Chiral Kinetic Theory by Wigner Function

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Outline

- Wigner functions in quantum mechanics
- Wigner functions in field theory: free complex scalar and Dirac field
- Wigner functions of chiral fermions in external electromagnetic field [J. Gao et al., 1203.0725; J. Gao, Q. Wang, 1504.07334]
- Chiral kinetic equation from 4D to 3D [J. Chen et al., 1210.8312; J. Gao, S. Pu, Q. Wang, 1704.00244]
- Wigner functions for massive fermions and applications [R. Fang, et al., 1604.04036; R. Fang, et al., 1611.04670]

Single-particle distribution function in classical theory

- Single particle distribution function in phase space $f(t,\mathbf{x},\mathbf{p})$

$$f(t,\mathbf{x},\mathbf{p})d^3xd^3p$$

particle number in volume element d^3xd^3p

- Macroscopic particle current

$$j^\mu(t,\mathbf{x}) = \int d^3p \frac{p^\mu}{E_p} f(t,\mathbf{x},\mathbf{p}) \tag{1}$$

- Macroscopic energy-momentum tensor current

$E_p = \sqrt{m^2+\mathbf{p}^2}$

$$T^{\mu\nu}(t,\mathbf{x}) = \int d^3p \frac{p^\mu p^\nu}{E_p} f(t,\mathbf{x},\mathbf{p}) \tag{2}$$

Heisenberg uncertainty principle

- Position and momentum of a particle cannot be determined simultaneously

$$\Delta x \cdot \Delta p \geqslant \frac{\hbar}{2} \tag{3}$$

- Quasi-classical approximation

$$l_{\mathrm{mfp}} \gg \lambda_{\mathrm{deBroglie}} \tag{4}$$

- Boltzmann equation

$$\begin{aligned} \frac{d}{dt} f(t,\mathbf{x},\mathbf{p}) &= \left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{E_p} \cdot \nabla + \mathbf{F} \cdot \nabla_{\mathbf{p}}\right) f(t,\mathbf{x},\mathbf{p}) = \mathcal{C}[f] \\ \mathcal{C}[f] &= \int_{124} d\tilde{\Gamma}_{1,2\rightarrow p,4}(f_1 f_2 F_p F_4 - F_1 F_2 f_p f_4) \end{aligned} \tag{5}$$

CM position and relative momentum

- Particle 1 with $\hat{\mathbf{x}}_1, \hat{\mathbf{p}}_1$, particle 2 with $\hat{\mathbf{x}}_2, \hat{\mathbf{p}}_2$

$$[\hat{x}_a^i, \hat{p}_b^j] = i\hbar\delta_{ab}\delta_{ij}, \quad [\hat{x}_a^i, \hat{x}_b^j] = 0, \quad [\hat{p}_a^i, \hat{p}_b^j] = 0, \quad (a, b = 1, 2)$$

- Position and momentum of the center of mass for Particles 1 and 2

$$\hat{\mathbf{X}} = \frac{1}{2}(\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2), \quad \hat{\mathbf{P}} = \hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2, \quad [\hat{X}^i, \hat{P}^j] = i\hbar\delta_{ij}$$

- Relative position and relative momentum of Particles 1 and 2

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2, \quad \hat{\mathbf{p}} = \frac{1}{2}(\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_2), \quad [\hat{x}^i, \hat{p}^j] = i\hbar\delta_{ij}$$

- CM position and relative momentum are commutable

$$[\hat{X}^i, \hat{p}^j] = 0$$

Definition of Wigner function

- Schroedinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(t,\mathbf{x})=H\Psi(t,\mathbf{x}),\quad H=-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{x}) \tag{6}$$

- Definition of Wigner function through wave function

$$W(t,\mathbf{x},\mathbf{p})=\frac{1}{(2\pi\hbar)^3}\int d^3y\ e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}}\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right) \tag{7}$$

- Wigner fucntion $W(t,\mathbf{x},\mathbf{p})$ in phase space is well defined

Property of Wigner function

- Properties of Wigner function

$$W^*(t,\mathbf{x},\mathbf{p}) \quad = \quad W(t,\mathbf{x},\mathbf{p}) \tag{8}$$

$$\begin{aligned}
\rho &= \Psi^*(t,\mathbf{x})\Psi(t,\mathbf{x}) = \int d^3p W(t,\mathbf{x},\mathbf{p}) \\
\mathbf{j} &= \frac{i\hbar}{2m}\Big(\Psi(t,\mathbf{x})\nabla\Psi^*(t,\mathbf{x}) - \Psi^*(t,\mathbf{x})\nabla\Psi(t,\mathbf{x})\Big) \\
&= \int d^3p \frac{\mathbf{p}}{m} W(t,\mathbf{x},\mathbf{p})
\end{aligned} \tag{9}$$

Property of Wigner function

- Proof of the current

$$\begin{aligned}
\int d^3p \frac{\mathbf{p}}{m} W(t,\mathbf{x},\mathbf{p}) &= \frac{1}{(2\pi\hbar)^3}\int d^3p \int d^3y \left(\frac{-i\hbar}{m}\right)\left(\nabla_{\mathbf{y}} e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}}\right) \\
&\quad \times \Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right) \\
&= i\frac{\hbar}{m}\int d^3y \frac{1}{(2\pi\hbar)^3}\int d^3p e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}} \\
&\quad \times \nabla_{\mathbf{y}}\left[\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right] \\
&= i\frac{\hbar}{2m}\int d^3y \delta^{(3)}(\mathbf{y}) \\
&\quad \times \left[\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\nabla_{\mathbf{x}}\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right. \\
&\quad \left. -\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\nabla_{\mathbf{x}}\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right]
\end{aligned}$$

$\overleftarrow{\nabla}_{\mathbf{y}} \to \overrightarrow{\nabla}_{\mathbf{y}}$

drop complete derivative in $\mathbf{y}$

$\nabla_{\mathbf{y}} \to \pm\frac{1}{2}\nabla_{\mathbf{x}}$

EOM for Wigner function

- Equation of motion for the Wigner function

$$\begin{aligned}\frac{\partial}{\partial t}W(t,\mathbf{x},\mathbf{p}) &= -\frac{1}{m}\mathbf{p}\cdot\nabla_{\mathbf{x}}W(t,\mathbf{x},\mathbf{p})+[\nabla_{\mathbf{x}}V(\mathbf{x})]\cdot\nabla_{\mathbf{p}}W(t,\mathbf{x},\mathbf{p})\\ &\quad+\sum_{n=1}\left(\frac{\hbar}{2}\right)^{2n}\frac{(-1)^n}{(2n+1)!}\\ &\quad\times(\nabla_{\mathbf{p}}\cdot\nabla_{\mathbf{x}})^{2n+1}V(\mathbf{x})W(t,\mathbf{x},\mathbf{p}) \qquad (10)\end{aligned}$$

- Proof.

$$\begin{aligned}\frac{\partial}{\partial t}W(t,\mathbf{x},\mathbf{p}) &= \frac{1}{(2\pi\hbar)^3}\int d^3y\, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}}\left[\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\partial_t\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right.\\ &\quad\left.+\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\partial_t\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right] \qquad (11)\end{aligned}$$

EOM for Wigner function

- Proof.

$$
\begin{aligned}
\frac{\partial}{\partial t}W &= \frac{1}{(2\pi\hbar)^3}\int d^3y\ e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}} \\
&\quad \left[\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\left(-\frac{i\hbar}{2m}\nabla_{\mathbf{x}}^2+\frac{i}{\hbar}V(\mathbf{x}+\frac{\mathbf{y}}{2})\right)\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right. \\
&\quad \left.+\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\left(\frac{i\hbar}{2m}\nabla_{\mathbf{x}}^2-\frac{i}{\hbar}V(\mathbf{x}-\frac{\mathbf{y}}{2})\right)\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right] \\
&= I(\nabla_{\mathbf{x}}^2)+I(V) \\
\partial_t\Psi^* &= \frac{i}{\hbar}H\Psi^* \\
\partial_t\Psi &= -\frac{i}{\hbar}H\Psi
\end{aligned}
$$

EOM for Wigner function

$$
\begin{aligned}
I(\nabla_{\mathbf{x}}^2) &= \frac{1}{(2\pi\hbar)^3}\left(-\frac{i\hbar}{2m}\right)\nabla_{\mathbf{x}}\cdot\int d^3y\ e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}} \\
&\quad \left[\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\nabla_{\mathbf{x}}\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\right. \\
&\quad \left.-\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\nabla_{\mathbf{x}}\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right] \\
&= \frac{1}{(2\pi\hbar)^3}\left(-\frac{i\hbar}{m}\right)\nabla_{\mathbf{x}}\cdot\int d^3y\ e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}} \\
&\quad \times\nabla_{\mathbf{y}}\left[\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right] \\
&= -\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{x}}W(t,\mathbf{x},\mathbf{p})
\end{aligned}
$$

EOM for Wigner function

$$
\begin{aligned}
I(V) &= \frac{1}{(2\pi\hbar)^3}\frac{i}{\hbar}\int d^3y\, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}}\left[V\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)-V\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right] \\
&\quad\times\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right) \\
&= \frac{1}{(2\pi\hbar)^3}\frac{i}{\hbar}\int d^3y\, e^{\frac{i\mathbf{p}\cdot\mathbf{y}}{\hbar}}\sum_{n=0}\frac{2}{(2n+1)!} \\
&\quad\times\left(\frac{1}{2}\mathbf{y}\cdot\nabla_{\mathbf{x}}\right)^{2n+1}V(\mathbf{x})\Psi\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\Psi^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right) \\
&= \sum_{n=0}\left(\frac{\hbar}{2}\right)^{2n}\frac{(-1)^n}{(2n+1)!}\left(\nabla_{\mathbf{p}}\cdot\nabla_{\mathbf{x}}\right)^{2n+1}V(\mathbf{x})W(t,\mathbf{x},\mathbf{p}) \\
&= [\nabla_{\mathbf{x}}V(\mathbf{x})]\cdot\nabla_{\mathbf{p}}W(t,\mathbf{x},\mathbf{p})+O(\hbar^2)
\end{aligned}
$$

Liouville equation for Wigner function

- Wigner function satisfies the Liouville equation at classical limit $\hbar = 0$,

$$\frac{\partial}{\partial t} W(t, \mathbf{x}, \mathbf{p}) + \underbrace{\frac{\mathbf{p}}{m}}_{\text{velocity}} \cdot \nabla_{\mathbf{x}} W(t, \mathbf{x}, \mathbf{p}) \underbrace{- [\nabla_{\mathbf{x}} V(\mathbf{x})]}_{\text{force}} \cdot \nabla_{\mathbf{p}} W(t, \mathbf{x}, \mathbf{p}) = 0 \tag{12}$$

- Quantum effect

$$-\sum_{n=1} \left(\frac{\hbar}{2}\right)^{2n} \frac{(-1)^n}{(2n+1)!} (\nabla_{\mathbf{p}} \cdot \nabla_{\mathbf{x}})^{2n+1} V(\mathbf{x}) W(t, \mathbf{x}, \mathbf{p}) \tag{13}$$

Wigner function: free complex scalar field

- Lagrangian and Euler-Lagrange equation

$$\begin{aligned} \mathscr{L} &= (\partial^\mu \phi^\dagger)(\partial_\mu \phi) - m^2 \phi^\dagger \phi \\ & (\partial^2 + m^2)\phi = (\partial^2 + m^2)\phi^\dagger = 0 \end{aligned} \quad (14)$$

- Current and energy-moment tensor

$$\begin{aligned} j^\mu &= i\phi^\dagger(\overrightarrow{\partial^\mu} - \overleftarrow{\partial^\mu})\phi \\ T^{\mu\nu} &= \frac{i^2}{2}\phi^\dagger(\overrightarrow{\partial^\mu} - \overleftarrow{\partial^\mu})(\overrightarrow{\partial^\nu} - \overleftarrow{\partial^\nu})\phi \end{aligned} \quad (15)$$

Wigner function: complex scalar field

- Definition and EOM

$$W(x,p) = 2\int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\left\langle \phi^\dagger\left(x+\frac{y}{2}\right)\phi\left(x-\frac{y}{2}\right)\right\rangle \quad (16)$$

$$\begin{aligned} &\text{quasi-on-shell} & \left(p^2-m^2-\frac{1}{4}\hbar^2\partial_x^2\right)W(x,p) &= 0 \\ &\text{Vlasov eq.} & p\cdot\partial_x W(x,p) &= 0 \quad (17)\end{aligned}$$

- Current and energy-moment tensor

$$\begin{aligned} j^\mu &= \int d^4p\, p^\mu W(x,p) \\ T^{\mu\nu} &= \int d^4p\, p^\mu p^\nu W(x,p) \quad (18)\end{aligned}$$

Wigner function: complex scalar field

- Decomposition of Wigner function

$$\mathscr{L} = W^{(+)}(x,p) + W^{(-)}(x,p) + W^{(0)}(x,p) \tag{19}$$

- where

$$\begin{aligned}
&\text{time-like } p \text{, positive energy} && W^{(+)}(x,p) \sim \Theta(p_0)\Theta(p^2) \\
&\text{time-like } p \text{, negative energy} && W^{(-)}(x,p) \sim \Theta(-p_0)\Theta(p^2) \\
&\text{space-like } p && W^{(0)}(x,p) \sim \Theta(-p^2)
\end{aligned} \tag{20}$$

Wigner function: complex scalar field

- If p is space like, $p^2 < 0$, we would have

$$\begin{aligned} \frac{1}{4}\hbar^2\partial_x^2 W(x,p) &= (p^2 - m^2)W(x,p) \\ \left(\frac{\hbar}{2m}\right)^2 \left|\partial_x^2 W(x,p)\right| &= \left|1 - \frac{p^2}{m^2}\right| |W(x,p)| > |W(x,p)| \quad (21) \end{aligned}$$

- which means that Wigner function has large fluctuation in the scale of Compton wave length. This arises from interference of wavepackets of postive and negative frequency, a quantum effect. We won't consider this case, instead we assume

$$\left(\frac{\hbar}{2m}\right)^2 \left|\partial_x^2 W(x,p)\right| \ll |W(x,p)| \quad (22)$$

Wigner function: complex scalar field

- So Wigner function is on mass-shell

$$\begin{aligned}(p^2-m^2)W(x,p) &= 0 \\ W(x,p) &\sim \delta(p^2-m^2) \end{aligned} \tag{23}$$

- We have

$$\begin{aligned} W(x,p) &= 2\delta(p^2-m^2)\left[\Theta(p_0)f(x,p)+\Theta(-p_0)\bar{f}(x,-p)\right] \\ j^\mu &= \int d^4p p^\mu W(x,p) = \int d^3p\frac{p^\mu}{E_p}\left[f(x,p)-\bar{f}(x,p)\right] \\ T^{\mu\nu} &= \int d^4p\, p^\mu p^\nu W(x,p) = \int d^3p\frac{p^\mu p^\nu}{E_p}\left[f(x,p)+\bar{f}(x,p)\right] \end{aligned} \tag{24}$$

Wigner function: complex scalar field

- Use second quantization to compute $W(x,p)$

$$\begin{aligned} \phi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_{\mathbf{p}}e^{-ip\cdot x}+b_{\mathbf{p}}^{\dagger}e^{ip\cdot x}) \\ &= \phi^{(+)}(x)+\phi^{(-)}(x) \end{aligned} \tag{25}$$

- We have

$$\begin{aligned} W^{(+)}(x,p) &= 2\int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\left\langle \phi^{(+)\dagger}\left(x+\frac{y}{2}\right)\phi^{(+)}\left(x-\frac{y}{2}\right)\right\rangle \\ W^{(-)}(x,p) &= 2\int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\left\langle \phi^{(-)\dagger}\left(x+\frac{y}{2}\right)\phi^{(-)}\left(x-\frac{y}{2}\right)\right\rangle \end{aligned} \tag{26}$$

Wigner function: complex scalar field

- Let's compute $W^{(+)}(x,p)$ and $W^{(-)}(x,p)$

$$
\begin{aligned}
W^{(+)}(x,p) &= 2\int \frac{d^4y}{(2\pi)^4}\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{1}{\sqrt{2E_{p1}}}\frac{1}{\sqrt{2E_{p2}}} \\
&\quad \times \exp\left[-ip\cdot y+ip_1\cdot(x+\frac{y}{2})-ip_2\cdot(x-\frac{y}{2})\right] \\
&\quad \times \left\langle a_{\mathbf{p}1}^{\dagger}a_{\mathbf{p}2}\right\rangle \\
&= 2\int \frac{d^4y}{(2\pi)^4}\frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_{p1}}\exp\left[-i(p-p_1)\cdot y\right]f_{\mathrm{BE}}(E_{p1}) \\
&= \frac{2}{(2\pi)^3}\Theta(p_0)\delta(p^2-m^2)f_{\mathrm{BE}}^{(+)}(E_p) \\
W^{(-)}(x,p) &= \frac{2}{(2\pi)^3}\Theta(-p_0)\delta(p^2-m^2)f_{\mathrm{BE}}^{(-)}(E_p) \qquad (27)
\end{aligned}
$$

$$
\left\langle :a_{\mathbf{p}}^{\dagger}a_{\mathbf{k}}:\right\rangle
$$
$$
=(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k})f_{\mathrm{BE}}^{(+)}(E_p)
$$

$$
f_{\mathrm{BE}}^{(+)}(E_p)=\frac{1}{e^{\beta(E_p-\mu)}-1}
$$
$$
f_{\mathrm{BE}}^{(-)}(E_p)=\frac{1}{e^{\beta(E_p+\mu)}-1}
$$

Wigner function: free Dirac field

- Lagrangian and EOM

$$\mathscr{L} = \bar{\psi}(i\gamma^{\rho}\partial_{\rho} - m)\psi \quad (28)$$

$$(i\gamma^{\rho}\partial_{\rho} - m)\psi = 0 \quad (29)$$

- Fermion number current, energy momentum tensor and angular momentum tensor

$$\begin{aligned} j^{\rho} &= \bar{\psi}\gamma^{\rho}\psi \\ T^{\rho\sigma} &= \frac{i}{2}\bar{\psi}\gamma^{\rho}(\overrightarrow{\partial^{\rho}} - \overleftarrow{\partial^{\rho}})\psi \\ M^{\rho\mu\nu} &= x^{\mu}T^{\rho\nu} - x^{\nu}T^{\rho\mu} + \frac{1}{2}\bar{\psi}\{\gamma^{\rho}, S^{\mu\nu}\}\psi \quad (30) \end{aligned}$$

$\{\gamma_0, \epsilon^{ijk}S^{ij}\} = \{\gamma_0, \Sigma_k\} = \frac{1}{2}\gamma^k\gamma_5$

$S^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]$

Wigner function: free Dirac field

- Definition of Wigner function

$$W_{\alpha\beta}(x,p) \;=\; \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \left\langle \bar{\psi}_\beta\left(x+\frac{y}{2}\right)\psi_\alpha\left(x-\frac{y}{2}\right)\right\rangle \quad (31)$$

- Fermion number current and energy momentum tensor

$$\begin{aligned} j^\rho &= \int d^4k \mathrm{Tr}[\gamma^\rho W(x,p)] \\ T^{\rho\sigma} &= \int d^4k \mathrm{Tr}[\gamma^\rho W(x,p)]p^\sigma \end{aligned} \quad (32)$$

Wigner function: free Dirac field

- EOM for Wigner function

$$\left[\gamma^{\rho}\left(\frac{1}{2}i\partial_{\rho}^{x}+p_{\rho}\right)-m\right]W(x,p) \quad = \quad 0 \tag{33}$$

$$\begin{aligned} 0 \quad &= \quad \left[\gamma^{\rho}\left(\frac{1}{2}i\partial_{\rho}^{x}+p_{\rho}\right)+m\right]\left[\gamma^{\sigma}\left(\frac{1}{2}i\partial_{\sigma}^{x}+p_{\sigma}\right)-m\right]W(x,p) \\ &= \quad \left[-\frac{1}{4}\partial_{x}^{2}+p^{2}-m^{2}+ip^{\sigma}\partial_{\sigma}^{x}\right]W(x,p) \\ &\Longrightarrow \quad \left\{\begin{array}{c}\left(-\frac{1}{4}\partial_{x}^{2}+p^{2}-m^{2}\right)W=0\\ p^{\sigma}\partial_{\sigma}^{x}W=0\end{array}\right. \end{aligned} \tag{34}$$

Wigner function: free Dirac field

- Proof of EOM for Wigner function

$$
\begin{aligned}
i\gamma^{\rho}_{ij}\frac{\partial}{\partial x^{\rho}}W_{jk}(x,p) &= i\gamma^{\rho}_{ij}\int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y} \\
&\quad \frac{\partial}{\partial x^{\rho}}\left\langle\bar{\psi}_k\left(x+\frac{y}{2}\right)\psi_j\left(x-\frac{y}{2}\right)\right\rangle \\
&= \int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y} \\
&\quad \times\left[2i\gamma^{\rho}_{ij}\left\langle\frac{\partial}{\partial y^{\rho}}\bar{\psi}_k\left(x+\frac{y}{2}\right)\psi_j\left(x-\frac{y}{2}\right)\right\rangle\right. \\
&\quad \left.+\left\langle\bar{\psi}_k\left(x+\frac{y}{2}\right)i\gamma^{\rho}_{ij}\frac{\partial}{\partial x^{\rho}}\psi_j\left(x-\frac{y}{2}\right)\right\rangle\right] \\
&= (-2p_{\rho}\gamma^{\rho}_{ij}+2m\delta_{ij})W_{jk}(x,p)
\end{aligned}
$$

Wigner function: free Dirac field

- For the first term, we have used

$$
\begin{aligned}
& e^{-ip\cdot y}2i\gamma_{ij}^{\rho}\left\langle \frac{\partial}{\partial y^{\rho}}\bar{\psi}_k\left(x+\frac{y}{2}\right)\psi_j\left(x-\frac{y}{2}\right)\right\rangle \\
\rightarrow\quad & -e^{-ip\cdot y}2p_{\rho}\gamma_{ij}^{\rho}\left\langle \bar{\psi}_k\left(x+\frac{y}{2}\right)\psi_j\left(x-\frac{y}{2}\right)\right\rangle \\
& +e^{-ip\cdot y}\left\langle \bar{\psi}_k\left(x+\frac{y}{2}\right)i\gamma_{ij}^{\rho}\frac{\partial}{\partial x^{\rho}}\psi_j\left(x-\frac{y}{2}\right)\right\rangle \\
\rightarrow\quad & (-2p_{\rho}\gamma_{ij}^{\rho}+m\delta_{ij})W_{jk}(x,p)
\end{aligned}
$$

Wigner function: free Dirac field

- Quantization for fermionc field

$$\begin{aligned}\psi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \\ &\quad \left[u_r(\mathbf{p})a_{\mathbf{p},r}e^{-ip\cdot x}+v_r(\mathbf{p})b^{\dagger}_{\mathbf{p},r}e^{ip\cdot x}\right] \\ &= \psi^{(+)}(x)+\psi^{(-)}(x) \qquad (35)\end{aligned}$$

- Decomposition of Wigner function ($s=\pm$)

$$W_{ij}^{(s)}(x,p) = \int \frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\left\langle \bar{\psi}_j^{(s)}\left(x+\frac{y}{2}\right)\psi_i^{(s)}\left(x-\frac{y}{2}\right)\right\rangle (36)$$

Wigner function: free Dirac field

- Calculate positive (negative) energy part

$$
\begin{aligned}
W_{ij}^{(+)}(x,p) &= \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \left\langle \bar{\psi}_j^{(+)}\left(x+\frac{y}{2}\right)\psi_i^{(+)}\left(x-\frac{y}{2}\right)\right\rangle \\
&= \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \int \frac{d^3p_1}{(2\pi)^3}\frac{1}{\sqrt{2E_{p1}}} \int \frac{d^3p_2}{(2\pi)^3}\frac{1}{\sqrt{2E_{p2}}} \\
&\quad \times \left\langle a_{\mathbf{p}1,r1}^{\dagger} a_{\mathbf{p}2,r2}\right\rangle e^{ip_1\cdot(x+y/2)} e^{-ip_2\cdot(x-y/2)} \\
&\quad \times u_{r2}(\mathbf{p}_2)\bar{u}_{r1}(\mathbf{p}_1) \\
&= \int \frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_{p1}} f_{\mathrm{FD}}(E_{p1}) \\
&\quad \times u_{r1}(\mathbf{p}_1)\bar{u}_{r1}(\mathbf{p}_1) \int \frac{d^4y}{(2\pi)^4} e^{-i(p-p_1)\cdot y}
\end{aligned}
$$

$\left\langle a_{\mathbf{p}1,r1}^{\dagger} a_{\mathbf{p}2,r2}\right\rangle$
$= (2\pi)^3\delta^{(3)}(\mathbf{p}_1-\mathbf{p}_2)$
$\delta_{r1,r2} f_{\mathrm{FD}}^{+}(E_{p1})$

Wigner function: free Dirac field

- Calculate positive (negative) energy part

$$\begin{aligned} W^{(+)}(x,p) &= \frac{1}{(2\pi)^3}\frac{1}{2E_p}\delta(p_0-E_p)f_{\mathrm{FD}}^{(+)}(E_p)\sum_r u_r(\mathbf{p})\bar{u}_r(\mathbf{p}) \\ &= \frac{1}{(2\pi)^3}\theta(p_0)\delta(p^2-m^2)f_{\mathrm{FD}}^{(+)}(E_p)\sum_r u_r(\mathbf{p})\bar{u}_r(\mathbf{p}) \\ W^{(-)}(x,p) &= \frac{1}{(2\pi)^3}\theta(-p_0)\delta(p^2-m^2)f_{\mathrm{FD}}^{(-)}(E_p)\sum_r v_r(-\mathbf{p})\bar{v}_r(-\mathbf{p}) \end{aligned} \tag{37}$$

Chiral fermions in EM field

- Right-handed and left-handed sectors are decoupled,

$$\mathscr{L} = \bar{\psi} i\gamma^{\rho} D_{\rho}\psi = \bar{\psi}_R i\gamma^{\rho} D_{\rho}\psi_R + \bar{\psi}_L i\gamma^{\rho} D_{\rho}\psi_L$$

$D_{\rho} = \partial_{\rho} + ieA_{\rho}$

covariant derivative

R and L projector

$\Lambda_{\pm} = \frac{1}{2}\left(1 \pm \gamma^5\right)$

$\Lambda_{\pm}^2 = \Lambda_{\pm}, \Lambda_{\pm}\Lambda_{\mp} = 0$

$\gamma^{\alpha}\Lambda_{\pm} = \Lambda_{\mp}\gamma^{\alpha}$

$$\begin{aligned} \psi_{R/L} &= \Lambda_{\pm}\psi \\ \bar{\psi}_{R/L} &= \psi^{\dagger}_{R/L}\gamma_0 = \psi^{\dagger}\Lambda_{\pm}\gamma_0 = \bar{\psi}\Lambda_{\mp} \\ \bar{\psi}_R i\gamma^{\rho}\partial_{\rho}\psi_L &= \bar{\psi}\Lambda_{-}\Lambda_{+} i\gamma^{\rho} D_{\rho}\psi = 0 \\ \bar{\psi}_L i\gamma^{\rho}\partial_{\rho}\psi_R &= \bar{\psi}\Lambda_{+}\Lambda_{-} i\gamma^{\rho} D_{\rho}\psi = 0 \end{aligned} \tag{38}$$

Chiral fermions in EM field

- Dirac equations in Weyl representation

$$\begin{aligned} \gamma^\rho D_\rho \psi &= 0 \\ \left(\begin{array}{cc} 0 & \sigma^\rho D_\rho \\ \bar{\sigma}^\rho D_\rho & 0 \end{array} \right) \left(\begin{array}{c} \chi_L \\ \chi_R \end{array} \right) &= 0 \rightarrow \left\{ \begin{array}{l} \sigma \cdot D\chi_R = 0 \\ \bar{\sigma} \cdot D\chi_L = 0 \end{array} \right. \end{aligned} \tag{39}$$

where

$$\begin{aligned} \psi &= \left(\begin{array}{c} \chi_L \\ \chi_R \end{array} \right), \quad \bar{\psi} = (\bar{\chi}_R, \bar{\chi}_L) \\ \gamma^\rho &= \left(\begin{array}{cc} 0 & \sigma^\rho \\ \bar{\sigma}^\rho & 0 \end{array} \right), \quad \sigma^\rho = (1, \boldsymbol{\sigma}), \ \bar{\sigma}^\rho = (1, -\boldsymbol{\sigma}) \\ \gamma_5 &= \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right), \quad \Lambda_R = \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right), \Lambda_L = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \end{aligned} \tag{40}$$

Chiral fermions in Pauli space

- Dirac equation for right-handed (left-handed) fermions

$$\begin{aligned} \mathscr{L} &= \chi_R^\dagger i\sigma\cdot D\chi_R + \chi_L^\dagger i\bar{\sigma}\cdot D\chi_L \\ \sigma\cdot D\chi_R &= 0,\ \ \chi_R^\dagger\sigma\cdot\overleftarrow{D}^\dagger = 0 \\ \bar{\sigma}\cdot D\chi_L &= 0,\ \ \chi_L^\dagger\bar{\sigma}\cdot\overleftarrow{D}^\dagger = 0 \end{aligned} \tag{41}$$

- Two-point Green function for right-handed fermions

$$S_{ab}(x_1,x_2) = \left\langle \chi_b^\dagger(x_2)\chi_a(x_1)\right\rangle \tag{42}$$

$$\sigma\cdot D_{x_1}S(x_1,x_2) = 0 \tag{43}$$

Equation for 2-point function

- Change of variables

$$X = \frac{1}{2}(x_1 + x_2),\ y = x_2 - x_1 \tag{44}$$

$$\partial_{x_1} = \frac{1}{2}\partial_X - \partial_y,\ \partial_{x_2} = \frac{1}{2}\partial_X + \partial_y \tag{45}$$

- Equation for 2-point function

$$\begin{aligned} 0 &= \sigma \cdot D_{x_1} S(x_1, x_2) = \sigma \cdot [\partial_{x_1} + \underbrace{ieA(x_1)]} S(x_1, x_2) \\ &\approx \sigma \cdot \Bigg[\frac{1}{2}\partial_X - \partial_y \\ &\qquad + \overbrace{ieA(X) - ie\frac{1}{2} y \cdot \partial_X A(X)}\Bigg] S(X, y) \end{aligned} \tag{46}$$

Gradient expansion

- Gauge invariant 2-point Green function with gauge link

$$\tilde{S}(X,y) \quad = \quad U(X,X-\frac{y}{2})S(X-\frac{y}{2},X+\frac{y}{2})U(X+\frac{y}{2},X) \tag{47}$$

$$U(x_1,x_2)=\mathcal{P}\exp\left[-ie\int_{x_2}^{x_1}dz\cdot A(z)\right] \tag{48}$$

- Weak field limit $ey\cdot A\ll 1$

$$\begin{aligned}\tilde{S}(X,y) &\simeq [1-iey\cdot A(X)]S(X,y)+O(e^2)\\ S(X,y) &\simeq [1+iey\cdot A(X)]\tilde{S}(X,y)+O(e^2)\end{aligned} \tag{49}$$

Equation for gauge invariant 2-point functions

- Equation for gauge invariant 2-point function

$$\begin{aligned}&\left[\frac{1}{2}\sigma\cdot\partial_X - \sigma\cdot\partial_y + \frac{1}{2}ie\sigma_\mu y_\nu F^{\mu\nu}\right.\\ &\left.+\frac{1}{2}ie(y\cdot A)(\sigma\cdot\partial_X) - ie(y\cdot A)(\sigma\cdot\partial_y)\right]\tilde{S}(X,y) = 0 \qquad (50)\end{aligned}$$

- Fourier transform

$$\begin{aligned}C(X,p_\mathrm{c}) &= \int d^4y e^{-ip_\mathrm{c}\cdot y}C(X,y),\\ C(X,y) &= \frac{1}{(2\pi)^4}\int d^4p_\mathrm{c}e^{ip_\mathrm{c}\cdot y}C(X,p_\mathrm{c}) \qquad (51)\end{aligned}$$

where p_c is the canonical momentum and $C = S, \tilde{S}$.

Equation for gauge invariant 2-point functions

- Equation for gauge invariant 2-point function

$$\begin{aligned}\left[\frac{1}{2}\sigma\cdot\partial_X-\sigma\cdot\partial_y+\frac{1}{2}ie\sigma_\mu y_\nu F^{\mu\nu}\right.\\ \left.+\frac{1}{2}ie(y\cdot A)(\sigma\cdot\partial_X)-ie(y\cdot A)(\sigma\cdot\partial_y)\right]\tilde{S}(X,y) &= 0 \end{aligned}\tag{52}$$

- Fourier transformation for y

$$\partial_y^\mu\rightarrow ip_\mathrm{c}^\mu$$

$$y^\mu\rightarrow i\frac{\partial}{\partial p_\mu^\mathrm{c}}$$

$$\begin{aligned}\left[\frac{1}{2}\sigma\cdot\partial_X-i\sigma\cdot p_\mathrm{c}-\frac{1}{2}e\sigma_\mu F^{\mu\nu}\partial_\nu^{p_\mathrm{c}}\right.\\ -\frac{1}{2}e(A\cdot\partial_{p_\mathrm{c}})(\sigma\cdot\partial_X)+ie(\sigma\cdot A)\\ \left.+ie(\sigma\cdot p_\mathrm{c})(A\cdot\partial_{p_\mathrm{c}})\right]\tilde{S}(X,p_\mathrm{c}) &= 0\end{aligned}\tag{53}$$

Equation for Wigner function

- In terms of kinetic momentum $p = p_c - eA$, we arrive at $[W_{R/L}(X,p) \equiv \tilde{S}_{R/L}(X,p)]$

$$\sigma \cdot \left(p + \frac{1}{2} i \nabla \right) W_R(X,p) = 0 \tag{54}$$

where $\nabla_\mu = \partial_\mu^X - eF_{\mu\nu}\partial_p^\nu$.

- Following the same method, we obtain its conjugate,

$$W_R(X,p) \left[-\sigma \cdot p + \frac{1}{2} i\sigma \cdot \overleftarrow{\nabla} \right] = 0. \tag{55}$$

- For left-handed, replace $\sigma^\mu \to \bar{\sigma}^\mu$.

Equation for Wigner function

- The Wigner functions for right- and left-handed fermions

$$\begin{aligned} W_R(x,p) &= \bar{\sigma}^\mu \mathscr{J}_\mu^+, \\ W_L(x,p) &= \sigma^\mu \mathscr{J}_\mu^-, \end{aligned} \tag{56}$$

where $\mathscr{J}_\mu^s$ can be extracted by taking traces

$$\mathscr{J}_\mu^+ = \frac{1}{2}\mathrm{Tr}\ (\sigma_\mu W_R),\ \mathscr{J}_\mu^- = \frac{1}{2}\mathrm{Tr}\ (\bar{\sigma}_\mu W_L). \tag{57}$$

Equation for Wigner function

- The set of equations for the vector component $\mathscr{J}^{s}_{\mu}(x,p)$ for chiral fermions (with chirality $s=\pm$)

$\nabla_{\mu}=\partial^{X}_{\mu}-eF_{\mu\nu}\partial^{\nu}_{p}$

$$\begin{aligned} p^{\mu}\mathscr{J}^{s}_{\mu}(x,p) &= 0 \\ \nabla^{\mu}\mathscr{J}^{s}_{\mu}(x,p) &= 0 \\ 2s(p^{\lambda}\mathscr{J}^{\rho}_{s}-p^{\rho}\mathscr{J}^{\lambda}_{s}) &= -\epsilon^{\mu\nu\lambda\rho}\nabla_{\mu}\mathscr{J}^{s}_{\nu} \end{aligned} \tag{58}$$

where $\mathscr{J}^{s}_{\mu}(x,p)$ are defined as

$$\mathscr{J}^{s}_{\mu}(x,p) = \frac{1}{2}[\mathscr{V}_{\mu}(x,p)+s\mathscr{A}_{\mu}(x,p)] \tag{59}$$

Equation for Wigner function

- Proof of Eq. (58). The equations of motion for right-handed fermions read

$$\begin{aligned} \sigma \cdot \left(p + \frac{1}{2} i \nabla \right) W_R(X,p) &= 0 \\ W_R(X,p) \left[-\sigma \cdot p + \frac{1}{2} i \sigma \cdot \overleftarrow{\nabla} \right] &= 0 \end{aligned} \tag{60}$$

or explicitly with $W_R(x,p) = \bar{\sigma}^\mu \mathscr{J}_\mu^+$

$$\begin{aligned} \sigma^\nu \bar{\sigma}^\mu \left[p_\nu + \frac{1}{2} i \nabla_\nu \right] \mathscr{J}_\mu^+ &= 0 \\ \bar{\sigma}^\mu \sigma^\nu \left[-p_\nu + \frac{1}{2} i \nabla_\nu \right] \mathscr{J}_\mu^+ &= 0 \end{aligned} \tag{61}$$

Equation for Wigner function

- Take the sum and difference of above equations to obtain

$$\begin{aligned}[] [\sigma^{\nu},\bar{\sigma}^{\mu}]p_{\nu}\mathscr{J}_{\mu}^{+}+\{\sigma^{\nu},\bar{\sigma}^{\mu}\}\frac{1}{2}i\nabla_{\nu}\mathscr{J}_{\mu}^{+} &= 0\\ \{\sigma^{\nu},\bar{\sigma}^{\mu}\}p_{\nu}\mathscr{J}_{\mu}^{+}+[\sigma^{\nu},\bar{\sigma}^{\mu}]\frac{1}{2}i\nabla_{\nu}\mathscr{J}_{\mu}^{+} &= 0 \end{aligned} \tag{62}$$

- Taking the trace and using $\mathrm{Tr}(\{\sigma^{\nu},\bar{\sigma}^{\mu}\})=g^{\mu\nu}$ and $\mathrm{Tr}([\sigma^{\nu},\bar{\sigma}^{\mu}])=0$, we obtain

$$\begin{aligned} p\cdot\mathscr{J}_{s} &= 0\\ \nabla\cdot\mathscr{J}_{s} &= 0 \end{aligned} \tag{63}$$

- Eq. (62) itself gives

$$2s(p^{\lambda}\mathscr{J}_{s}^{\rho}-p^{\rho}\mathscr{J}_{s}^{\lambda})=-\epsilon^{\mu\nu\lambda\rho}\nabla_{\mu}\mathscr{J}_{\nu}^{s} \tag{64}$$

- End of proof

Solving equation perturbatively

- We further assume that the space-time derivative ∂_x and the field strength tensor $F_{\mu\nu}$ are small quantities at the same order and can be used as power expansion parameters (similar to the Knudsen number expansion in hydrodynamics).
- Then $\mathscr{J}^s_\mu(x,p)$ can be expanded as

$$\mathscr{J}^s_\mu(x,p) = \mathscr{J}^s_{\mu,(0)}(x,p) + \mathscr{J}^s_{\mu,(1)}(x,p) + \cdots \,, \tag{65}$$

where the subscripts $(0), (1), ...$ denote orders in the power expansion in $(\partial_x)^n$ and $(F_{\mu\nu})^n$.

Solving equation perturbatively

- Note that $\mathscr{J}^{s}_{\mu,(n)}$ are related to $\mathscr{J}^{s}_{\mu,(n-1)}$ via last Eq. of (58) for $n \geq 1$, so it serves as a recursive relation which can be used to solve $\mathscr{J}^{s}_{\mu,(n)}$ order by order

$$2s(p^{\lambda}\mathscr{J}_{s}^{\rho(n)} - p^{\rho}\mathscr{J}_{s}^{\lambda(n)}) = -\epsilon^{\mu\nu\lambda\rho}\nabla_{\mu}\mathscr{J}^{s}_{\nu(n-1)} \tag{66}$$

- The zeroth or leading order can be obtained by taking ensemble average,

$$\mathscr{J}^{\rho}_{(0)s}(x,p) = p^{\rho} f_s \delta(p^2) \tag{67}$$

where $(p_0 \equiv u \cdot p)$

$$f_s(x,p) = \frac{2}{(2\pi)^3}[\Theta(p_0) f_{\mathrm{FD}}(p_0 - \mu_s) + \Theta(-p_0) f_{\mathrm{FD}}(-p_0 + \mu_s)] \tag{68}$$

Solving equation perturbatively

- Check the first equation

$$p \cdot \mathscr{J}_{(0)s}(x,p) = p^2\delta(p^2)f_s = 0 \tag{69}$$

- Let us check $\nabla \cdot \mathscr{J}_{s}^{(0)} = 0$. Before we do it, we note that the space-time derivative ∂_x is through $T(x)$, $u(x)$, $\mu(x)$ and $\mu_5(x)$,

$$\partial_\sigma^x = \partial_\sigma T \frac{\partial}{\partial T} + \partial_\sigma u_\rho \frac{\partial}{\partial u_\rho} + \partial_\sigma \mu \frac{\partial}{\partial \mu} + \partial_\sigma \mu_5 \frac{\partial}{\partial \mu_5}. \tag{70}$$

Solving equation perturbatively

- Substituting $\mathscr{J}^{\rho}_{(0)s}(x,p) = p^{\rho} f_s \delta(p^2)$ into $\nabla\cdot\mathscr{J}_{s}^{(0)} = 0$, we have

$$\begin{aligned}
\nabla_{\rho}\mathscr{J}^{\rho}_{(0)s} &= \delta(p^2) p^{\rho}\nabla_{\rho} f_s \\
&= \delta(p^2)\frac{\partial f}{\partial(\beta\cdot p)}\left\{-\bar{p}^{\rho}\left[\partial_{\rho}\frac{\mu}{T}+\frac{E_{\rho}}{T}\right]-(u\cdot p)u^{\rho}\partial_{\rho}\frac{\mu}{T}\right. \\
&\quad +sp^{\rho}\partial_{\rho}\frac{\mu_5}{T}+(u\cdot p)\frac{\bar{p}^{\mu}}{T}[(u\cdot\partial)u_{\mu}-\partial_{\mu}\ln T] \\
&\quad +\frac{\bar{p}^2}{T^2}\left[(u\cdot\partial)T+\frac{1}{3}T\Delta^{\rho\sigma}\partial_{\rho}u_{\sigma}\right] \\
&\quad +\frac{1}{2T}\left(\bar{p}^{\rho}\bar{p}^{\sigma}-\frac{1}{3}\bar{p}^2\Delta^{\rho\sigma}\right) \\
&\quad \left.\times\left[\Delta^{\sigma\alpha}\Delta^{\rho\beta}\left(\partial_{\alpha}u_{\beta}+\partial_{\beta}u_{\alpha}-\frac{2}{3}\Delta_{\alpha\beta}\Delta^{\rho\sigma}\partial_{\rho}u_{\sigma}\right)\right]\right\} \\
&= 0, \qquad (71)
\end{aligned}$$

$\nabla_{\mu} = \partial^{x}_{\mu} - eF_{\mu\nu}\partial^{\nu}_{p}$

$p^{\rho} = \bar{p}^{\rho} + (p\cdot u)u^{\rho}$

$\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu}$

Solving equation perturbatively

- We have used

$$\begin{aligned}
\partial^{\sigma}u^{\rho} &= \Delta^{\sigma\alpha}\Delta^{\rho\beta}\left(\partial_{\alpha}u_{\beta}+\partial_{\beta}u_{\alpha}-\frac{2}{3}\Delta_{\alpha\beta}\Delta^{\rho\sigma}\partial_{\rho}u_{\sigma}\right)\\
&\quad +\frac{1}{2}[u^{\sigma}(u\cdot\partial)u^{\rho}+u^{\rho}(u\cdot\partial)u^{\sigma}]+\frac{1}{3}\Delta^{\sigma\rho}(\partial\cdot u)\\
&\quad -\frac{1}{2}(\partial^{\sigma}u^{\rho}-\partial^{\rho}u^{\sigma}).\\
\nabla_{\rho}f_{s} &= \frac{\partial f}{\partial(\beta\cdot p)}\left[\frac{1}{2}p^{\sigma}(\partial_{\rho}\beta_{\sigma}+\partial_{\sigma}\beta_{\rho})\right.\\
&\quad \left.-\frac{1}{2}p^{\sigma}\epsilon_{\rho\sigma\tau\kappa}\tilde{\Omega}^{\tau\kappa}-\partial_{\rho}\frac{\mu}{T}-\frac{E_{\rho}}{T}+s\partial_{r}\frac{\mu_{5}}{T}\right], \qquad (72)
\end{aligned}$$

with

$$\tilde{\Omega}^{\xi\eta}=\frac{1}{2}\epsilon^{\xi\eta\nu\sigma}\Omega_{\nu\sigma},\ \Omega_{\nu\sigma}=\frac{1}{2}(\partial_{\nu}u_{\sigma}-\partial_{\sigma}u_{\nu}). \qquad (73)$$

Static-equilibrium conditions

The static-equilibrium conditions must be fulfilled

$$\Delta^{\sigma\alpha}\Delta^{\rho\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\Delta_{\alpha\beta}\Delta^{\rho\sigma}\partial_\rho u_\sigma\right) = 0, \tag{74}$$

$$T\Delta^{\sigma\rho}\partial_\rho \frac{\mu}{T} + eE^\sigma = 0, \tag{75}$$

$$u^\rho\partial_\rho u^\sigma - \Delta^{\sigma\rho}\partial_\rho \ln T = 0, \tag{76}$$

$$\partial_\sigma \frac{\mu_5}{T} = 0, \quad u^\sigma\partial_\sigma \frac{\mu}{T} = 0, \tag{77}$$

$$u^\sigma\partial_\sigma T + \frac{1}{3}T\Delta^{\rho\sigma}\partial_\rho u_\sigma = 0. \tag{78}$$

The conditions (74,75,76) mean $\pi^{\mu\nu} = \nu^\mu = \nu_5^\mu = h^\mu = 0$. The conditions (75,76,77) are not independent due to the relation (80). Eq. (78) is satisfied by ideal hydro.

Static-equilibrium conditions

EM tensor, fermion number and axial charge current are

$$\begin{aligned} T^{\mu\nu} &= \epsilon u^{\mu} u^{\nu} - (P+\Pi)\Delta^{\mu\nu} \\ &\quad + h^{\mu} u^{\nu} + h^{\nu} u^{\mu} + \pi^{\mu\nu}, \\ j^{\mu} &= n u^{\mu} + \nu^{\mu} \\ j_5^{\mu} &= n_5 u^{\mu} + \nu_5^{\mu} \end{aligned} \tag{79}$$

where h^{μ}, Π, $\pi^{\mu\nu}$ ν^{μ} and ν_5^{μ} are the heat flux, bulk pressure, shear viscous tensor and diffusion currents, respectively. A relation from ideal hydro,

$$\begin{aligned} & \Delta^{\nu\mu}\partial_{\mu}\frac{1}{T} + \frac{1}{T}(u\cdot\partial)u^{\nu} \\ = \; & \frac{\Delta^{\nu\mu}}{\epsilon+P}\left[n\left(\partial_{\mu}\frac{\mu}{T} + \frac{eE_{\mu}}{T}\right) + n_5\partial_{\mu}\frac{\mu_5}{T}\right] \end{aligned} \tag{80}$$

Simplified version of static-equilibrium conditions

With constant temperature, the above conditions are reduced to following simplified form

For $\mu_5(x)$, see Wu, Hou, Ren (2016)

$$\begin{aligned} &\partial_\sigma \mu_5 = 0, \quad \partial_\sigma T = 0 \\ &\partial^\rho u^\sigma + \partial^\sigma u^\rho = 0 \\ &\partial_\sigma \mu = -QE_\sigma \end{aligned} \tag{81}$$

Like Killing condition
Becattini (2012)
Becattini, Bucciantini, Grossi, Tinti (2015)
Becattini, Grossi (2015)

Finding the 1st order solution

Contracting p_λ with

$$2s(p^\lambda \mathscr{J}_s^\rho - p^\rho \mathscr{J}_s^\lambda) = -\epsilon^{\mu\nu\lambda\rho}\nabla_\mu \mathscr{J}_\nu^s \tag{82}$$

and using $p \cdot \mathscr{J}^s(x,p) = 0$, we obtain

$$\epsilon^{\mu\nu\lambda\rho} p_\lambda \nabla_\mu \mathscr{J}_\nu^s = -2sp^2 \mathscr{J}_s^\rho . \tag{83}$$

The general form of $\mathscr{J}_s^\rho$ is

$$\mathscr{J}_s^\mu = p^\mu f_s \delta(p^2) + \mathscr{X}_s^\mu \delta(p^2) + \frac{s}{2p^2}\epsilon^{\mu\nu\rho\sigma} p_\nu \nabla_\rho \mathscr{J}_\sigma^s , \tag{84}$$

where $\mathscr{H}_s^\mu$ should satisfy $p \cdot \mathscr{X}_s = 0$.

Finding the 1st order solution

Substitute the general form (84) into Eq. (82) and obtain

$$(p_\mu \mathscr{X}_\nu - p_\nu \mathscr{X}_\mu)\delta(p^2) = \frac{s}{2}\epsilon_{\mu\nu\lambda\rho}p^\lambda(\nabla^\rho f_s)\delta(p^2), \tag{85}$$

where we have neglected the second or higher order terms. Now we evaluate $\nabla^\rho f_s$ in Eq. (85) as

$$\begin{aligned}\nabla_\rho f_s &= \frac{df_s}{dp_0}\left(p^\sigma\partial_\rho u_\sigma - \partial_\rho\mu - s\partial_\rho\mu_5 - eF_{\rho\xi}u^\xi\right)\\ &= \frac{df_s}{dp_0}\left[\frac{1}{2}p^\sigma(\partial_\rho u_\sigma + \partial_\sigma u_\rho) + p^\sigma\Omega_{\rho\sigma} - \partial_\rho\mu - s\partial_\rho\mu_5 - eF_{\rho\xi}u^\xi\right]\\ &= -\frac{1}{2}\frac{df_s}{dp_0}p^\sigma\epsilon_{\rho\sigma\tau\kappa}\tilde{\Omega}^{\tau\kappa}, \end{aligned}\tag{86}$$

where we have used the conditions (81).

Finding the 1st order solution

Inserting Eq. (86) into Eq. (85) we obtain

$$\begin{aligned}
(p_\mu \mathscr{X}_\nu - p_\nu \mathscr{X}_\mu)\delta(p^2) &= \frac{s}{4} p^\lambda p_\sigma \frac{df_s}{dp_0} \epsilon_{\rho\lambda\mu\nu} \epsilon^{\rho\sigma\tau\kappa} \tilde{\Omega}_{\tau\kappa} \delta(p^2) \\
&= -\frac{s}{4} \frac{df_s}{dp_0} \delta(p^2) \tilde{\Omega}_{\tau\kappa} [\delta_\mu^\tau \delta_\nu^\kappa p^2 + \delta_\mu^\kappa p_\nu p^\tau + \delta_\nu^\tau p_\mu p^\kappa \\
&\quad -\delta_\mu^\tau p_\nu p^\kappa - \delta_\mu^\kappa \delta_\nu^\tau p^2 - \delta_\nu^\kappa p_\mu p^\tau] \\
&= -\frac{s}{2} \frac{df_s}{dp_0} \delta(p^2) [p_\mu \tilde{\Omega}_{\nu\kappa} p^\kappa - p_\nu \tilde{\Omega}_{\mu\kappa} p^\kappa], \qquad (87)
\end{aligned}$$

where we have used $p^2\delta(p^2) = 0$. So we obtain

$$\mathscr{X}_\mu = -\frac{s}{2} \frac{df_s}{dp_0} \tilde{\Omega}_{\mu\lambda} p^\lambda. \qquad (88)$$

Finding the 1st order solution

The last term of Eq. (84) is of the first order if we replace $\mathscr{J}_\sigma^s$ with the zeroth order value $p_\sigma f_s \delta(p^2)$, it becomes

$$\begin{aligned} \frac{s}{2p^2}\epsilon^{\mu\nu\rho\sigma} p_\nu \nabla_\rho \mathscr{J}_\sigma^s &= \frac{s}{2p^2}\epsilon^{\mu\nu\rho\sigma} p_\nu \nabla_\rho [p_\sigma f_s \delta(p^2)] \\ &= -\frac{s}{p^2} e\tilde{F}^{\mu\nu} p_\nu f_s \delta(p^2). \end{aligned} \tag{89}$$

Perturbative solution to the 1st order

Perturbation in $(\partial_\mu^x)^n$ and $(F_{\mu\nu})^n$. The solution at the 0-th and 1-st order

$$\begin{aligned}
\mathscr{J}^{\rho}_{(0)s}(x,p) &= p^\rho f_s \delta(p^2) \\
\mathscr{J}^{\rho}_{(1)s}(x,p) &= -\frac{s}{2}\tilde{\Omega}^{\rho\beta} p_\beta \frac{df_s}{dp_0}\delta(p^2) - \frac{s}{p^2} Q \tilde{F}^{\rho\lambda} p_\lambda f_s \delta(p^2)
\end{aligned}$$

$$\tilde{\Omega}^{\rho\beta} = \frac{1}{2}\epsilon^{\rho\beta\mu\nu}\Omega_{\mu\nu} \qquad \tilde{F}^{\rho\lambda} = \frac{1}{2}\epsilon^{\rho\lambda\mu\nu}F_{\mu\nu}$$

$$\Omega_{\mu\nu} = \frac{1}{2}(\partial_\mu u_\nu - \partial_\nu u_\mu)$$

where f_s is the distribution function $(\mu_s = \mu + s\mu_5,\ p_0 = p\cdot u\ )$,

$$f_s(x,p) = \frac{2}{(2\pi)^3}\left[\underbrace{\Theta(p_0) f_{\rm FD}(p_0-\mu_s)}_{\text{fermion part}} + \underbrace{\Theta(-p_0) f_{\rm FD}(-p_0+\mu_s)}_{\text{anti-fermion part}}\right]$$

$f_{\rm FD}(x) = \frac{1}{e^{\beta x}+1}$, Fermi-Dirac

Currents from Wigner function

We can compute the left- and right-handed currents,

$$\omega^{\rho} = \frac{1}{2}\epsilon^{\rho\sigma\alpha\beta} u_{\sigma}\partial_{\alpha}u_{\beta}$$

$$\begin{aligned} j_s^{\mu} &= \int d^4p\, \mathscr{J}_s^{\mu} \\ &= n_s u^{\mu} + \xi_{B,s}B^{\mu} + \xi_s \omega^{\mu}, \end{aligned} \tag{90}$$

The results for n_s, $\xi_{B,s}$ and ξ_s are

$$\mu_s = \mu + \mu_5$$

$$\begin{aligned} n_s &= \frac{1}{6}T^2\mu_s + \frac{1}{6\pi^2}\mu_s^3, \\ \xi_{B,s} &= \frac{es}{4\pi^2}\mu_s, \\ \xi_s &= s\frac{1}{12}T^2 + \frac{s}{4\pi^2}\mu_s^2. \end{aligned} \tag{91}$$

Currents from Wigner function

Proof of Eq. (90).

$$\begin{aligned}
j_s^{\rho(0)} &= \int d^4p\, \overset{(p\cdot u)u^\rho}{p^\rho} f_s \delta(p^2) = u^\rho \int d^4p p_0 f_s \delta(p^2) \\
n_s &= \int d^4p p_0 f_s \delta(p^2) \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left[f_{\mathrm{FD}}(E_p - \mu_s) - f_{\mathrm{FD}}(E_p + \mu_s) \right] \\
&= \frac{1}{2\pi^2} \int_0^\infty dE_p E_p^2 \left[f_{\mathrm{FD}}(E_p - \mu_s) - f_{\mathrm{FD}}(E_p + \mu_s) \right] \\
&= \frac{1}{6} T^2 \mu_s + \frac{1}{6\pi^2} \mu_s^3 \qquad (92)
\end{aligned}$$

Currents from Wigner function

$$
\begin{aligned}
j_s^{\rho(1)}(B) &= \int d^4p\, \mathscr{J}_s^{\mu(1)}(B) \qquad \nearrow (u\cdot p)u_\lambda \\
&= -se\int d^4p \frac{1}{p^2}\tilde{F}^{\rho\lambda} p_\lambda f_s \delta(p^2) = \xi_{B,s}B^\rho \\
\xi_{B,s} &= -se\int d^4p \frac{p_0}{p^2} f_s\delta(p^2) = -\frac{1}{2}se\int d^4p \frac{df_s}{dp_0}\delta(p^2) \\
&= -\frac{1}{2}se\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{E_p}\left[\frac{d}{dE_p}f_{\mathrm{FD}}(E_p-\mu_s) - \frac{d}{dE_p}f_{\mathrm{FD}}(E_p+\mu_s)\right] \\
&= se\frac{1}{4\pi^2}\int_0^\infty dE_p\left[f_{\mathrm{FD}}(E_p-\mu_s) - f_{\mathrm{FD}}(E_p+\mu_s)\right] \\
&= \frac{es}{4\pi^2}\mu_s, \qquad (93)
\end{aligned}
$$

Currents from Wigner function

$$
\begin{aligned}
j_s^{\rho(1)}(\omega) &= \int d^4p\, \mathscr{J}_s^{\mu(1)}(\omega) \\
&= -\frac{s}{2}\tilde{\Omega}^{\rho\lambda}\int d^4p\, p\, {\color{red}\overset{\nearrow\,(u\cdot p)u_\lambda}{p_\lambda}}\frac{df_s}{dp_0}\delta(p^2) = \xi_s\omega^\mu \\
\xi_s &= -\frac{s}{2}\int d^4p\, p p_0\frac{df_s}{dp_0}\delta(p^2) \\
&= -\frac{s}{2}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\left[\frac{d}{dE_p}f_{\mathrm{FD}}(E_p-\mu_s)+\frac{d}{dE_p}f_{\mathrm{FD}}(E_p+\mu_s)\right] \\
&= s\frac{1}{2\pi^2}\int_0^\infty dE_p E_p\left[f_{\mathrm{FD}}(E_p-\mu_s)+f_{\mathrm{FD}}(E_p+\mu_s)\right] \\
&= s\frac{1}{12}T^2+\frac{s}{4\pi^2}\mu_s^2, \qquad (94)
\end{aligned}
$$

CME and CVE

we can obtain the total fermion number and axial charge currents,

$$\begin{aligned} j^{\mu} &= \sum_{s=\pm} j_s^{\mu} = nu^{\mu} + \xi_B B^{\mu} + \xi\omega^{\mu}, & (95) \\ j_5^{\mu} &= \sum_{s=\pm} s j_s^{\mu} = n_5 u^{\mu} + \xi_{B5} B^{\mu} + \xi_5 \omega^{\mu}, & (96) \end{aligned}$$

where n and n_5 are the total fermion number and axial charge densities and given by

$$\begin{aligned} n &= n_+ + n_- = \frac{1}{3}\left[\mu T^2 + \frac{1}{\pi^2}(\mu^3 + 3\mu\mu_5^2)\right], \\ n_5 &= n_+ - n_- = \frac{1}{3}\left[\mu_5 T^2 + \frac{1}{\pi^2}(\mu_5^3 + 3\mu_5\mu^2)\right]. & (97) \end{aligned}$$

CME and CVE

The chiral magnetic and vortical conductivities (ξ_B, ξ) and the magnetic and vortical conductivities from the axial charge current (ξ_{B5}, ξ_5) are given by

$$\begin{aligned}
\xi_B &= \xi_{B+} + \xi_{B-} = \frac{e}{2\pi^2}\mu_5, \\
\xi &= \xi_+ + \xi_- = \frac{1}{\pi^2}\mu\mu_5, \\
\xi_{B5} &= \xi_{B+} - \xi_{B-} = \frac{e}{2\pi^2}\mu, \\
\xi_5 &= \xi_+ - \xi_- = \frac{1}{6}T^2 + \frac{1}{2\pi^2}\left(\mu^2 + \mu_5^2\right). \qquad (98)
\end{aligned}$$

Energy-momentum tensor

The total energy-momentum tensor is given by

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{2}\int d^4p \sum_{s=\pm}(p^\mu \mathscr{J}_s^\nu + p^\nu \mathscr{J}_s^\mu) \\ &= (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu} \\ &\quad +n_5(u^\mu\omega^\nu + u^\nu\omega^\mu) + \frac{1}{2}e\xi(u^\mu B^\nu + u^\nu B^\mu), \end{aligned} \tag{99}$$

where the energy density and pressure are

$$\begin{aligned} \epsilon &= \frac{7\pi^2}{60}T^4 + \frac{1}{2}(\mu^2 + \mu_5^2)T^2 + \frac{1}{4\pi^2}(\mu^4 + \mu_5^4 + 6\mu^2\mu_5^2), \\ P &= \frac{1}{3}\epsilon. \end{aligned} \tag{100}$$

Conservation laws

We can verify the conservation laws for j^μ and j_5^μ as

$$\partial_\mu j^\mu = 0, \tag{101}$$

and

$$\partial_\mu j_5^\mu = -\frac{e^2}{2\pi^2} E \cdot B. \tag{102}$$

We can also verify the energy-momentum conservation in the background field,

$$\partial_\rho T^{\rho\sigma} = eF^{\sigma\alpha} j_\alpha. \tag{103}$$

Derivation of conservation laws

We take divergence of j_s^ρ,

$$\xi_{B,s} = \frac{es}{4\pi^2}\mu_s$$

$$\xi_s = s\frac{1}{12}T^2 + \frac{s}{4\pi^2}\mu_s^2$$

$$\begin{aligned}
\partial_\rho j_s^\rho &= \partial_\rho(n_s u^\rho + \xi_{B,s}B^\rho + \xi_s\omega^\rho) \\
&= u^\rho\partial_\rho n_s + B^\rho\partial_\rho\xi_{B,s} + \xi_{B,s}\partial_\rho B^\rho + \omega^\rho\partial_\rho\xi_s \\
&= \frac{\partial\xi_{B,s}}{\partial\mu}B^\rho\partial_\rho\mu + 2\xi_{B,s}(E\cdot\omega) + \frac{\partial\xi_s}{\partial\mu}\omega^\rho\partial_\rho\mu \\
&= -\frac{se^2}{4\pi^2}(E\cdot B), \qquad (104)
\end{aligned}$$

where we have used $\partial\cdot u = 0$, $\partial\cdot\omega = 0$, $\partial\cdot B = 2E\cdot\omega$ and $e(\partial\xi_s/\partial\mu) = 2\xi_{B,s}$.

Derivation of conservation laws

We verify the energy-momentum conservation (103),

$$\begin{aligned}
\partial_\rho T^{\rho\sigma} &= -\partial^\sigma P + (u^\rho\omega^\sigma + u^\sigma\omega^\rho)\partial_\rho n_5 + n_5\partial_\rho(u^\rho\omega^\sigma + u^\sigma\omega^\rho) \\
&\quad +\frac{1}{2}e\xi\partial_\rho(u^\rho B^\sigma + u^\sigma B^\rho) + \frac{1}{2}e(u^\rho B^\sigma + u^\sigma B^\rho)\partial_\rho\xi \\
&= -\partial^\sigma\mu\left.\frac{\partial P}{\partial\mu}\right|_T + u^\sigma\omega^\rho\partial_\rho\mu\left.\frac{\partial n_5}{\partial\mu}\right|_T + e\xi(E\cdot\omega)u^\sigma \\
&\quad +e\frac{\mu_5}{2\pi^2}(u^\rho B^\sigma + u^\sigma B^\rho)\partial_\rho\mu \\
&= enE^\rho - e\xi(E\cdot\omega)u^\sigma - e^2\xi_B(E\cdot B)u^\sigma \\
&= eF^{\rho\sigma}j_\sigma - e\xi F^{\rho\sigma}\omega_\sigma - e^2\xi_B F^{\rho\sigma}B_\sigma \\
&\quad -e\xi(E\cdot\omega)u^\nu - e^2\xi_B(E\cdot B)u^\sigma \\
&= eF^{\rho\sigma}j_\sigma
\end{aligned} \tag{105}$$

$$n_5 = \frac{1}{3}\left[\mu_5 T^2 + \frac{1}{\pi^2}(\mu_5^3 + 3\mu_5\mu^2)\right]$$

$$\xi = \frac{1}{\pi^2}\mu\mu_5$$

$$2\xi = \partial n_5/\partial\mu$$

$$j^\mu = nu^\mu + \xi_B B^\mu + \xi\omega^\mu$$

Derivation of conservation laws

where we have used

$$\begin{aligned}
u\cdot\partial\omega^{\mu} &= u\cdot\partial B^{\mu}=\partial\cdot\omega=0,\\
\partial\cdot B &= 2E\cdot\omega,\\
\omega^{\rho}\partial_{\rho}u_{\sigma} &= \epsilon_{\rho\sigma\alpha\beta}u^{\alpha}\omega^{\beta}\omega^{\rho}=0,\\
B^{\rho}\partial_{\rho}u_{\sigma} &= \epsilon_{\rho\sigma\alpha\beta}u^{\alpha}\omega^{\beta}B^{\rho}=0,\\
u^{\sigma}\omega^{\rho}\partial_{\rho}\mu\left.\frac{\partial n_5}{\partial\mu}\right|_{T} &= -2e\xi(E\cdot\omega)u^{\sigma},\\
F^{\rho\sigma}B_{\sigma} &= -(E\cdot B)u^{\rho},\\
F^{\rho\sigma}\omega_{\sigma} &= -(E\cdot\omega)u^{\rho}+\epsilon^{\rho\sigma\alpha\beta}u_{\alpha}\omega_{\sigma}B_{\beta}\\
&= -(E\cdot\omega)u^{\rho}. \qquad (106)
\end{aligned}$$

Derivation of Covariant Chiral Kinetic Equation from Wigner function

- CCKE is a phase-space equation for classical point particles (chiral fermions)
- One should reproduce $j^{\mu}(x)$, $j_5^{\mu}(x)$, $T^{\mu\nu}(x)$ obtained by Wigner function from velocity $\frac{dx^{\mu}}{d\tau}$ and $\frac{dp^{\mu}}{d\tau}$ in CCKE, which incorporate CME, CVE, chiral anomaly etc. All $j^{\mu}(x)$, $j_5^{\mu}(x)$, $T^{\mu\nu}(x)$ are observables which are results of corresponding phase-space quantities after integration over momentum.
- There is freedom to add terms to a phase-space quantity as long as its momentum integral is vanishing, like gauge freedom.
- Reference: J. Gao, S. Pu and Q. Wang, arXiv:1704.00244.

Derivation of CCKE

We insert the Wigner function solutions into

$$\nabla_\mu[\mathscr{J}^\mu_{(0)s} + \mathscr{J}^\mu_{(1)s}] \quad = \quad 0$$

The zero-th order is evaluated as

$$\begin{aligned}\nabla_\mu[p^\mu f_s \delta(p^2)] &= (\partial_x^\mu - QF^{\mu\nu}\partial_\nu^p)[p^\mu f_s \delta(p^2)] \\ &= \delta(p^2) p^\mu \nabla_\mu f_s\end{aligned}$$

where we used $p^\mu \nabla_\mu \delta(p^2) = 0$ and $\nabla_\mu p^\mu = 0$. The first order can also be evaluated similarly.

Derivation of CCKE

We combine the 0-th and 1st order contribution we obtain

$$
\begin{aligned}
& \nabla_\mu[\mathscr{J}^\mu_{(0)s} + \mathscr{J}^\mu_{(1)s}] \\
= \quad & \delta(p^2)\left[p^\mu \nabla_\mu f_s + sQ\frac{1}{p^2}\Omega^{\mu\lambda} p_\lambda \tilde{F}_{\mu\kappa} p^\kappa f_s' \right. \\
& \left. -\frac{s}{2}\tilde{\Omega}^{\mu\lambda} p_\lambda (\nabla_\mu f_s') - sQ\frac{1}{p^2}\tilde{F}^{\mu\lambda} p_\lambda (\nabla_\mu f_s)\right] = 0
\end{aligned}
$$

Derivation of CCKE

Further simplification gives

$\nabla^\mu \equiv \partial_x^\mu - QF^{\mu\nu}\partial_\nu^p$

$$\begin{aligned}
&\left[\left(p^\mu - sQ\frac{1}{p^2}\tilde{F}^{\mu\lambda}p_\lambda - s\frac{p_0}{p^2}\tilde{\Omega}^{\mu\lambda}p_\lambda + \frac{s}{2}\tilde{\Omega}^{\mu\lambda}u_\lambda\right)\nabla_\mu f_s\right.\\
&\left.+\left(-\frac{s}{2}\tilde{\Omega}^{\mu\lambda}p_\lambda\Omega_{\mu\nu} + sQ\frac{1}{p^2}\Omega^{\mu\lambda}p_\lambda\tilde{F}_{\mu\kappa}p^\kappa u_\nu\right)\partial_p^\nu f_s\right]\delta(p^2) \quad = \quad 0,
\end{aligned}$$

which can be cast into the form

$$\delta(p^2)\left(\frac{dx^\mu}{d\tau}\partial_\mu^x f_s + \frac{dp^\mu}{d\tau}\partial_\mu^p f_s\right) = 0$$

Derivation of CCKE

Here $dx^\mu/d\tau$ and $dp^\mu/d\tau$ are given by

$$\begin{aligned}
m_0\frac{dx^\mu}{d\tau} &= p^\mu - sQ\frac{1}{p^2}\tilde{F}^{\mu\lambda}p_\lambda - s\frac{p_0}{p^2}\tilde{\Omega}^{\mu\lambda}p_\lambda + \frac{s}{2}\tilde{\Omega}^{\mu\lambda}u_\lambda, \\
m_0\frac{dp^\mu}{d\tau} &= QF^{\mu\nu}p_\nu + sQ^2\frac{1}{4p^2}F^{\nu\lambda}\tilde{F}_{\nu\lambda}p^\mu \\
&\quad + sQ\frac{1}{p^2}\Omega_{\nu\lambda}p^\lambda\tilde{F}^{\nu\kappa}p_\kappa u^\mu - \frac{s}{8}\Omega^{\nu\lambda}\tilde{\Omega}_{\nu\lambda}p^\mu \\
&\quad + \frac{1}{2}sQF^{\mu\nu}\tilde{\Omega}_{\nu\lambda}u^\lambda - sQ\frac{p_0}{p^2}F^{\mu\nu}\tilde{\Omega}_{\nu\lambda}p^\lambda
\end{aligned} \tag{107}$$

Derivation of CCKE

We can write $dx^\mu/d\tau$ and $dp^\mu/d\tau$ in a different form

$$\begin{aligned}
m_0\frac{dx^\mu}{d\tau} &= p^\mu - sQ\frac{1}{p^2}\tilde{F}^{\mu\lambda}p_\lambda + s\left(\frac{1}{2}-\frac{p_0^2}{p^2}\right)\omega^\mu + s\frac{p_0}{p^2}(p\cdot\omega)u^\mu \\
&\quad +X^\mu \\
m_0\frac{dp^\mu}{d\tau} &= QF^{\mu\nu}p_\nu + sQ^2\frac{p^\mu}{4p^2}F^{\nu\lambda}\tilde{F}_{\nu\lambda} \\
&\quad +\frac{1}{2}sQ(E\cdot\omega)u^\mu - sQ\frac{1}{p^2}(p\cdot\omega)(p\cdot E)u^\mu + sQ\frac{1}{p^2}p_0(p\cdot\omega)E^\mu \\
&\quad +Y^\mu
\end{aligned}$$

Freedom of adding more terms

The new terms X^μ and Y^μ can be added

$$\begin{aligned}
\frac{dx^\mu}{d\tau} \leftarrow X^\mu &= sC_1(p,u)\omega^\mu + sC_2(p,u)(p\cdot\omega)u^\mu \\
&+sC_3(p,u)(p\cdot\omega)\bar{p}^\mu \\
\frac{dp^\mu}{d\tau} \leftarrow Y^\mu &= -sQ[C_1(p,u)(\omega\cdot E) + C_3(p,u)(p\cdot\omega)(p\cdot E)]u^\sigma \\
&+sQC_4(p,\omega)\bar{p}^\sigma
\end{aligned}$$

which satisfy the equation

$$X^\sigma\partial^x_\sigma f_s + Y^\sigma\partial^p_\sigma f_s = 0$$

Freedom of adding more terms

We assume following forms for these unknown functions

$$\begin{aligned}
C_1(p,u) &= C_{10} + C_{11}\frac{p_0^2}{p^2} \\
C_2(p,u) &= C_{20}\frac{p_0}{p^2} + C_{21}\frac{1}{p_0} \\
C_3(p,u) &= C_{30}\frac{1}{p^2} \\
C_4(p,\omega) &= C_{40}(\omega\cdot E)\frac{1}{p_0} + C_{41}\frac{1}{p^2 p_0}(p\cdot\omega)(p\cdot E)
\end{aligned}$$

Coefficients restricted by
conservation of charge
and energy-momentum

where $\{C_{10}, C_{11}, C_{20}, C_{21}, C_{30}, C_{40}, C_{41}\}$ are dimensionless constants to be determined.

Coefficients restricted by
matching power and force

Constraints from currents

The currents for chiral fermions with chirality $s = \pm 1$ are given by,

$$\begin{aligned} j_s^\mu &= \int d^4p\delta(p^2)m_0\frac{dx^\mu}{d\tau}f_s \\ &= j_s^\mu(\mathrm{EM}) + j_s^\mu(\omega) \end{aligned}$$

where for the current from EM field

$$\begin{aligned} j_s^\mu(\mathrm{EM}) &= -sQ\int d^4p\delta(p^2)\frac{1}{p^2}\tilde{F}^{\mu\lambda}p_\lambda f_s = \xi_B^s B^\mu \\ j^\mu(\mathrm{EM}) &= (\xi_B^+ + \xi_B^-)B^\mu = \frac{Q}{2\pi^2}\mu_5 B^\mu \\ j_5^\mu(\mathrm{EM}) &= (\xi_B^+ - \xi_B^-)B^\mu = \frac{Q}{2\pi^2}\mu B^\mu \end{aligned}$$

Constraints from currents

For the current from vorticity

$$\begin{aligned}
j_s^\mu(\omega) &= \left(C_{10}-\frac{1}{2}C_{11}+\frac{1}{2}C_{30}+1\right)\xi_s\omega^\mu \\
j^\mu(\omega) &= j_+^\mu(\omega)+j_-^\mu(\omega)\rightarrow\frac{1}{\pi^2}\mu\mu_5\omega^\mu \\
j_5^\mu(\omega) &= j_+^\mu(\omega)-j_-^\mu(\omega) \\
&\rightarrow \left[\frac{1}{6}T^2+\frac{1}{2\pi^2}(\mu^2+\mu_5^2)\right]\omega^\mu
\end{aligned}$$

$\xi_s=s\frac{1}{12}T^2+\frac{s}{4\pi^2}\mu_s^2$

which gives the constraints

$$C_{10}-\frac{1}{2}C_{11}+\frac{1}{2}C_{30}=0$$

Constraints from stress tensor

The energy mometum tensor in the relativistic chiral kinetic theory can be obtained by

$$\begin{aligned}
T^{\rho\sigma} &= \frac{1}{2}m_0 \int d^4p\delta(p^2)\sum_s p^{(\rho}\frac{dx^{\sigma)}}{d\tau}f_s \\
&= T^{\rho\sigma}(\mathrm{EM}) + T^{\rho\sigma}(\omega) \\
&= \frac{1}{2}Q\xi u^{(\rho}B^{\sigma)} + \left(\frac{1}{2}C_{10} - \frac{1}{4}C_{11} + \frac{1}{4}C_{30} + \frac{1}{4}C_{20} - \frac{1}{6}C_{21} + \frac{3}{4}\right) n_5 u^{(\rho}\omega^{\sigma)} \\
&= \frac{1}{2}Q\xi u^{(\rho}B^{\sigma)} + n_5 u^{(\rho}\omega^{\sigma)}
\end{aligned}$$

which gives the constraints

$$\frac{1}{2}C_{10} - \frac{1}{4}C_{11} + \frac{1}{4}C_{30} + \frac{1}{4}C_{20} - \frac{1}{6}C_{21} + \frac{3}{4} = 1$$

Constraints

We combine two constraints from conservation laws

$$\begin{aligned} C_{10}-\frac{1}{2}C_{11}+\frac{1}{2}C_{30} &= 0 \\ C_{20}-\frac{2}{3}C_{21} &= 1 \end{aligned}$$

We see that C_{20} and C_{21} cannot all be zero.

Chiral kinetic equation: from 4D to 3D

With CCKE in 4D, we can obtain its 3D version by integrating over p_0,

$$\begin{aligned} I &= \int dp_0 \delta(p^2) \left[\frac{dx^\sigma}{d\tau} \partial_\sigma^x f_s + \frac{dp^\rho}{d\tau} \partial_\rho^p f_s \right] \\ &= \underbrace{I_{x0}}_{\sigma=0} + \underbrace{I_x}_{\sigma=i} + \underbrace{I_{p0}}_{\rho=0} + \underbrace{I_p}_{\rho=i} \end{aligned}$$

Each term has three contributions

$$I_j = I_j(0) + I_j(\mathrm{EM}) + I_j(\omega)$$

where $j = x0, p0, x, p$.

Extract $d\mathbf{p}/d\tau$

For an on-shell particle the energy is not an independent phase space variable, its rate $dE_p/d\tau$ from I_{p0} can be determined by

$$\frac{dE_p}{d\tau} = \frac{\mathbf{p}}{|\mathbf{p}|} \cdot \frac{d\mathbf{p}}{d\tau}$$

So in derivation of the 3D chiral kinetic equation from the 4D one, the p_0 degree of freedom is fixed and is not a kinematic variable in the 3D kinetic equation.

We can extract the EM field contribution to $d\mathbf{p}/d\tau$ from $I_p(\mathrm{EM})$ by matching $dE_p/d\tau$,

$$\frac{d\mathbf{p}}{d\tau}(\mathrm{EM}) = Q\left(\mathbf{E} + \frac{\mathbf{p}}{|\mathbf{p}|} \times \mathbf{B}\right) + sQ^2(\mathbf{E}\cdot\mathbf{B})\mathbf{p}\frac{1}{2|\mathbf{p}|^3}$$

Extract $d\mathbf{p}/d\tau$

The energy rate from the vorticity can be obtained from $I_{p0}(\omega)$,

$$\begin{aligned}\frac{dE_p}{d\tau}(\omega) &= -sQ\left(C_{30}+1\right)\frac{1}{2|\mathbf{p}|}(\mathbf{E}\cdot\boldsymbol{\omega}) \\ &\quad +sQ(C_{30}+1)\frac{3}{2|\mathbf{p}|^3}(\mathbf{p}\cdot\omega)(\mathbf{p}\cdot\mathbf{E})\end{aligned}$$

We can compare it with $d\mathbf{p}/d\tau$ extracted from $I_p(\omega)$,

$$\begin{aligned}\frac{d\mathbf{p}}{d\tau}(\omega) &= sQ\left[\frac{1}{|\mathbf{p}|^2}(\mathbf{p}\cdot\boldsymbol{\omega})\mathbf{E}-C_{40}\frac{1}{|\mathbf{p}|^2}(\boldsymbol{\omega}\cdot\mathbf{E})\mathbf{p}\right. \\ &\quad \left.-C_{41}\frac{2}{|\mathbf{p}|^4}(\mathbf{p}\cdot\omega)(\mathbf{p}\cdot\mathbf{E})\mathbf{p}\right]\end{aligned}$$

Extract $d\mathbf{p}/d\tau$

Enforcing $\frac{dE_p}{d\tau} = \frac{\mathbf{p}}{|\mathbf{p}|} \cdot \frac{d\mathbf{p}}{d\tau}$, the coefficients must satisfy

$$\begin{aligned} C_{40} &= \frac{1}{2}(C_{30}+1) \\ C_{41} &= \frac{1}{2} - \frac{3}{4}(C_{30}+1) \end{aligned}$$

So in this case we obtain $d\mathbf{p}/d\tau$ from vorticity as

$$\begin{aligned} \frac{d\mathbf{p}}{d\tau}(\omega) &= sQ\left[\frac{1}{|\mathbf{p}|^2}(\mathbf{p}\cdot\boldsymbol{\omega})\mathbf{E} - (C_{30}+1)\frac{1}{2|\mathbf{p}|^2}(\boldsymbol{\omega}\cdot\mathbf{E})\mathbf{p} \right. \\ &\left. +(1+3C_{30})\frac{1}{2|\mathbf{p}|^4}(\mathbf{p}\cdot\boldsymbol{\omega})(\mathbf{p}\cdot\mathbf{E})\mathbf{p}\right] \end{aligned}$$

Extract $dx_0/d\tau$ and $d\mathbf{x}/d\tau$

From from I_{x0} and I_x we obtain

$$\begin{aligned}\frac{dx_0}{d\tau} &= 1+\mathscr{C}_B sQ(\mathbf{\Omega}\cdot\mathbf{B})+\left(4-\frac{2}{3}C_{21}\right)s|\mathbf{p}|(\mathbf{\Omega}\cdot\boldsymbol{\omega}),\\ \frac{d\mathbf{x}}{d\tau} &= \hat{\mathbf{p}}+sQ\mathbf{B}(\hat{\mathbf{p}}\cdot\mathbf{\Omega})+\mathscr{C}_E sQ(\mathbf{E}\times\mathbf{\Omega})\\ &\quad +s\left(1-\frac{1}{2}C_{30}\right)\frac{\boldsymbol{\omega}}{|\mathbf{p}|}+3sC_{30}(\mathbf{\Omega}\cdot\boldsymbol{\omega})\mathbf{p},\end{aligned}$$

where we have used the constraints about C_{10}, C_{11}, C_{20},C_{21} and C_{30}.

CKE in 3D

[Son,Yamamoto 2012,2013; Stephanov,Yin 2012; Chen,Pu,Wang,Wang 2012; Chen,Son,Stephanov,Yee,Yin 2014; Kharzeev,Stephanov,Yee 2016; Hidaka,Pu,Yang 2016; Mueller,Venugopalan 2017]

$$\begin{aligned}
\frac{dx_0}{d\tau} &= 1+\mathscr{C}_B sQ(\boldsymbol{\Omega}\cdot\mathbf{B})+\left(4-\frac{2}{3}C_{21}\right)s|\mathbf{p}|(\boldsymbol{\Omega}\cdot\omega)\\
\frac{d\mathbf{x}}{d\tau} &= \hat{\mathbf{p}}+sQ\mathbf{B}(\hat{\mathbf{p}}\cdot\boldsymbol{\Omega})+\mathscr{C}_E sQ(\mathbf{E}\times\boldsymbol{\Omega})\\
&\quad+\left(1-\frac{1}{2}C_{30}\right)s\frac{\omega}{|\mathbf{p}|}+3C_{30}s(\boldsymbol{\Omega}\cdot\omega)\mathbf{p}\\
\frac{d\mathbf{p}}{d\tau} &= Q\left(\mathbf{E}+\frac{\mathbf{p}}{|\mathbf{p}|}\times\mathbf{B}\right)+sQ^2(\mathbf{E}\cdot\mathbf{B})\mathbf{p}\frac{1}{2|\mathbf{p}|^3}\\
&\quad+sQ\frac{1}{|\mathbf{p}|^2}\left[(\mathbf{p}\cdot\omega)\mathbf{E}-\frac{1}{2}(C_{30}+1)(\omega\cdot\mathbf{E})\mathbf{p}\right.\\
&\quad\left.+\frac{1}{2}(1+3C_{30})\frac{1}{|\mathbf{p}|^2}(\mathbf{p}\cdot\omega)(\mathbf{p}\cdot\mathbf{E})\mathbf{p}\right]
\end{aligned}$$

Coeffcients in 3D-CKE

- CKE in 3D is not uniquely determined due to free coefficients $\{C_{21}, C_{30}, \mathscr{C}_B, \mathscr{C}_E\}$
- $\mathscr{C}_B, \mathscr{C}_E = 1$ or 2. The freedom to choose $\mathscr{C}_B$ and $\mathscr{C}_E$ is because the integration over **p** of their corresponding terms are all vanishing, we can make choices as to keep or drop them following some physical reasonings. The $\mathscr{C}_E = 1$ is consistent to the previous result.
- With the coefficients $\{C_{21}, C_{30}, \mathscr{C}_B, \mathscr{C}_E\} = \{0, 0, 1, 1\}$ we reproduce our previous result of [Chen,Pu,Wang,Wang, PRL 110, 262301(2013)].

Coefficients in 3D-CKE

Another possible choice $C_{30} = 2/3$. In this case the vorticity terms in $d\mathbf{x}/d\tau$ read

$$\frac{d\mathbf{x}}{d\tau}(\omega) = \frac{s}{|\mathbf{p}|}(\hat{\mathbf{p}} \cdot \boldsymbol{\omega})\hat{\mathbf{p}} + \frac{s}{|\mathbf{p}|}\frac{2}{3}\boldsymbol{\omega}$$

When calculating the vorticity contribution to the current by the integration over $\mathbf{p}$ for $d\mathbf{x}/d\tau$ times the distribution function, one can verify that the first term of contributes to $1/3$ of the chiral vortical effect while the second term contributes to the rest $2/3$. In comparison to the result of [Kharzeev,Stephanov,Yee, 1612.01674], the $1/3$ contribution corresponds to that from the spin-vorticity coupling energy, while the rest $2/3$ contribution corresponds to that from the magnetization current.

Summary of CCKE from Wigner function

- The CCKE is derived from 4D Wigner function by an improved perturbative method under the static equilibrium conditions. The CKE in 3D can be obtained by intergation over the time component of the 4-momentum.
- There is freedom to add more terms to the CCKE allowed by conservation laws.
- In the derivation of the 3-dimensional equation, there is also freedom to choose coefficients of some terms in $dx_0/d\tau$ and $d\mathbf{x}/d\tau$ whose 3-mometum integrals are vanishing.
- So the 3-dimensional chiral kinetic equation derived from the CCKE is not uniquely determined in our curre
- nt approach which is a near-equilibrium approach.
- To go beyond the current approach, one needs a new way of building up the CKE in 3D from the CCKE or directly from the covariant equation for the Wigner function.

Wigner functions for massive fermions and applications

- This is last lecture. I would like to thank Defu and XuGuang for their effort and time and for perfect organization. Also many thanks to IOPP-CCNU for sponsoring such a nice summer school which benefit the community especially young students. I myself benefit from it too since it is a good chance for me to sort out my thread of thought again and to deepen my understanding through re-thinking.
- Build up the 1st order solution to Wigner functions for massive fermions
- Two applications: (1) The polarization for massive fermions; (2) Pseudoscalar condensation induced by chiral anomaly and vorticity for massive fermions.

Equation for Wigner function

The Wigner function satisfies the following equation of motion,

$$(\gamma_\mu K^\mu - m)W(x,p) = 0, \tag{108}$$

where the operator K^μ is given by

$$K^\mu = p_{\mathrm{W}}^\mu + i\hbar\frac{1}{2}\nabla^\mu, \tag{109}$$

with

$$\begin{aligned} p_{\mathrm{W}}^\mu &= p^\mu - \hbar\frac{1}{2}Qj_1(\Delta)F^{\mu\nu}\partial_\nu^p \\ \nabla^\mu &= \partial_x^\mu - Qj_0(\Delta)F^{\mu\nu}\partial_\nu^p \end{aligned} \tag{110}$$

$\Delta \equiv \frac{1}{2}\hbar\partial_p \cdot \partial_x$, where ∂_x acts only on $F^{\mu\nu}$

$j_0(x)$ and $j_1(x)$ are spherical Bessel functions

Vasak, Gyulassy, Elze, Ann. Phys.173,462(1987)

Elze, Gyulassy, Vasak, Nucl. Phys. B276,706(1986)

Wigner Function in 4D

Wigner function decomposition in 16 generators of Clifford algebra

$$W(x,p) \;=\; \frac{1}{4}\Big[\underbrace{\mathscr{F}}_{\text{scalar}} + i\gamma_5\underbrace{\mathscr{P}}_{\text{p-scalar}} + \gamma^\mu\underbrace{\mathscr{V}_\mu}_{\text{vector}} + \gamma_5\gamma^\mu\underbrace{\mathscr{A}_\mu}_{\text{axial-vector}} + \frac{1}{2}\sigma^{\mu\nu}\underbrace{\mathscr{S}_{\mu\nu}}_{\text{tensor}}\Big]$$

Currents and energy-momentum tensor can be obtained from Wigner function components

$$j^\mu = \int d^4p\mathscr{V}_\mu, \quad j_5^\mu = \int d^4p\mathscr{A}_\mu, \quad T^{\mu\nu} = \frac{1}{2}\int d^4p p^{(\mu}\mathscr{V}^{\nu)}$$

Vasak, Gyulassy, Elze, Ann. Phys.173,462(1987)

Elze, Gyulassy, Vasak, Nucl. Phys. B276,706(1986)

Equation for Wigner function

If $F^{\mu\nu}$ is a constant we have simpler forms of these operators

$$\begin{aligned} p_{\mathrm{W}}^{\mu} &= p^{\mu} \\ \nabla^{\mu} &= \partial_x^{\mu} - QF^{\mu\nu}\partial_{\nu}^{p} \\ K^{\mu} &= p^{\mu} + i\hbar\frac{1}{2}\nabla^{\mu} \end{aligned} \tag{111}$$

The equation for Wigner function has the familiar form

$$\left[\gamma\cdot\left(p+\frac{1}{2}i\hbar\nabla\right)-m\right]W(x,p)=0 \tag{112}$$

Equation for Wigner function

For constant field, in terms of component, Eq. (112) can be rewritten as

$$\begin{aligned}
K^{\mu}\mathscr{V}_{\mu}-m\mathscr{F} &= 0,\\
K^{\mu}\mathscr{A}_{\mu}+im\mathscr{P} &= 0,\\
K_{\mu}\mathscr{F}+iK^{\nu}\mathscr{S}_{\nu\mu}-m\mathscr{V}_{\mu} &= 0,\\
iK_{\mu}\mathscr{P}+\frac{1}{2}\epsilon_{\mu\beta\nu\sigma}K^{\beta}\mathscr{S}^{\nu\sigma}+m\mathscr{A}_{\mu} &= 0,\\
-i(K_{\mu}\mathscr{V}_{\nu}-K_{\nu}\mathscr{V}_{\mu})-\epsilon_{\mu\nu\alpha\beta}K^{\alpha}\mathscr{A}^{\beta}-m\mathscr{S}_{\mu\nu} &= 0. \qquad (113)
\end{aligned}$$

Equation for Wigner function

The real parts are

$$
\begin{aligned}
p^{\mu}\mathscr{V}_{\mu}-m\mathscr{F} &= 0,\\
p^{\mu}\mathscr{A}_{\mu} &= 0,\\
p_{\mu}\mathscr{F}-\frac{1}{2}\hbar\nabla^{\nu}\mathscr{S}_{\nu\mu}-m\mathscr{V}_{\mu} &= 0,\\
-\frac{1}{2}\hbar\nabla_{\mu}\mathscr{P}+\frac{1}{2}\epsilon_{\mu\beta\nu\sigma}p^{\beta}\mathscr{S}^{\nu\sigma}+m\mathscr{A}_{\mu} &= 0,\\
\frac{1}{2}\hbar(\nabla_{\mu}\mathscr{V}_{\nu}-\nabla_{\nu}\mathscr{V}_{\mu})-\epsilon_{\mu\nu\alpha\beta}p^{\alpha}\mathscr{A}^{\beta}-m\mathscr{S}_{\mu\nu} &= 0.
\end{aligned}
\tag{114}
$$

Equation for Wigner function

The imaginary parts are

$$\begin{aligned}
\hbar\nabla^{\mu}\mathscr{V}_{\mu} &= 0, \\
\frac{1}{2}\hbar\nabla^{\mu}\mathscr{A}_{\mu}+m\mathscr{P} &= 0, \\
\frac{1}{2}\hbar\nabla_{\mu}\mathscr{F}+p^{\nu}\mathscr{S}_{\nu\mu} &= 0, \\
p_{\mu}\mathscr{P}+\frac{1}{4}\hbar\epsilon_{\mu\beta\nu\sigma}\nabla^{\beta}\mathscr{S}^{\nu\sigma} &= 0, \\
(p_{\mu}\mathscr{V}_{\nu}-p_{\nu}\mathscr{V}_{\mu})+\frac{1}{2}\hbar\epsilon_{\mu\nu\alpha\beta}\nabla^{\alpha}\mathscr{A}^{\beta} &= 0. \qquad (115)
\end{aligned}$$

Equation for Wigner function

For massless case, $m = 0$, we have two sets of equations which are decoupled, one set is about $(\mathscr{V}^{\mu}, \mathscr{A}^{\mu})$, another set is about $(\mathscr{F}^{\mu}, \mathscr{P}, \mathscr{S}^{\mu\nu})$.

Wigner function: the 0-th order

At leading order of electromagnetic interaction, the gauge link in the Wigner function can be set to 1, we have

$$W_{\alpha\beta}(x,p)=\int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\left\langle\bar{\psi}_\beta(x+\frac{y}{2})\psi_\alpha(x-\frac{y}{2})\right\rangle . \tag{116}$$

The quantized field

$$\begin{aligned}\psi(x) &= \frac{1}{\sqrt{\Omega}}\sum_{\mathbf{k},s}\frac{1}{\sqrt{2E_k}}[a(\mathbf{k},s)u(\mathbf{k},s)e^{-ik\cdot x}+b^\dagger(\mathbf{k},s)v(\mathbf{k},s)e^{ik\cdot x}]\\ \bar{\psi}(x) &= \frac{1}{\sqrt{\Omega}}\sum_{\mathbf{k},s}\frac{1}{\sqrt{2E_k}}[a^\dagger(\mathbf{k},s)\bar{u}(\mathbf{k},s)e^{ik\cdot x}+b(\mathbf{k},s)\bar{v}(\mathbf{k},s)e^{-ik\cdot x}]\end{aligned}$$

Wigner function: the 0-th order

At leading or the 0-th order

$$\begin{aligned} W_{\alpha\beta}(x,p) &= \frac{1}{(2\pi)^3}\delta(p^2-m^2) \\ & \left\{\theta(p^0)\sum_s f_{\mathrm{FD}}(E_p-\mu_s)u_\alpha(\mathbf{p},s)\bar{u}_\beta(\mathbf{p},s)\right. \\ & \left.-\theta(-p^0)\sum_s f_{\mathrm{FD}}(E_p+\mu_s)v_\alpha(-\mathbf{p},s)\bar{v}_\beta(-\mathbf{p},s)\right\}, \end{aligned} \tag{117}$$

where we have used

$$\begin{aligned} \left\langle a^\dagger(\mathbf{p},s)a(\mathbf{p},s)\right\rangle &= f_{\mathrm{FD}}(E_p-\mu_s) \\ \left\langle b^\dagger(-\mathbf{p},s)b(-\mathbf{p},s)\right\rangle &= f_{\mathrm{FD}}(E_p+\mu_s) \end{aligned} \tag{118}$$

Wigner function: the 0-th order

We extract the scalar component as

$$\begin{aligned}
\mathscr{F}_{(0)} &= \mathrm{Tr}[W] = m\delta(p^2 - m^2)\sum_s f_s \\
\mathscr{V}^{\mu}_{(0)} &= \mathrm{Tr}[\gamma^\mu W] = p^\mu \delta(p^2 - m^2)\sum_s f_s \\
f_s &= \frac{2}{(2\pi)^3}\left[\theta(p^0) f_{\mathrm{FD}}(p_0 - \mu_s)\right. \\
&\quad \left. +\theta(-p^0) f_{\mathrm{FD}}(-p_0 + \mu_s)\right] \qquad (119)
\end{aligned}$$

where we have used

$$\begin{aligned}
\bar{u}(\mathbf{p}, s)u(\mathbf{p}, s) &= 2m \\
\bar{v}(-\mathbf{p}, s)v(-\mathbf{p}, s) &= -2m \\
\bar{u}(\mathbf{p}, s)\gamma^\mu u(\mathbf{p}, s) &= 2(E_p, \mathbf{p}) \\
\bar{v}(-\mathbf{p}, s)\gamma^\mu v(-\mathbf{p}, s) &= 2(E_p, -\mathbf{p})
\end{aligned}$$

Wigner function: the 0-th order

For the axial vector component, we obtain

$$\begin{aligned}\mathscr{A}^{\mu}_{(0)} &= \mathrm{Tr}[\gamma^{\mu}\gamma^{5}W] \\ &= m\left[\theta(p_0)n^{\mu}(\mathbf{p},\mathbf{n})-\theta(-p_0)n^{\mu}(-\mathbf{p},-\mathbf{n})\right] \\ &\quad \times\delta(p^2-m^2)\sum_{s}sf_s \qquad (120)\end{aligned}$$

where we have used

$$\begin{aligned}\bar{u}(\mathbf{p},s)\gamma^{\mu}\gamma^{5}u(\mathbf{p},s) &= 2msn^{\mu}(\mathbf{p},\mathbf{n}) \\ \bar{v}(-\mathbf{p},s)\gamma^{\mu}\gamma^{5}v(-\mathbf{p},s) &= 2msn^{\mu}(-\mathbf{p},-\mathbf{n}) \\ n^{\mu}(\mathbf{p},\mathbf{n})=\Lambda^{\mu}{}_{\nu}(-\mathbf{v}_p)n^{\nu}(\mathbf{0},\mathbf{n}) &= \left(\frac{\mathbf{n}\cdot\mathbf{p}}{m},\mathbf{n}+\frac{(\mathbf{n}\cdot\mathbf{p})\mathbf{p}}{m(m+E_p)}\right)\end{aligned}$$

Wigner function: the 0-th order

One can check that $n^\mu(\mathbf{p},\mathbf{n})$ satisfies $n^2 = -1$ and $n \cdot p = 0$, so it behaves like a spin 4-vector up to a factor of 1/2. For Pauli spinors χ_s and $\chi_{s'}$ in $u(\mathbf{p},s)$ and $v(-\mathbf{p},s')$ respectively, we have $\chi_s^\dagger \boldsymbol{\sigma} \chi_s = s\mathbf{n}$ and $\chi_{s'}^\dagger \boldsymbol{\sigma} \chi_{s'} = -s'\mathbf{n}$. We can take the massless limit by setting $\mathbf{n} = \hat{\mathbf{p}}$, then

$$\begin{aligned} mn^\mu(\mathbf{p},\mathbf{n}) &\rightarrow (|\mathbf{p}|,\mathbf{p}) \\ mn^\mu(-\mathbf{p},-\mathbf{n}) &\rightarrow (|\mathbf{p}|,-\mathbf{p}) \end{aligned}$$

This way we can recover the previous results for massless case,

$$\mathscr{A}^\mu_{(0)} \rightarrow p^\mu \delta(p^2) \sum s f_s$$

Wigner function: the 1-st order for axial vector component

We now propose the form for the axial component at the first order for massive fermions based on the solution in massless case,

$$\begin{aligned}\mathscr{A}^{\alpha}_{(1)}(x,p) &= -\frac{1}{2}\hbar\tilde{\Omega}^{\alpha\sigma}p_{\sigma}\sum_{s}\frac{df_{s}}{dp_{0}}\delta(p^{2}-m^{2}) \\ & -Q\hbar\tilde{F}^{\alpha\lambda}p_{\lambda}(\sum_{s}f_{s})\frac{\delta(p^{2}-m^{2})}{p^{2}-m^{2}} \qquad (121)\end{aligned}$$

where the first term is induced by the vorticity and EM field. We can check that $p\cdot A_{(1)}(x,p)=0$ is satisfied.

Spin tensor and axial vector component

The spin tensor density is defined by

$$M^{\alpha\beta}(x) \;=\; \psi^{\dagger}(x)\frac{1}{2}\sigma^{\alpha\beta}\psi(x)=\frac{1}{2}\mathrm{Tr}\left[\gamma_{0}\sigma^{\alpha\beta}\psi(x)\bar{\psi}(x)\right]. \tag{122}$$

In terms of the Wigner function,

$$\left\langle M^{\alpha\beta}(x)\right\rangle \;=\; \frac{1}{2}\int d^{4}p\mathrm{Tr}\left[\gamma_{0}\sigma^{\alpha\beta}W(x,p)\right]. \tag{123}$$

If we take $\alpha\beta=ij$ (spatial indices), we have a simple relation

$$M^{ij}(x,p) \;=\; -\frac{1}{2}\epsilon^{ijk}\mathscr{A}_{k}(x,p)=\frac{1}{2}\epsilon^{ijk}\mathscr{A}^{k}(x,p), \tag{124}$$

So we can regard the axial vector component as spin density in phase space. This is a good approximation at non-relativistic limit $E_p \approx m$ (for Λ hyperon at RHIC energy).

Spin tensor and axial vector component

In relativistic case we have to put a Lorentz factor in accordance to Pauli-Lubanski pseudovector,

$$\begin{aligned} S^{\mu} &= -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}M_{\nu\rho}p_{\sigma} \\ S^{i} &\rightarrow \frac{p_0}{2m}\epsilon^{ijk}M_{jk} = \frac{p_0}{4m}\epsilon^{ijk}\epsilon^{jkl}\mathscr{A}^{l}(x,p) \\ &= \frac{E_p}{2m}\mathscr{A}^{i}(x,p) \end{aligned} \tag{125}$$

Polarization density

Polarization density from axial vector current,

$$
\begin{aligned}
\Pi &= \int d^4p \frac{E_p}{2m} \mathscr{A}^{\alpha}_{(1)}(x,p) \\
&= -\frac{1}{4m}\hbar \int d^4p E_p \tilde{\Omega}^{\alpha\sigma} p_\sigma \sum_s \frac{df_s}{dp_0}\delta(p^2-m^2) \\
&\quad -\frac{1}{2m} Q\hbar \int d^4p E_p \tilde{F}^{\alpha\lambda} p_\lambda (\sum_s f_s) \frac{\delta(p^2-m^2)}{p^2-m^2} \\
&= \hbar\beta \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left\{ \left(\frac{E_p}{2m}\omega^\alpha + Q\frac{1}{2m}B^\alpha \right) f_{\rm FD}(E_p-\mu)[1-f_{\rm FD}(E_p-\mu)] \right. \\
&\quad \left. + \left(\frac{E_p}{2m}\omega^\alpha - Q\frac{1}{2m}B^\alpha \right) f_{\rm FD}(E_p+\mu)[1-f_{\rm FD}(E_p+\mu)] \right\} \qquad (126)
\end{aligned}
$$

Polarization density and fermion number density

- Polarization density in phase space

$+$: fermions

$-$: anti-fermions

$$\Pi^{\mu}_{\pm}(x,p) = \hbar\beta\left(\frac{E_p}{2m}\omega^{\alpha}+Q\frac{1}{2m}B^{\alpha}\right) \times f_{\mathrm{FD}}(E_p\mp\mu)[1-f_{\mathrm{FD}}(E_p\mp\mu)] \qquad (127)$$

- Polarization density in coordinate space

$$\Pi^{\mu}_{\pm}(x) = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\Pi^{\mu}_{\pm}(x,p) \qquad (128)$$

- Number density for fermions and anti-fermions in phase space and coordinate space

$$\rho_{\pm}(x,p)=2f_{\mathrm{FD}}(E_p\mp\mu),\ \rho_{\pm}(x)=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\rho_{\pm}(x,p)$$

Polarization density and fermion number density

Mean polarization per fermion in phase space and coordinate space

$$\begin{aligned}\overline{\Pi}^{\mu}_{\pm}(x,p) &= \frac{\Pi^{\mu}_{\pm}(x,p)}{\rho_{\pm}(x,p)} \\ \overline{\Pi}^{\mu}_{\pm}(x,p) &= \frac{\Pi^{\mu}_{\pm}(x)}{\rho_{\pm}(x)}\end{aligned}$$

Polarization on freezeout hypersurface

In order to derive the freezeout formula for non-isotropic distribution, we cannot use $p^\mu \rightarrow (u \cdot p)u^\mu$, then the polarization density becomes

$$\begin{aligned}
\Pi^\mu_\pm(x,p) &= \hbar\beta\frac{1}{E_p}\left(\frac{1}{2m}\tilde{\Omega}^{\alpha\sigma}p_\sigma \pm Q\frac{1}{2m}B^\alpha\right) \\
&\quad \times f_{\mathrm{FD}}(E_p \mp \mu)[1 - f_{\mathrm{FD}}(E_p \mp \mu)] \\
&\rightarrow \\
\frac{d\Pi^\mu_\pm(x,p)}{d^3p} &= \hbar\frac{\beta}{2mE_p}\int d\Sigma_\lambda p^\lambda \left(\tilde{\Omega}^{\alpha\sigma}p_\sigma \pm QB^\alpha\right) \\
&\quad \times f_{\mathrm{FD}}(E_p \mp \mu)[1 - f_{\mathrm{FD}}(E_p \mp \mu)] \\
\frac{d\rho_\pm(x,p)}{d^3p} &= \frac{2}{E_p}\int d\Sigma_\lambda p^\lambda f_{\mathrm{FD}}(E_p \mp \mu) \\
P_\pm &= \frac{d\Pi^\mu_\pm(x,p)/d^3p}{d\rho_\pm(x,p)/d^3p}
\end{aligned}$$

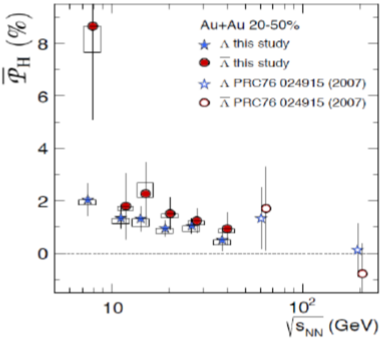
integration over hypersuface

STAR data for Λ polarization

- At each energy, a positive polarization for Λ and $\overline{\Lambda}$ at 1.1-3.6σ level. The polarizations decrease with energies. On average over all data,

$$\mathscr{P}_{\Lambda} = (1.08 \pm 0.15)\%$$
$$\mathscr{P}_{\bar{\Lambda}} = (1.38 \pm 0.30)\%$$

- Systematic uncertainties are smaller than statistical ones and are mainly from estimated combinatoric background of proton-pion pairs.
- Other small systematic uncertainties in the overall scale: a) Λ decay parameter α_H (2%); b) the reaction-plane resolution (2%); c) detector efficiency corrections (3.5%)
- The data contain both primary and those feed-down contributions from heavier particles. The effect of feed-down is about 20% difference between the polarization of primary and all hyperons.



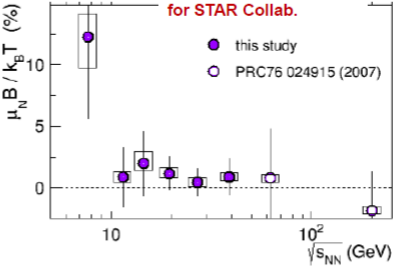
STAR collab., 1701.06657

Can we extract B from data?

- STAR data at low energies

  P ≈ 1-8% from 7.7 to 62.4 GeV

  ΔP ($\bar{\Lambda}$- Λ) ≈ 0.03% - 0.2%
- P_Λ is anti-parallel to B due to negative magnetic moment
- Magnetic field that leads to



Coutesy of Isaac Upsal for STAR Collab.

$$\Delta P \sim \frac{1}{2}\beta\frac{B^\alpha}{m_\Lambda} \qquad \approx O(1)$$

$$\times \sum_{e=\pm}\boxed{\frac{\int d^3p f^e_{\mathrm{FD}}(1-f^e_{\mathrm{FD}})}{\int d^3p f^e_{\mathrm{FD}}}}$$

$$\sim \beta\frac{B^\alpha}{m_\Lambda} \Longrightarrow B \sim T m_\Lambda \Delta P \sim \boxed{(0.1 \sim 0.01)m_\pi^2}$$

[Pang et al. 2016; Becattini et al. 2016 Shuryak 2016]

➡ too large for low energy HIC in freezeout scenario.

- From vorticity, there is more Pauli blocking effect for fermions than anti-fermions in lower energy HIC

Largest vorticity ever observed

- The fluid vorticity may be estimated from the data using the hydrodynamic relation with a systematic uncertainty of a factor of 2, mostly due to uncertainties in the temperature

$$\boxed{\begin{aligned}\omega &\sim k_B T(\mathscr{P}_\Lambda + \mathscr{P}_{\bar{\Lambda}})/\hbar \\ &\approx (9 \pm 1) \times 10^{21}\ \mathrm{s}^{-1}\end{aligned}}$$

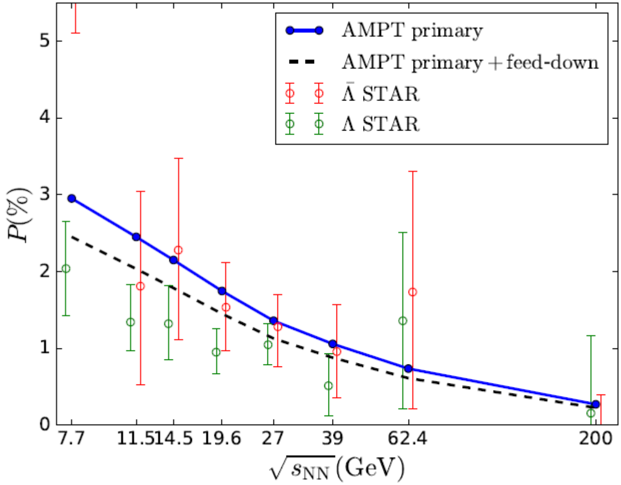
STAR Collab., 1701.06657;
Becattini et al., 1610.02506;
Pang et al., PRC 94, 024904(2016);
Aristova, Frenklakh, Gorsky, Kharzeev, JHEP(2016);

- This far surpasses the vorticity of all other known fluids

solar subsurface flow	$10^{-7}\ \mathrm{s}^{-1}$
large scale terrestrial atmospheric patterns	$10^{-7} - 10^{-5}\ \mathrm{s}^{-1}$
Great Red Spot of Jupiter	$10^{-4}\ \mathrm{s}^{-1}$
supercell tornado cores	$10^{-1}\ \mathrm{s}^{-1}$
rotating, heated soap bubbles	$100\ \mathrm{s}^{-1}$
turbulent flow in bulk superfluid He-II	$150\ \mathrm{s}^{-1}$
superfluid nanodroplets	$10^{7}\ \mathrm{s}^{-1}$

Our latest results by AMPT model

Li, Pang, QW, Xia, 1704.01507



Pseudoscalar condensation induced by chiral anomaly and vorticity

In quantum electrodynamics the anomalous nonconservation of the chiral or axial vector current can be written as

$$\begin{aligned}\partial_\mu j_5^\mu &= -2mP - \frac{Q^2}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} \\ \partial_\mu j_{5,\pi}^\mu &= f_\pi m_\pi^2 \phi_\pi - \frac{Q_e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}\end{aligned}$$

Here fermion current, $j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$, $P = -i\bar{\psi}\gamma_5\psi$. For the pion current, we have $j_{5,\pi}^\mu = \bar{\psi}\gamma^\mu\gamma_5(\sigma_3/2)\psi$, $P_\pi = -i\bar{\psi}\gamma_5(\sigma_3/2)\psi$, where $\psi = (u,d)^T$ and $\bar{\psi} = (\bar{u}, \bar{d})$. We have PCAC hypothesis, $2m_q P_\pi = -f_\pi m_\pi^2 \phi_\pi$

Pseudoscalar condensation induced by chiral anomaly and vorticity

There is an equation that relates the pseudoscalar to the axial vector component which is of special interest,

$$\hbar\nabla^{\mu}\mathscr{A}_{\mu} = -2m\mathscr{P}$$

An interesting observation of the above equation is that the pseudoscalar component $\mathscr{P}$ is of quantum origin since it is proportional to the Planck constant $\hbar$.

Pseudoscalar condensation induced by chiral anomaly and vorticity

We obtain the pseudoscalar condensate [R. Fang, et al., 1611.04670]

$$\begin{aligned} P &= \frac{1}{4\pi^2}\hbar^2 Q^2(E\cdot B)\frac{1}{m}[C_1(\beta m,\beta\mu)-1] \\ &\quad +\frac{1}{4\pi^2}\hbar^2 Q(E\cdot\omega)\beta m C_2(\beta m,\beta\mu). \end{aligned} \tag{129}$$

From the small mass behavior $C_1(\beta m,\beta\mu)-1\sim(\beta m)^2$ and $C_2(\beta m,\beta\mu)\sim O(1)$, the pseudoscalar is proportional to the fermion mass.

Pseudoscalar condensation induced by chiral anomaly and vorticity

We can also obtain the pion condensate at finite T and μ,

$$\begin{aligned} P_\pi &= \frac{N_C}{24\pi^2}\hbar^2 Q_e^2(E\cdot B)\frac{1}{m_q}[C_1(\beta m_q,\beta\mu_q)-1] \\ &\quad +\frac{N_C}{8\pi^2}\hbar^2 Q_e(E\cdot\omega)\beta m_q C_2(\beta m_q,\beta\mu_q) \end{aligned}$$

In vacuum at zero temperature and chemical potentials, both functions C_1 and C_2 are vanishing and we obtain

$$P_\pi^{\mathrm{vac}} = -\frac{\hbar^2}{8\pi^2 m_q} Q_e^2(E\cdot B), \tag{130}$$

which is consistent with the result derived in the NJL model and chiral perturbation theory [Cao, Huang, Phys. Lett. B757 (2016) 1].