}) - ip_2\cdot(x-\frac{y}{2})\right] \times \left\langle : a^\dagger_{\mathbf{p}1} a_{\mathbf{p}2} : \right\rangle$$

$$\left\langle : a^\dagger_{\mathbf{p}} a_{\mathbf{k}} : \right\rangle = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k}) f^{(+)}_{\mathrm{BE}}(E_p)$$

$$= 2\int \frac{d^4y}{(2\pi)^4}\frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_{p1}}\exp\left[-i(p-p_1)\cdot y\right] f_{\mathrm{BE}}(E_{p1})$$

$$f^{(+)}_{\mathrm{BE}}(E_p) = \frac{1}{e^{\beta(E_p-\mu)}-1}$$

$$f^{(-)}_{\mathrm{BE}}(E_p) = \frac{1}{e^{\beta(E_p+\mu)}-1}$$

$$= \frac{2}{(2\pi)^3}\Theta(p_0)\delta(p^2-m^2) f^{(+)}_{\mathrm{BE}}(E_p)$$

$$W^{(-)}(x,p) = \frac{2}{(2\pi)^3}\Theta(-p_0)\delta(p^2-m^2) f^{(-)}_{\mathrm{BE}}(E_p)$$

Wigner function: free Dirac field

- Lagrangian and EOM

$$\mathscr{L} = \bar{\psi}(i\gamma^{\rho}\partial_{\rho} - m)\psi$$

$$(i\gamma^{\rho}\partial_{\rho} - m)\psi = 0$$

- Fermion number current, energy momentum tensor and angular momentum tensor

$$j^{\rho} = \bar{\psi}\gamma^{\rho}\psi$$

$$T^{\rho\sigma} = \frac{i}{2}\bar{\psi}\gamma^{\rho}(\overrightarrow{\partial^{\rho}} - \overleftarrow{\partial^{\rho}})\psi$$

$$M^{\rho\mu\nu} = x^{\mu}T^{\rho\nu} - x^{\nu}T^{\rho\mu} + \frac{1}{2}\bar{\psi}\{\gamma^{\rho}, S^{\mu\nu}\}\psi$$

$\{\gamma_0, \epsilon^{ijk}S^{ij}\} = \{\gamma_0, \Sigma_k\} = \frac{1}{2}\gamma^{k}\gamma_5$

$S^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]$

Wigner function: free Dirac field

- Definition of Wigner function

$$W_{\alpha\beta}(x,p) = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \left\langle \bar{\psi}_\beta\left(x+\frac{y}{2}\right)\psi_\alpha\left(x-\frac{y}{2}\right)\right\rangle$$

- Fermion number current and energy momentum tensor

$$j^\rho = \int d^4k \mathrm{Tr}[\gamma^\rho W(x,p)]$$
$$T^{\rho\sigma} = \int d^4k \mathrm{Tr}[\gamma^\rho W(x,p)]p^\sigma$$

Wigner function: free Dirac field

- EOM for Wigner function

$$\left[\gamma^{\rho}\left(\frac{1}{2}i\partial_{\rho}^{x}+p_{\rho}\right)-m\right]W(x,p) = 0$$

$$\begin{aligned} 0 &= \left[\gamma^{\rho}\left(\frac{1}{2}i\partial_{\rho}^{x}+p_{\rho}\right)+m\right]\left[\gamma^{\sigma}\left(\frac{1}{2}i\partial_{\sigma}^{x}+p_{\sigma}\right)-m\right]W(x,p) \\ &= \left[-\frac{1}{4}\partial_{x}^{2}+p^{2}-m^{2}+ip^{\sigma}\partial_{\sigma}^{x}\right]W(x,p) \\ &\Longrightarrow \left\{ \begin{array}{c} \left(-\frac{1}{4}\partial_{x}^{2}+p^{2}-m^{2}\right)W=0 \\ p^{\sigma}\partial_{\sigma}^{x}W=0 \end{array} \right. \end{aligned}$$

Wigner function: free Dirac field

- Proof of EOM for Wigner function

$$
\begin{aligned}
i\gamma_{ij}^{\rho}\frac{\partial}{\partial x^{\rho}}W_{jk}(x,p) &= i\gamma_{ij}^{\rho}\int\frac{d^{4}y}{(2\pi)^{4}}e^{-ip\cdot y} \\
&\quad \frac{\partial}{\partial x^{\rho}}\left\langle\bar{\psi}_{k}\left(x+\frac{y}{2}\right)\psi_{j}\left(x-\frac{y}{2}\right)\right\rangle \\
&= \int\frac{d^{4}y}{(2\pi)^{4}}e^{-ip\cdot y} \\
&\quad \times\left[2i\gamma_{ij}^{\rho}\left\langle\frac{\partial}{\partial y^{\rho}}\bar{\psi}_{k}\left(x+\frac{y}{2}\right)\psi_{j}\left(x-\frac{y}{2}\right)\right\rangle\right. \\
&\quad \left.+\left\langle\bar{\psi}_{k}\left(x+\frac{y}{2}\right)i\gamma_{ij}^{\rho}\frac{\partial}{\partial x^{\rho}}\psi_{j}\left(x-\frac{y}{2}\right)\right\rangle\right] \\
&= (-2p_{\rho}\gamma_{ij}^{\rho}+2m\delta_{ij})W_{jk}(x,p)
\end{aligned}
$$

Wigner function: free Dirac field

- For the first term, we have used

$$
\begin{aligned}
& e^{-ip\cdot y}2i\gamma^{\rho}_{ij}\left\langle \frac{\partial}{\partial y^{\rho}}\bar{\psi}_{k}\left(x+\frac{y}{2}\right)\psi_{j}\left(x-\frac{y}{2}\right)\right\rangle \\
\rightarrow\quad & -e^{-ip\cdot y}2p_{\rho}\gamma^{\rho}_{ij}\left\langle \bar{\psi}_{k}\left(x+\frac{y}{2}\right)\psi_{j}\left(x-\frac{y}{2}\right)\right\rangle \\
& +e^{-ip\cdot y}\left\langle \bar{\psi}_{k}\left(x+\frac{y}{2}\right)i\gamma^{\rho}_{ij}\frac{\partial}{\partial x^{\rho}}\psi_{j}\left(x-\frac{y}{2}\right)\right\rangle \\
\rightarrow\quad & (-2p_{\rho}\gamma^{\rho}_{ij}+m\delta_{ij})W_{jk}(x,p)
\end{aligned}
$$

Wigner function: free Dirac field

- Quantization for fermionc field

$$\begin{aligned}\psi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \\ &\quad \left[u_r(\mathbf{p})a_{\mathbf{p},r}e^{-ip\cdot x}+v_r(\mathbf{p})b^\dagger_{\mathbf{p},r}e^{ip\cdot x}\right] \\ &= \psi^{(+)}(x)+\psi^{(-)}(x)\end{aligned}$$

- Decomposition of Wigner function ($s=\pm$)

$$W_{ij}^{(s)}(x,p) = \int \frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\left\langle \bar{\psi}_j^{(s)}\left(x+\frac{y}{2}\right)\psi_i^{(s)}\left(x-\frac{y}{2}\right)\right\rangle$$

Wigner function: free Dirac field

- Calculate positive (negative) energy part

$$
\begin{aligned}
W_{ij}^{(+)}(x,p) &= \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \left\langle \bar{\psi}_j^{(+)}\left(x+\frac{y}{2}\right)\psi_i^{(+)}\left(x-\frac{y}{2}\right)\right\rangle \\
&= \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \int \frac{d^3p_1}{(2\pi)^3}\frac{1}{\sqrt{2E_{p1}}} \int \frac{d^3p_2}{(2\pi)^3}\frac{1}{\sqrt{2E_{p2}}} \\
&\quad \times \left\langle a_{\mathbf{p}1,r1}^{\dagger} a_{\mathbf{p}2,r2}\right\rangle e^{ip_1\cdot(x+y/2)} e^{-ip_2\cdot(x-y/2)} \\
&\quad \times u_{r2}(\mathbf{p}_2)\bar{u}_{r1}(\mathbf{p}_1) \\
&= \int \frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_{p1}} f_{\mathrm{FD}}(E_{p1}) \\
&\quad \times u_{r1}(\mathbf{p}_1)\bar{u}_{r1}(\mathbf{p}_1) \int \frac{d^4y}{(2\pi)^4} e^{-i(p-p_1)\cdot y}
\end{aligned}
$$

$\left\langle a_{\mathbf{p}1,r1}^{\dagger} a_{\mathbf{p}2,r2}\right\rangle$
$= (2\pi)^3\delta^{(3)}(\mathbf{p}_1-\mathbf{p}_2)$
$\delta_{r1,r2} f_{\mathrm{FD}}^{+}(E_{p1})$

Wigner function: free Dirac field

- Calculate positive (negative) energy part

$$
\begin{aligned}
W^{(+)}(x,p) &= \frac{1}{(2\pi)^3}\frac{1}{2E_p}\delta(p_0-E_p)f_{\mathrm{FD}}^{(+)}(E_p)\sum_r u_r(\mathbf{p})\bar{u}_r(\mathbf{p}) \\
&= \frac{1}{(2\pi)^3}\theta(p_0)\delta(p^2-m^2)f_{\mathrm{FD}}^{(+)}(E_p)\sum_r u_r(\mathbf{p})\bar{u}_r(\mathbf{p}) \\
W^{(-)}(x,p) &= \frac{1}{(2\pi)^3}\theta(-p_0)\delta(p^2-m^2)f_{\mathrm{FD}}^{(-)}(E_p)\sum_r v_r(-\mathbf{p})\bar{v}_r(-\mathbf{p})
\end{aligned}
$$

Chiral fermions in EM field

- Right-handed and left-handed sectors are decoupled,

$$\mathscr{L} = \bar{\psi} i\gamma^\rho D_\rho \psi = \bar{\psi}_R i\gamma^\rho D_\rho \psi_R + \bar{\psi}_L i\gamma^\rho D_\rho \psi_L$$

$D_\rho = \partial_\rho + ieA_\rho$

covariant derivative

R and L projector

$\Lambda_\pm = \frac{1}{2}\left(1 \pm \gamma^5\right)$

$\Lambda_\pm^2 = \Lambda_\pm , \Lambda_\pm \Lambda_\mp = 0$

$\gamma^\alpha \Lambda_\pm = \Lambda_\mp \gamma^\alpha$

$$\begin{aligned} \psi_{R/L} &= \Lambda_\pm \psi \\ \bar{\psi}_{R/L} &= \psi_{R/L}^\dagger \gamma_0 = \psi^\dagger \Lambda_\pm \gamma_0 = \bar{\psi}\Lambda_\mp \\ \bar{\psi}_R i\gamma^\rho \partial_\rho \psi_L &= \bar{\psi}\Lambda_- \Lambda_+ i\gamma^\rho D_\rho \psi = 0 \\ \bar{\psi}_L i\gamma^\rho \partial_\rho \psi_R &= \bar{\psi}\Lambda_+ \Lambda_- i\gamma^\rho D_\rho \psi = 0 \end{aligned}$$

Chiral fermions in EM field

- Dirac equations in Weyl representation

$$\begin{aligned} \gamma^\rho D_\rho \psi &= 0 \\ \left(\begin{array}{cc} 0 & \sigma^\rho D_\rho \\ \bar{\sigma}^\rho D_\rho & 0 \end{array}\right)\left(\begin{array}{c} \chi_L \\ \chi_R \end{array}\right) &= 0 \rightarrow \left\{\begin{array}{l} \sigma \cdot D\chi_R = 0 \\ \bar{\sigma} \cdot D\chi_L = 0 \end{array}\right. \end{aligned}$$

where

$$\begin{aligned} \psi &= \left(\begin{array}{c} \chi_L \\ \chi_R \end{array}\right), \quad \bar{\psi} = (\bar{\chi}_L, \bar{\chi}_R) \\ \gamma^\rho &= \left(\begin{array}{cc} 0 & \sigma^\rho \\ \bar{\sigma}^\rho & 0 \end{array}\right), \quad \sigma^\rho = (1, \boldsymbol{\sigma}),\ \bar{\sigma}^\rho = (1, -\boldsymbol{\sigma}) \\ \gamma_5 &= \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right), \quad \Lambda_R = \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right), \quad \Lambda_L = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right) \end{aligned}$$

Chiral fermions in Pauli space

- Dirac equation for right-handed (left-handed) fermions

$$
\begin{aligned}
\mathscr{L} &= \chi_R^\dagger i\sigma\cdot D\chi_R + \chi_L^\dagger i\bar{\sigma}\cdot D\chi_L \\
\sigma\cdot D\chi_R &= 0,\ \ \chi_R^\dagger \sigma\cdot \overleftarrow{D}^\dagger = 0 \\
\bar{\sigma}\cdot D\chi_L &= 0,\ \ \chi_L^\dagger \bar{\sigma}\cdot \overleftarrow{D}^\dagger = 0
\end{aligned}
$$

- Two-point Green function for right-handed fermions

$$
S_{ab}(x_1,x_2) = \left\langle \chi_b^\dagger(x_2)\chi_a(x_1)\right\rangle
$$

$$
\sigma\cdot D_{x_1} S(x_1,x_2) = 0
$$

Gradient expansion

- Center and distance of two points

$$X = \frac{1}{2}(x_1 + x_2),\ y = x_2 - x_1$$

$$\partial_{x_1} = \frac{1}{2}\partial_X - \partial_y,\ \partial_{x_2} = \frac{1}{2}\partial_X + \partial_y$$

- Two-point Green function for right-handed fermions

$$S_{ab}(x_1, x_2) \quad = \quad \left\langle \chi_b^\dagger(x_2)\chi_a(x_1) \right\rangle$$

$$\sigma \cdot D_{x_1} S(x_1, x_2) = 0$$

Gradient expansion

- Equation for 2-point function

$$\begin{aligned}\sigma\cdot D_{x_1}S(x_1,x_2) &= \sigma\cdot[\partial_{x_1}+ieA(x_1)]S(x_1,x_2)\\ &\approx \sigma\cdot\left[\frac{1}{2}\partial_X-\partial_y\right.\\ &\qquad \left.+ieA(X)-ie\frac{1}{2}y\cdot\partial_X A(X)\right]S(X,y)\end{aligned}$$

- Gauge invariant 2-point Green function with gauge link

$$\tilde{S}(X,y)=U(X,X-\frac{y}{2})S(X-\frac{y}{2},X+\frac{y}{2})U(X+\frac{y}{2},X)$$

$$U(x_1,x_2)=\mathcal{P}\exp\left[-ie\int_{x_2}^{x_1}dz\cdot A(z)\right]$$

Gradient expansion

- Weak background field

$$\tilde{S}(X,y) \simeq [1 - iey \cdot A(X)]S(X,y) + O(e^2)$$
$$S(X,y) \simeq [1 + iey \cdot A(X)]\tilde{S}(X,y) + O(e^2)$$

- Equation for gauge invariant 2-point function

$$\left[\tilde{S}(X,y)\frac{1}{2}\sigma \cdot \partial_X - \sigma \cdot \partial_y + \frac{1}{2}ie\sigma_\mu y_\nu F^{\mu\nu} + \frac{1}{2}ie(y \cdot A)(\sigma \cdot \partial_X) - ie(y \cdot A)(\sigma \cdot \partial_y)\right]\tilde{S}(X,y) = 0$$

Wigner Function in 4D

- Gauge invariant Wigner operator/function in backgroud EM fields

$$W(x,p) = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \left\langle \overline{\psi}_\beta \left(x+\frac{1}{2}y\right) U\left(A, x+\frac{1}{2}y, x-\frac{1}{2}y\right) \psi_\alpha\left(x-\frac{1}{2}y\right)\right\rangle$$

Heinz, 1983

Vasak, Gyulassy, Elze, 1986,1987

- Dirac equation in EM fields

$$[i\gamma^\mu D_\mu(x) - m]\psi(x) = 0, \quad \overline{\psi}(x)\left[i\gamma^\mu D_\mu^\dagger(x) - m\right] = 0$$

- In homogeneous EM fields, quantum kinetic equation for Wigner function of massless fermions can be derived from Dirac Eq.

$$\left[\gamma_\mu \left(p^\mu + \frac{1}{2} i \nabla^\mu\right) - m\right] W(x,p) = 0$$

$\nabla^\mu \equiv \partial_x^\mu - QF^{\mu\nu}\partial_\nu^p$

phase-space derivative

Wigner Function in 4D

- Wigner function decomposition in 16 generators of Clifford algebra

$$W(x,p) = \frac{1}{4}\left[\underset{\text{scalar}}{\mathscr{F}} + i\gamma_5 \underset{\text{p-scalar}}{\mathscr{P}} + \gamma^\mu \underset{\text{vector}}{\mathscr{V}_\mu} + \gamma_5\gamma^\mu \underset{\text{axial-vector}}{\mathscr{A}_\mu} + \frac{1}{2}\sigma^{\mu\nu} \underset{\text{tensor}}{\mathscr{S}_{\mu\nu}}\right]$$

- Currents and energy-momentum tensor can be obtained from Wigner function components

$$j^\mu = \int d^4p \mathscr{V}_\mu, \quad j_5^\mu = \int d^4p \mathscr{A}_\mu, \quad T^{\mu\nu} = \frac{1}{2}\int d^4p\, p^{(\mu}\mathscr{V}^{\nu)}$$

Vasak, Gyulassy, Elze, Ann. Phys.173,462(1987)

Elze, Gyulassy, Vasak, Nucl. Phys. B276,706(1986)

Equations for Wigner function components

Two sets of equations for WF components are derived from QKE for WF which are decoupled from each other. One set of equations is for the vector component $\mathscr{J}_{\mu}^{s}(x,p)$ for chiral fermions (with chirality $s=\pm$),

$$
\begin{aligned}
p^{\mu}\mathscr{J}_{\mu}^{s}(x,p) &= 0 \\
\nabla^{\mu}\mathscr{J}_{\mu}^{s}(x,p) &= 0 \\
2s(p^{\lambda}\mathscr{J}_{s}^{\rho}-p^{\rho}\mathscr{J}_{s}^{\lambda}) &= -\epsilon^{\mu\nu\lambda\rho}\nabla_{\mu}\mathscr{J}_{\nu}^{s}
\end{aligned}
$$

where $\mathscr{J}_{\mu}^{s}(x,p)$ are defined as

$$
\mathscr{J}_{\mu}^{s}(x,p) = \frac{1}{2}[\mathscr{V}_{\mu}(x,p)+s\mathscr{A}_{\mu}(x,p)]
$$

Perturbative solution

Perturbation in $(\partial_\mu^x)^n$ and $(F_{\mu\nu})^n$. The solution at the 0-th and 1-st order

$$\mathscr{J}^\rho_{(0)s}(x,p) = p^\rho f_s \delta(p^2)$$

$$\mathscr{J}^\rho_{(1)s}(x,p) = -\frac{s}{2}\tilde{\Omega}^{\rho\beta} p_\beta \frac{df_s}{dp_0}\delta(p^2) - \frac{s}{p^2} Q \tilde{F}^{\rho\lambda} p_\lambda f_s \delta(p^2)$$

$\tilde{\Omega}^{\rho\beta} = \frac{1}{2}\epsilon^{\rho\beta\mu\nu}\Omega_{\mu\nu}$ $\tilde{F}^{\rho\lambda} = \frac{1}{2}\epsilon^{\rho\lambda\mu\nu}F_{\mu\nu}$

$\Omega_{\mu\nu} = \frac{1}{2}(\partial_\mu u_\nu - \partial_\nu u_\mu)$

where f_s is the distribution function ($\mu_s = \mu + s\mu_5$, $p_0 = p\cdot u$),

$$f_s(x,p) = \frac{2}{(2\pi)^3}\left[\Theta(p_0) f_{\rm FD}(p_0 - \mu_s) + \Theta(-p_0) f_{\rm FD}(-p_0 + \mu_s)\right]$$

fermion part — anti-fermion part

$f_{\rm FD}(x) = \frac{1}{e^{\beta x}+1}$, Fermi-Dirac

Static-equilibrium conditions

The Wigner function solution is obtained under static equilibrium conditions

$$\underbrace{\Delta^{\sigma\alpha}\Delta^{\rho\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\Delta_{\alpha\beta}\Delta^{\rho\sigma}\partial_\rho u_\sigma\right) = 0}$$

$$T\Delta^{\sigma\rho}\partial_\rho\frac{\mu}{T} + QE^\sigma = 0$$

$$u^\rho\partial_\rho u^\sigma - \Delta^{\sigma\rho}\partial_\rho \ln T = 0$$

$$u^\sigma\partial_\sigma T + \frac{1}{3}T\Delta^{\rho\sigma}\partial_\rho u_\sigma = 0$$

$$\partial_\sigma\frac{\mu_5}{T} = 0, \quad u^\sigma\partial_\sigma\frac{\mu}{T} = 0$$

Like Killing condition
Becattini (2012)
Becattini, Bucciantini, Grossi, Tinti (2015)
Becattini, Grossi (2015)

Simplified conditions for constant temperature

With constant temperature, the above conditions are reduced to following simplified form

$$\partial_\sigma \mu_5 = 0, \quad \partial_\sigma T = 0$$
$$\partial^\rho u^\sigma + \partial^\sigma u^\rho = 0$$
$$\partial_\sigma \mu = -QE_\sigma$$

Like Killing condition
Becattini (2012)
Becattini, Bucciantini, Grossi, Tinti (2015)
Becattini, Grossi (2015)

Derivation of CCKE

We insert the Wigner function solutions into

$$\nabla_\mu[\mathscr{J}^\mu_{(0)s} + \mathscr{J}^\mu_{(1)s}] = 0$$

The zero-th order is evaluated as

$$\begin{aligned}\nabla_\mu[p^\mu f_s \delta(p^2)] &= (\partial^\mu_x - QF^{\mu\nu}\partial^p_\nu)[p^\mu f_s \delta(p^2)] \\ &= \delta(p^2) p^\mu \nabla_\mu f_s\end{aligned}$$

where we used $p^\mu \nabla_\mu \delta(p^2) = 0$ and $\nabla_\mu p^\mu = 0$. The first order can also be evaluated similarly.

Derivation of CCKE

We combine the 0-th and 1st order contribution we obtain

$$
\begin{aligned}
& \nabla_\mu[\mathscr{J}^\mu_{(0)s} + \mathscr{J}^\mu_{(1)s}] \\
= \quad & \delta(p^2)\Bigg[p^\mu \nabla_\mu f_s + sQ\frac{1}{p^2}\Omega^{\mu\lambda}p_\lambda \tilde{F}_{\mu\kappa}p^\kappa f_s' \\
& -\frac{s}{2}\tilde{\Omega}^{\mu\lambda}p_\lambda(\nabla_\mu f_s') - sQ\frac{1}{p^2}\tilde{F}^{\mu\lambda}p_\lambda(\nabla_\mu f_s)\Bigg] = 0
\end{aligned}
$$

Derivation of CCKE

Further simplification gives

$\nabla^\mu \equiv \partial_x^\mu - QF^{\mu\nu}\partial_\nu^p$

$$\left[\left(p^\mu - sQ\frac{1}{p^2}\tilde{F}^{\mu\lambda}p_\lambda - s\frac{p_0}{p^2}\tilde{\Omega}^{\mu\lambda}p_\lambda + \frac{s}{2}\tilde{\Omega}^{\mu\lambda}u_\lambda\right)\nabla_\mu f_s + \left(-\frac{s}{2}\tilde{\Omega}^{\mu\lambda}p_\lambda\Omega_{\mu\nu} + sQ\frac{1}{p^2}\Omega^{\mu\lambda}p_\lambda\tilde{F}_{\mu\kappa}p^\kappa u_\nu\right)\partial_p^\nu f_s\right]\delta(p^2) = 0,$$

which can be cast into the form

$$\delta(p^2)\left(\frac{dx^\mu}{d\tau}\partial_\mu^x f_s + \frac{dp^\mu}{d\tau}\partial_\mu^p f_s\right) = 0$$

Derivation of CCKE

Here $dx^\mu/d\tau$ and $dp^\mu/d\tau$ are given by

$$
\begin{aligned}
m_0\frac{dx^\mu}{d\tau} &= p^\mu - sQ\frac{1}{p^2}\tilde{F}^{\mu\lambda}p_\lambda + s\left(\frac{1}{2}-\frac{p_0^2}{p^2}\right)\omega^\mu + s\frac{p_0}{p^2}(p\cdot\omega)u^\mu \\
&\quad + X^\mu \\
m_0\frac{dp^\mu}{d\tau} &= QF^{\mu\nu}p_\nu + sQ^2\frac{p^\mu}{4p^2}F^{\nu\lambda}\tilde{F}_{\nu\lambda} \\
&\quad + \frac{1}{2}sQ(E\cdot\omega)u^\mu - sQ\frac{1}{p^2}(p\cdot\omega)(p\cdot E)u^\mu + sQ\frac{1}{p^2}p_0(p\cdot\omega)E^\mu \\
&\quad + Y^\mu
\end{aligned}
$$

Freedom of adding more terms

The new terms X^μ and Y^μ can be added

$$
\begin{aligned}
\frac{dx^\mu}{d\tau} \leftarrow X^\mu &= sC_1(p,u)\omega^\mu + sC_2(p,u)(p\cdot\omega)u^\mu \\
&\quad + sC_3(p,u)(p\cdot\omega)\bar{p}^\mu \\
\frac{dp^\mu}{d\tau} \leftarrow Y^\mu &= -sQ[C_1(p,u)(\omega\cdot E) + C_3(p,u)(p\cdot\omega)(p\cdot E)]u^\sigma \\
&\quad + sQC_4(p,\omega)\bar{p}^\sigma
\end{aligned}
$$

which satisfy the equation

$$X^\sigma\partial_\sigma^x f_s + Y^\sigma\partial_\sigma^p f_s = 0$$

Freedom of adding more terms

We assume following forms for these unknown functions

$$
\begin{aligned}
C_1(p,u) &= C_{10} + C_{11}\frac{p_0^2}{p^2} \\
C_2(p,u) &= C_{20}\frac{p_0}{p^2} + C_{21}\frac{1}{p_0} \\
C_3(p,u) &= C_{30}\frac{1}{p^2} \\
C_4(p,\omega) &= C_{40}(\omega\cdot E)\frac{1}{p_0} + C_{41}\frac{1}{p^2 p_0}(p\cdot\omega)(p\cdot E)
\end{aligned}
$$

Coefficients restricted by conservation of charge and energy-momentum

where $\{C_{10}, C_{11}, C_{20}, C_{21}, C_{30}, C_{40}, C_{41}\}$ are dimensionless constants to be determined.

Coefficients restricted by matching power and force

Constraints from currents

The currents for chiral fermions with chirality $s = \pm 1$ are given by,

$$\begin{aligned} j_s^\mu &= \int d^4p\delta(p^2)m_0\frac{dx^\mu}{d\tau}f_s \\ &= j_s^\mu(\mathrm{EM}) + j_s^\mu(\omega) \end{aligned}$$

where for the current from EM field

$$\begin{aligned} j_s^\mu(\mathrm{EM}) &= -sQ\int d^4p\delta(p^2)\frac{1}{p^2}\tilde{F}^{\mu\lambda}p_\lambda f_s = \xi_B^s B^\mu \\ j^\mu(\mathrm{EM}) &= (\xi_B^+ + \xi_B^-)B^\mu = \frac{Q}{2\pi^2}\mu_5 B^\mu \\ j_5^\mu(\mathrm{EM}) &= (\xi_B^+ - \xi_B^-)B^\mu = \frac{Q}{2\pi^2}\mu B^\mu \end{aligned}$$

Constraints from currents

For the current from vorticity

$$
\begin{aligned}
j_s^\mu(\omega) &= \left(C_{10} - \frac{1}{2}C_{11} + \frac{1}{2}C_{30} + 1\right)\xi_s\omega^\mu \\
j^\mu(\omega) &= j_+^\mu(\omega) + j_-^\mu(\omega) \to \frac{1}{\pi^2}\mu\mu_5\omega^\mu \\
j_5^\mu(\omega) &= j_+^\mu(\omega) - j_-^\mu(\omega) \\
&\to \left[\frac{1}{6}T^2 + \frac{1}{2\pi^2}(\mu^2 + \mu_5^2)\right]\omega^\mu
\end{aligned}
$$

which gives the constraints

$$
C_{10} - \frac{1}{2}C_{11} + \frac{1}{2}C_{30} = 0
$$

Constraints from stress tensor

The energy mometum tensor in the relativistic chiral kinetic theory can be obtained by

$$
\begin{aligned}
T^{\rho\sigma} &= \frac{1}{2} m_0 \int d^4 p \delta(p^2) \sum_s p^{(\rho} \frac{dx^{\sigma)}}{d\tau} f_s \\
&= T^{\rho\sigma}(\mathrm{EM}) + T^{\rho\sigma}(\omega) \\
&= \frac{1}{2} Q \xi u^{(\rho} B^{\sigma)} + \left(\frac{1}{2} C_{10} - \frac{1}{4} C_{11} + \frac{1}{4} C_{30} + \frac{1}{4} C_{20} - \frac{1}{6} C_{21} + \frac{3}{4} \right) n_5 u^{(\rho} \omega^{\sigma)} \\
&= \frac{1}{2} Q \xi u^{(\rho} B^{\sigma)} + n_5 u^{(\rho} \omega^{\sigma)}
\end{aligned}
$$

which gives the constraints

$$
\frac{1}{2} C_{10} - \frac{1}{4} C_{11} + \frac{1}{4} C_{30} + \frac{1}{4} C_{20} - \frac{1}{6} C_{21} + \frac{3}{4} = 1
$$

Constraints

We combine two constraints from conservation laws

$$
\begin{aligned}
C_{10} - \frac{1}{2} C_{11} + \frac{1}{2} C_{30} &= 0 \\
C_{20} - \frac{2}{3} C_{21} &= 1
\end{aligned}
$$

We see that C_{20} and C_{21} cannot all be zero.

Chiral kinetic equation: from 4D to 3D

With CCKE in 4D, we can obtain its 3D version by integrating over p_0,

$$
\begin{aligned}
I &= \int dp_0 \delta(p^2) \left[\frac{dx^\sigma}{d\tau} \partial_\sigma^x f_s + \frac{dp^\rho}{d\tau} \partial_\rho^p f_s \right] \\
&= \underset{\sigma=0}{I_{x0}} + \underset{\sigma=i}{I_x} + \underset{\rho=0}{I_{p0}} + \underset{\rho=i}{I_p}
\end{aligned}
$$

Each term has three contributions

$$
I_j = I_j(0) + I_j(\mathrm{EM}) + I_j(\omega)
$$

where $j = x0, p0, x, p$.

Extract $d\mathbf{p}/d\tau$

For an on-shell particle the energy is not an independent phase space variable, its rate $dE_p/d\tau$ from I_{p0} can be determined by

$$\frac{dE_p}{d\tau} = \frac{\mathbf{p}}{|\mathbf{p}|} \cdot \frac{d\mathbf{p}}{d\tau}$$

So in derivation of the 3D chiral kinetic equation from the 4D one, the p_0 degree of freedom is fixed and is not a kinematic variable in the 3D kinetic equation. We can extract the EM field contribution to $d\mathbf{p}/d\tau$ from $I_p(\mathrm{EM})$

$$\frac{d\mathbf{p}}{d\tau}(\mathrm{EM}) = Q\left(\mathbf{E} + \frac{\mathbf{p}}{|\mathbf{p}|} \times \mathbf{B}\right) + sQ^2(\mathbf{E}\cdot\mathbf{B})\mathbf{p}\frac{1}{2|\mathbf{p}|^3}$$

Extract $d\mathbf{p}/d\tau$

The energy rate from the vorticity can be obtained from $I_{p0}(\omega)$,

$$\begin{aligned} \frac{dE_p}{d\tau}(\omega) &= -sQ\left(C_{30}+1\right)\frac{1}{2|\mathbf{p}|}(\mathbf{E}\cdot\boldsymbol{\omega}) \\ &\quad +sQ(C_{30}+1)\frac{3}{2|\mathbf{p}|^3}(\mathbf{p}\cdot\boldsymbol{\omega})(\mathbf{p}\cdot\mathbf{E}) \end{aligned}$$

We can compare it with $d\mathbf{p}/d\tau$ extracted from $I_p(\omega)$,

$$\begin{aligned} \frac{d\mathbf{p}}{d\tau}(\omega) &= sQ\left[\frac{1}{|\mathbf{p}|^2}(\mathbf{p}\cdot\boldsymbol{\omega})\mathbf{E} - C_{40}\frac{1}{|\mathbf{p}|^2}(\boldsymbol{\omega}\cdot\mathbf{E})\mathbf{p} \right. \\ &\quad \left. -C_{41}\frac{2}{|\mathbf{p}|^4}(\mathbf{p}\cdot\boldsymbol{\omega})(\mathbf{p}\cdot\mathbf{E})\mathbf{p}\right] \end{aligned}$$

Extract $d\mathbf{p}/d\tau$

Enforcing $\frac{dE_p}{d\tau} = \frac{\mathbf{p}}{|\mathbf{p}|} \cdot \frac{d\mathbf{p}}{d\tau}$, the coefficients must satisfy

$$\begin{aligned} C_{40} &= \frac{1}{2}(C_{30}+1) \\ C_{41} &= \frac{1}{2} - \frac{3}{4}(C_{30}+1) \end{aligned}$$

So in this case we obtain $d\mathbf{p}/d\tau$ from vorticity as

$$\begin{aligned} \frac{d\mathbf{p}}{d\tau}(\omega) &= sQ\left[\frac{1}{|\mathbf{p}|^2}(\mathbf{p}\cdot\omega)\mathbf{E} - (C_{30}+1)\frac{1}{2|\mathbf{p}|^2}(\omega\cdot\mathbf{E})\mathbf{p} \right. \\ & \left. +(1+3C_{30})\frac{1}{2|\mathbf{p}|^4}(\mathbf{p}\cdot\omega)(\mathbf{p}\cdot\mathbf{E})\mathbf{p}\right] \end{aligned}$$

Extract $dx_0/d\tau$ and $d\mathbf{x}/d\tau$

From from I_{x0} and I_x we obtain

$$
\begin{aligned}
\frac{dx_0}{d\tau} &= 1+\mathscr{C}_B sQ(\boldsymbol{\Omega}\cdot\mathbf{B})+\left(4-\frac{2}{3}C_{21}\right)s|\mathbf{p}|(\boldsymbol{\Omega}\cdot\boldsymbol{\omega}),\\
\frac{d\mathbf{x}}{d\tau} &= \hat{\mathbf{p}}+sQ\mathbf{B}(\hat{\mathbf{p}}\cdot\boldsymbol{\Omega})+\mathscr{C}_E sQ(\mathbf{E}\times\boldsymbol{\Omega})\\
&\quad +s\left(1-\frac{1}{2}C_{30}\right)\frac{\boldsymbol{\omega}}{|\mathbf{p}|}+3sC_{30}(\boldsymbol{\Omega}\cdot\boldsymbol{\omega})\mathbf{p},
\end{aligned}
$$

where where we have used the constraints about C_{10}, C_{11}, C_{20},C_{21} and C_{30}.

CKE in 3D

[Son,Yamamoto 2012,2013; Stephanov,Yin 2012; Chen,Pu,Wang,Wang 2012; Chen,Son,Stephanov,Yee,Yin 2014; Kharzeev,Stephanov,Yee 2016; Hidaka,Pu,Yang 2016; Mueller,Venugopalan 2017]

$$
\begin{aligned}
\frac{dx_0}{d\tau} &= 1 + \mathscr{C}_B sQ(\mathbf{\Omega}\cdot\mathbf{B}) + \left(4 - \frac{2}{3}C_{21}\right) s|\mathbf{p}|(\mathbf{\Omega}\cdot\boldsymbol{\omega}) \\
\frac{d\mathbf{x}}{d\tau} &= \hat{\mathbf{p}} + sQ\mathbf{B}(\hat{\mathbf{p}}\cdot\mathbf{\Omega}) + \mathscr{C}_E sQ(\mathbf{E}\times\mathbf{\Omega}) \\
&\quad + \left(1 - \frac{1}{2}C_{30}\right) s\frac{\boldsymbol{\omega}}{|\mathbf{p}|} + 3C_{30}s(\mathbf{\Omega}\cdot\boldsymbol{\omega})\mathbf{p} \\
\frac{d\mathbf{p}}{d\tau} &= Q\left(\mathbf{E} + \frac{\mathbf{p}}{|\mathbf{p}|}\times\mathbf{B}\right) + sQ^2(\mathbf{E}\cdot\mathbf{B})\mathbf{p}\frac{1}{2|\mathbf{p}|^3} \\
&\quad + sQ\frac{1}{|\mathbf{p}|^2}\left[(\mathbf{p}\cdot\boldsymbol{\omega})\mathbf{E} - \frac{1}{2}(C_{30}+1)(\boldsymbol{\omega}\cdot\mathbf{E})\mathbf{p}\right. \\
&\quad \left. + \frac{1}{2}(1+3C_{30})\frac{1}{|\mathbf{p}|^2}(\mathbf{p}\cdot\boldsymbol{\omega})(\mathbf{p}\cdot\mathbf{E})\mathbf{p}\right]
\end{aligned}
$$

Coeffcients in 3D-CKE

- CKE in 3D is not uniquely determined due to free coefficients $\{C_{21}, C_{30}, \mathscr{C}_B, \mathscr{C}_E\}$
- $\mathscr{C}_B, \mathscr{C}_E = 1$ or 2. The freedom to choose $\mathscr{C}_B$ and $\mathscr{C}_E$ is because the integration over **p** of their corresponding terms are all vanishing, we can make choices as to keep or drop them following some physical reasonings. The $\mathscr{C}_E = 1$ is consistent to the previous result.
- With the coefficients $\{C_{21}, C_{30}, \mathscr{C}_B, \mathscr{C}_E\} = \{0, 0, 1, 1\}$ we reproduce our previosu result of [Chen,Pu,Wang,Wang, PRL 110, 262301(2013)].

Coefficients in 3D-CKE

Another possible choice of C_{30} is $C_{30} = 2/3$. In this case the vorticity terms in $d\mathbf{x}/d\tau$ read

$$\frac{d\mathbf{x}}{d\tau}(\omega) = \frac{s}{|\mathbf{p}|}(\hat{\mathbf{p}}\cdot\boldsymbol{\omega})\hat{\mathbf{p}} + \frac{s}{|\mathbf{p}|}\frac{2}{3}\boldsymbol{\omega}$$

When calculating the vorticity contribution to the current by the integration over $\mathbf{p}$ for $d\mathbf{x}/d\tau$ times the distribution function, one can verify that the first term of contributes to $1/3$ of the chiral vortical effect while the second term contributes to the rest $2/3$. In comparison to the result of [Kharzeev,Stephanov,Yee, 1612.01674], the $1/3$ contribution corresponds to that from the spin-vorticity coupling energy, while the rest $2/3$ contribution corresponds to that from the magnetization current.

Summary

- The CCKE is derived from 4D Wigner function by an improved perturbative method under the static equilibrium conditions. The CKE in 3D can be obtained by intergation over the time component of the 4-momentum.
- There is freedom to add more terms to the CCKE allowed by conservation laws.
- In the derivation of the 3-dimensional equation, there is also freedom to choose coefficients of some terms in $dx_0/d\tau$ and $d\mathbf{x}/d\tau$ whose 3-mometum integrals are vanishing.
- So the 3-dimensional chiral kinetic equation derived from the CCKE is not uniquely determined in our current approach.
- To go beyond the current approach, one needs a new way of building up the CKE in 3D from the CCKE or directly from the covariant equation for the Wigner function.