CERN, March 23, 2017

Holography and the quantum renormalization group

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Bibliography

Ongoing work with:

Francesco Nitti, Lukas Witkowski and Leandro Pimenta (APC, Paris)

Published work in ArXiv:1611.05493

Based on earlier work:

- with Francesco Nitti and Wenliang Li ArXiv:1401.0888
- with Vassilis Niarchos ArXiv:1205.6205

Introduction

• The Wilsonian RG is controlled by first order flow equations of the form

$$\frac{dg_i}{dt} = \beta_i(g_i) \quad , \quad t = \log \mu$$

• Despite our knowledge, there are many aspects of QFT RG flows (in the most symmetric case: unitary relativistic QFTs), that are still not understood.

♠ It is not known if the end-points of RG flows in 4d are fixed points or include other exotic possibilities (limit circles or “chaotic” behavior)

♠ This is correlated with the potential symmetry of scale invariant theories: are they always conformally invariant? (CFTs)?

• In 2d, the answer to this question is ”yes”.

♠ Although in 4d this has been analyzed also recently, there are still loopholes in the argument.

El Showk+Rychkov+Nakayama, Luty+Polchinski+Rattazzi,

Dymarsky+Komargodksi+Schwimmer+Theisen+Farnsworth+Luty+Prilepina

♠ In 2d it is a folk-theorem that the strong version of the c-theorem is expected to exclude limit cycles and other exotic behavior in Unitary Relativistic QFTs.

Zamolodchikov

♠ In 4d, we have the weak form of the a-theorem, proved recently.

Komargodski+Schwimmer

♠ We also have a perturbative proof of the strong version, but with important subtleties.

Osborn, Jack+Osborn

♠ The relation between flows of couplings (β-functions) and the trace of the stress tensor is subtle.

Osborn, Fortin+Grinstein+Stergiou

♠ The subtleties include the possibility that limit cycles appear, but they are artifacts that can be "redefined away". They involve rotations in the space of coupling constants.

Fortin+Grinstein+Stergiou, Luty+Polchinski+Rattazzi, Nakayama

♠ The description of this effect in holography is literally a "gauge artifact" (because the global symmetry is gauged).

• The folk-theorem between the strong version of the a-theorem and the appearance of limit cycles has at least one important loop hole:

If the β-functions have branch singularities away from the UV fixed point, then a limit cycle can be compatible with the strong version of the a/c-theorem.

Curtright+Zachos

• If it ever happens, this can only happen "beyond perturbation theory".

Holography and Quantum RG

• Enter holography as a means of probing strong coupling behavior.

• Holography provides a neat description of RG Flows.

• It also gives a natural a-function and the strong version of the a-theorem holds.

♠ But...the relevant equations that are converted into RG equations are second order!

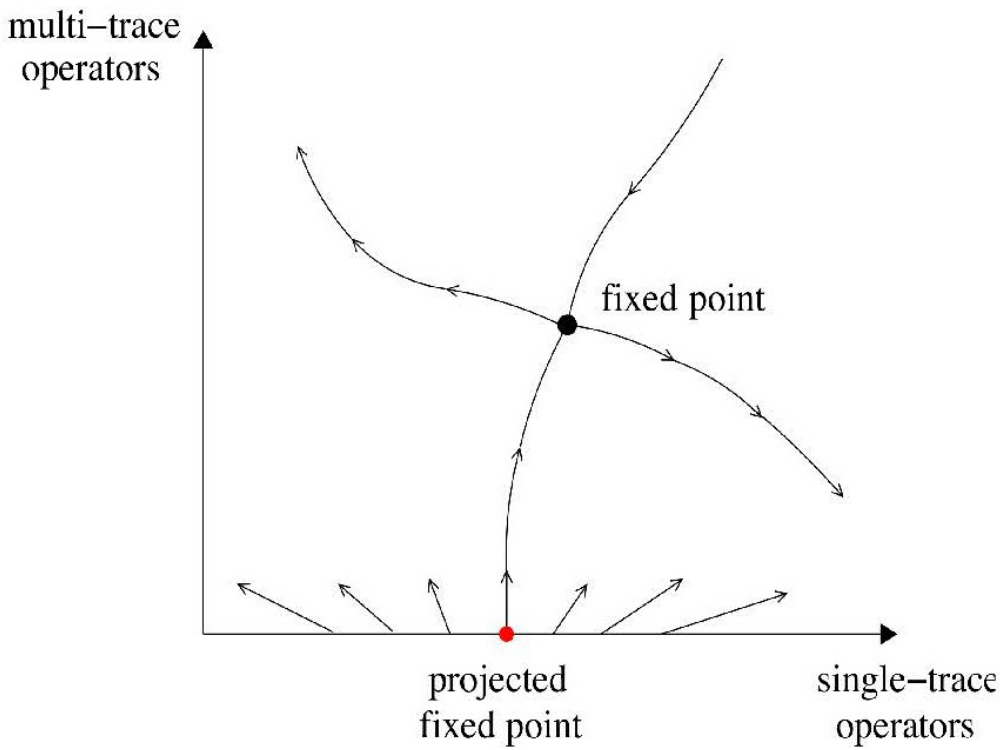
• It is known for some time that the Hamilton-Jacobi formalism in holography gives first order RG-equations.

de Boer+Verlinde[2], Skenderis+Townsend, Gursoy+Kiritsis+Nitti, Papadimitriou, Kiritsis+Li+Nitti

• This would imply that (conceptually at least) holographic RG flows are very similar to (perturbative) QFT flows.

• S. S. Lee has argued that by projecting the RG flow on single trace couplings (as required in holography) turns the RG equation into a second order flow equation.

• He called this equation the quantum RG equation.



• In theories with a holographic dual this equation is expected to match the holographic RG equations.

The Goal

• Sporadic investigations of holographic RG flows have indicated that they have the structure that we see in QFT.

• We would like to do a systematic study of holographic RG flows and see to what extend and when the second order nature of the bulk equations matters.

• We would like investigate whether there are holographic RG flows that do not match the standard QFT intuition.

The strategy

• Review of the holographic RG flows.

• Understanding the space of solutions.

• Standard RG flows start a maximum of the bulk potential and end at a nearby minimum.

• We find exotic holographic RG flows:

♠ "Bouncing flows" : the β-function has branch cuts.

♠ "Skipping flows" : the theory bypasses the next fixed point.

♠ "Irrelevant vev flows" : the theory flows between two minima of the bulk potential.

• Outlook

Holographic RG flows: the setup

• For simplicity and clarity I will consider the bulk theory to contain only the metric and a single scalar (Einstein-dilaton gravity), dual to the stress tensor $T_{\mu\nu}$ and a scalar operator O of a dual QFT.

• The two derivative action (after field redefinitions) is

$$S_{bulk} = M^{d-1} \int d^{d+1}x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right] + S_{GH}$$

• We assume $V(\phi)$ is analytic everywhere except at $\phi = \pm\infty$.

• We will consider the AdS regime: ($V < 0$ always) and look for solutions with d-dimensional Poincaré invariance.

$$ds^2 = du^2 + e^{2A(u)} dx_\mu dx^\mu \quad , \quad \phi(u)$$

• The Einstein equations give:

$$2(d-1)\ddot{A} + \dot{\phi}^2 = 0 \quad , \quad d(d-1)\dot{A}^2 - \frac{1}{2}\dot{\phi}^2 + V(\phi) = 0$$

• There are three integration constants in the equations above.

• The Einstein equations can be turned to first order equations using the "superpotential" (no-supersymmetry here).

$$\dot{A} = -\frac{1}{2(d-1)}W(\phi) \quad , \quad \dot{\phi} = W'(\phi) \quad , \quad dot = \frac{d}{du}$$

$$-\frac{d}{4(d-1)}W(\phi)^2 + \frac{1}{2}W(\phi)'^2 = V(\phi) \quad , \quad ' = \frac{d}{d\phi}$$

• This map fails ONLY where $\dot{\phi} = 0$.

• These equations have the same number of integration constants. In particular there is a continuous one-parameter family of $W(\phi)$.

• Given a $W(\phi)$, $A(u)$ and $\phi(u)$ can be found by integrating the first order flow equations.

• The two integration constants will be later interpreted as couplings of the dual QFT.

• The third integration constant hidden in the superpotential equation controls the vev of the operator dual to ϕ.

• Therefore:

RG flows are in one-to one correspondence with the solutions of the "superpotential equation".

$$-\frac{d}{4(d-1)}W(\phi)^2+\frac{1}{2}W(\phi)'^2=V(\phi)$$

• This is the key equation I will be addressing in the rest of this talk.

Regularity

- One key point: out of all solutions W, typically one only gives rise to a regular bulk solution. (and more generally a discrete number*).

- All others have bulk singularities and are therefore unacceptable* (holographic) classical solutions.

- This reduces the number of (continuous) integration constants from 3 to 2.

- This has a natural interpretation in the dual QFT: the theory determines it possible vevs (we exclude flat directions).

- The remaining first order equations are now the first order RG equations for the coupling and the space-time volume.

- Now we can favorably compare with QFT RG Flows.

Holographic RG Flows

• A QFT with a (relevant) scalar operator $O(x)$ that drives a flow, has two parameters: the scale factor of a flat metric, and the $O(x)$ coupling constant.

• These two parameters, generically correspond to the two integration constants of the first order bulk equations.

• Since ϕ is interpreted as a running coupling and A is the log of the RG energy scale, the holographic β-function is

$$\dot{A} = -\frac{1}{2(d-1)} W(\phi) \quad , \quad \dot{\phi} = W'(\phi)$$

$$\frac{d\phi}{dA} = -\frac{1}{2(d-1)} \frac{d}{d\phi} \log W(\phi) \equiv \beta(\phi) \sim \frac{1}{C}\frac{d}{d\phi} C(\phi)$$

• $C \sim 1/W^{d-1}$ is the (holographic) C-function for the flow.

Girardello+Petrini+Porrati+Zaffaroni, Freedman+Gubser+Pilch+Warner

• $W(\phi)$ is the non-derivative part of the Schwinger source functional of the dual QFT =on-shell bulk action.

de Boer+Verlinde[2]

$$S_{on-shell} = \int d^dx\sqrt{\gamma}\ W(\phi) + \cdots \Big|_{u\to u_{UV}}$$

• The renormalized action is given by

$$S_{renorm} = \int d^dx\sqrt{\gamma}\ (W(\phi) - W_{ct}(\phi)) + \cdots \Big|_{u\to u_{UV}} =$$

$$= constant \int d^dx\ e^{dA(u_0) - \frac{1}{2(d-1)}\int_{\phi_{UV}}^{\phi_0} d\tilde{\phi}\frac{W'}{W}} + \cdots$$

• The statement that $\frac{dS_{renorm}}{du_0} = 0$ is equivalent to the RG invariance of the renormalized Schwinger functional.

• It is also equivalent to the RG equation for ϕ.

• We can prove that

$$T_\mu{}^\mu = \beta(\phi)\ \langle O\rangle$$

• The Legendre transform of S_{renorm} is the (quantum) effective potential for the vev of the QFT operator O.

Detour: The local RG

- The holographic RG can be generalized straightforwardly to the local RG

$$\dot\phi = W' - f'\ R + \frac{1}{2}\left(\frac{W}{W'}f'\right)'(\partial\phi)^2 + \left(\frac{W}{W'}f'\right)\Box\phi + \cdots$$

$$\dot\gamma_{\mu\nu} = -\frac{W}{d-1}\gamma_{\mu\nu} - \frac{1}{d-1}\left(f\ R + \frac{W}{2W'}f'(\partial\phi)^2\right)\gamma_{\mu\nu} +$$

$$+2f\ R_{\mu\nu} + \left(\frac{W}{W'}f' - 2f''\right)\partial_\mu\phi\partial_\nu\phi - 2f'\nabla_\mu\nabla_\nu\phi + \cdots$$

Kiritsis+Li+Nitti

- $f(\phi)$, $W(\phi)$ are solutions of

$$-\frac{d}{4(d-1)}W^2 + \frac{1}{2}W'^2 = V \quad , \quad W'\ f' - \frac{d-2}{2(d-1)}W\ f = 1$$

- Like in 2d σ-models we may use it to define "geometric" RG flows.

General properties of the superpotential

- From the superpotential equation we obtain a bound:

$$W(\phi)^2 = -\frac{4(d-1)}{d}V(\phi) + \frac{2(d-1)}{d}W'^2 \geq -\frac{4(d-1)}{d}V(\phi) \equiv B^2(\phi) > 0$$

- Because of the $(u, W) \to (-u, -W)$ symmetry we can fix the flow (and sign of W) so that we flow from $u = -\infty$ (UV) to $u = \infty$ (IR). This implies that:

$$W > 0 \quad \text{always} \quad \text{so} \quad W \geq B$$

- The holographic "a-theorem":

$$\frac{dW}{du} = \frac{dW}{d\phi}\,\frac{d\phi}{du} = W'^2 \geq 0$$

so that the a-function any decreasing function of W always decreases along the flow, ie. W is positive and increases.

• The inequality now can be written directly in terms of W:

$$W(\phi) \quad \geq \quad B(\phi) \quad \equiv \quad \sqrt{-\frac{4(d-1)}{d}V(\phi)}$$

• The maxima of V are minima of B and the minima of V are maxima of B.

• The bulk potential provides a lower boundary for W and therefore for the associated flows.

• Regularity of the flow=regularity of the curvature and other invariants of the bulk theory:

A flow is regular iff W, V remain finite during the flow.

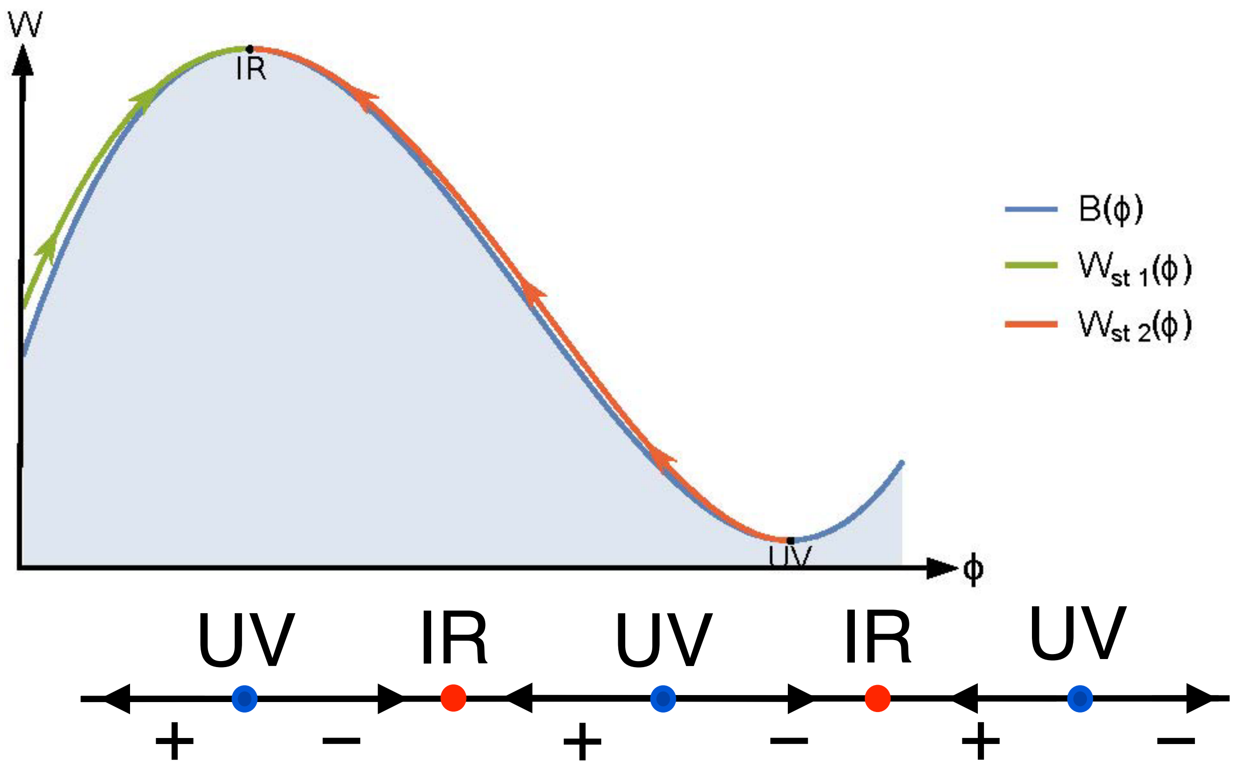
• As V is assumed finite for ϕ finite. The same can be proven for W.

Therefore singular flows end up at $\phi \to \pm\infty$

.

The standard holographic RG flows

• The standard lore says that the maxima of the potential correspond to UV fixed points, the minima to IR fixed points, and the flow from a maximum is to the next minimum.

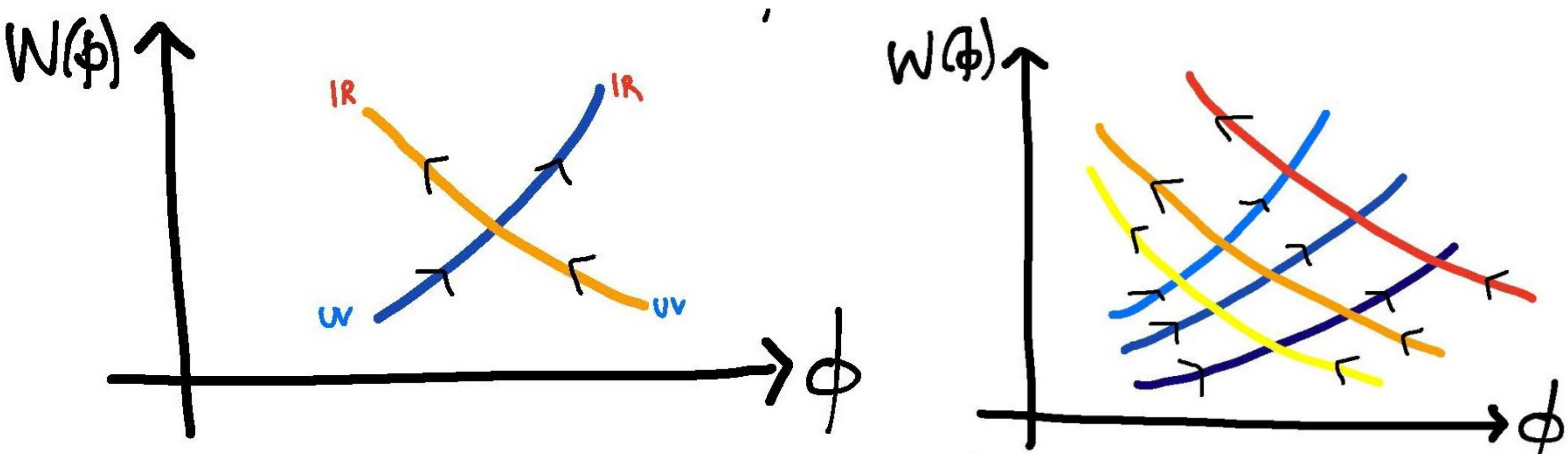


• The real story is a bit more complicated.

More flow rules

• At every point away from the $B(\phi)$ boundary ($W > B$) always two solutions pass:

$$W' = \pm\sqrt{2V + \frac{d}{2(d-1)}W^2} = \pm\sqrt{\frac{d}{2(d-1)}\left(W^2 - B^2\right)}$$



The critical points of W

- On the boundary $W = B$, we obtain $W' = 0$ and only one solution exists.

- The critical $(W' = 0)$ points of W come in three kinds:

♠ $W = B$ at non-extremum of the potential (generic).

♠ Maxima of V (minima of B) (non-generic)

♠ Minima of V (maxima of B) (non-generic)

The maxima of V

- We will examine solutions for W near a maximum of V.

- We put the maximum at $\phi = 0$.

- When $V'(0) = 0$, $W''(0)$ is finite.

$$V(\phi) = -\frac{1}{\ell^2}\left[d(d-1) - \frac{m^2\ell^2}{2}\phi^2 + \mathcal{O}(\phi^3)\right]$$

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2} \quad , \quad m^2\ell^2 \quad < 0 \quad , \quad \Delta_+ \geq \Delta_- \geq 0$$

- We set (locally) $\ell = 1$ from now on.

- If $W'(0) \neq 0$ there is one solution (per branch) off the critical curve,

- If $W'(0) = 0$ there are two classes of solutions:

• A continuous family of solutions (the W_- family)

$$W_- = 2(d-1) + \frac{\Delta_-}{2}\phi^2 + \cdots + C\phi^{\frac{d}{\Delta_-}}[1+\cdots] + \mathcal{O}(C^2)$$

• The solution for ϕ and A corresponding to this, is the standard UV source flow:

$$\phi(u) = \alpha\, e^{\Delta_- \; u} + \cdots + \frac{\Delta_-}{d}\, C\, e^{\Delta_+ \; u} + \cdots \quad , \quad e^A = e^{u-A_0} + \cdots \quad , \quad u \to -\infty$$

• the solution describes the UV region ($u \to -\infty$) with a perturbation by a relevant operator of dimension $\Delta_+ < d$.

• The source is α. It is not part of W.

• C determines the vev: $\langle O \rangle \sim C\ \alpha^{\frac{\Delta_+}{\Delta_-}}$.

• The near-boundary AdS is an attractor of all these solutions.

• A single isolated solution W_+ also arriving at $W(0) = B(0)$

$$W_+ = 2(d-1) + \frac{\Delta_+}{2}\phi^2 + \mathcal{O}(\phi^3) \quad , \quad \Delta_+ > \Delta_-$$

• Always $W''_+ > W''_-$.

• The associated solution for ϕ, A is

$$\phi(u) = \alpha \; e^{\Delta_+ \; u} + \cdots \quad , \quad e^A = e^{-u+A_0} + \cdots$$

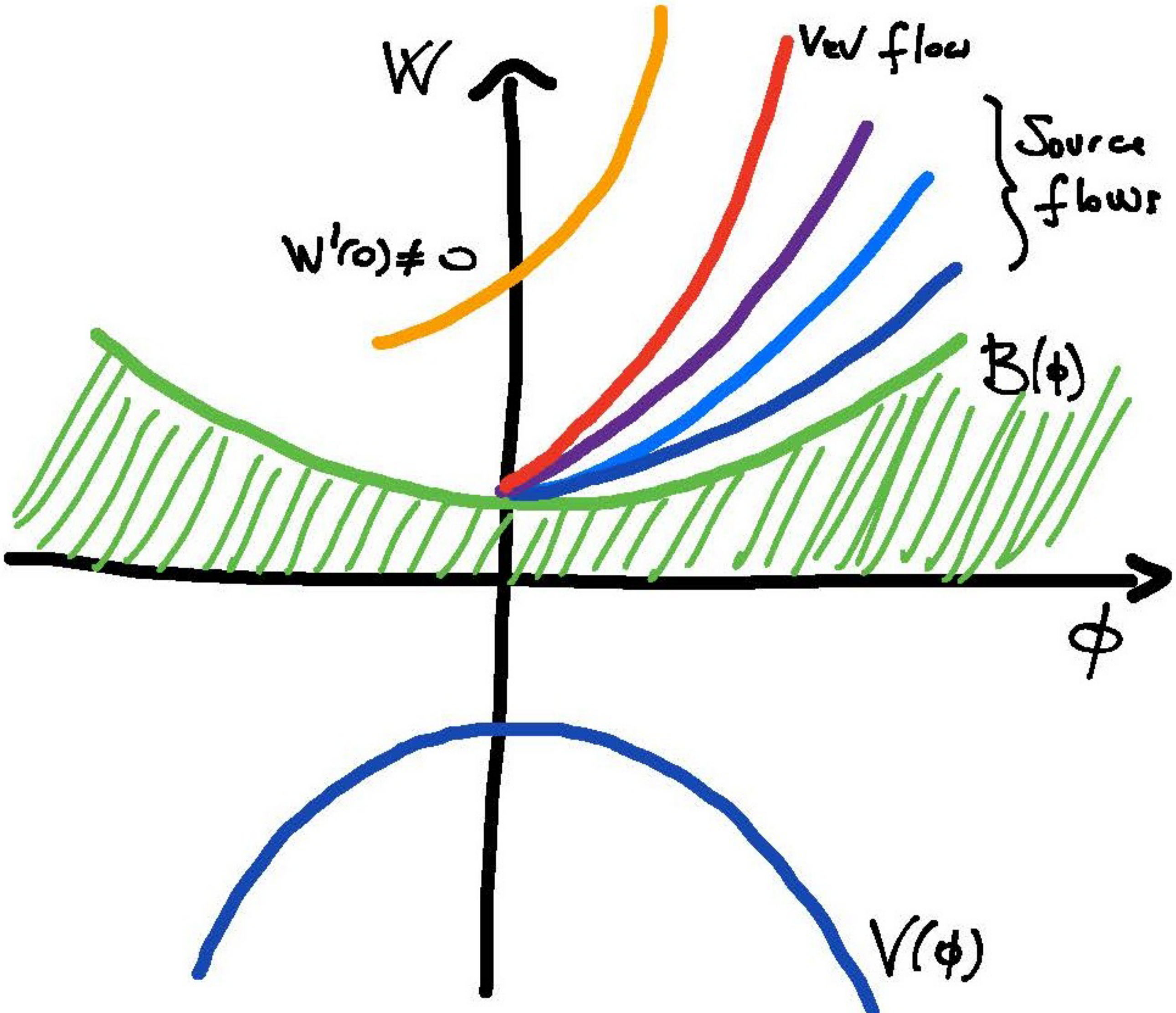
• This is a vev flow ie. the source is zero.

$$\langle O \rangle = (2\Delta_+ - d) \; \alpha$$

• The value of the vev is NOT determined by the superpotential equation.

• It can be reached in a appropriately defined limit $C \to \infty$ of the W_- family.

• The whole class of solutions exists both from the left of $\phi = 0$ and from the right.



RETURN1 RETURN2

The BF bound

- The BF bound can be written as

$$\frac{4(d-1)}{d}\ \frac{V''(0)}{V(0)} \leq 1$$

- If a solution for W near $\phi = 0$ exists, then the BF bound is automatically satisfied as it can be written

$$\left(\frac{4(d-1)}{d}\frac{W''(0)}{W(0)} - 1\right)^2 \geq 0$$

- When BF is violated, although there is no (real) W, there exists a UV-regular solution for the flow: $\phi(u)$, $A(u)$.

- This solution is unstable against linear perturbations (and corresponds to a non-unitary CFT).

The minima of V

- We expand the potential near the minimum:

$$V(\phi) = -\frac{1}{\ell^2}\left[d(d-1) - \frac{m^2\ell^2}{2}\phi^2 + \mathcal{O}(\phi^3)\right] \quad , \quad \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2}$$

$$m^2 > 0 \quad , \quad \Delta_+ > 0 \quad , \quad \Delta_- < 0$$

- There are solutions with $W'(0) \neq 0$. These are solutions that do not stop at the minimum.

- There are two isolated solutions with $W'(0) = 0$.

$$W_\pm(\phi) = \frac{1}{\ell}\left[2(d-1) + \frac{\Delta_\pm}{2}\phi^2 + \mathcal{O}(\phi^3)\right],$$

- No continuous parameter here as it generates a singularity.

- Although the solutions look similar, their interpretation is very different. W_+ has a local minimum while W_- has a local maximum.

• The W_- solution:

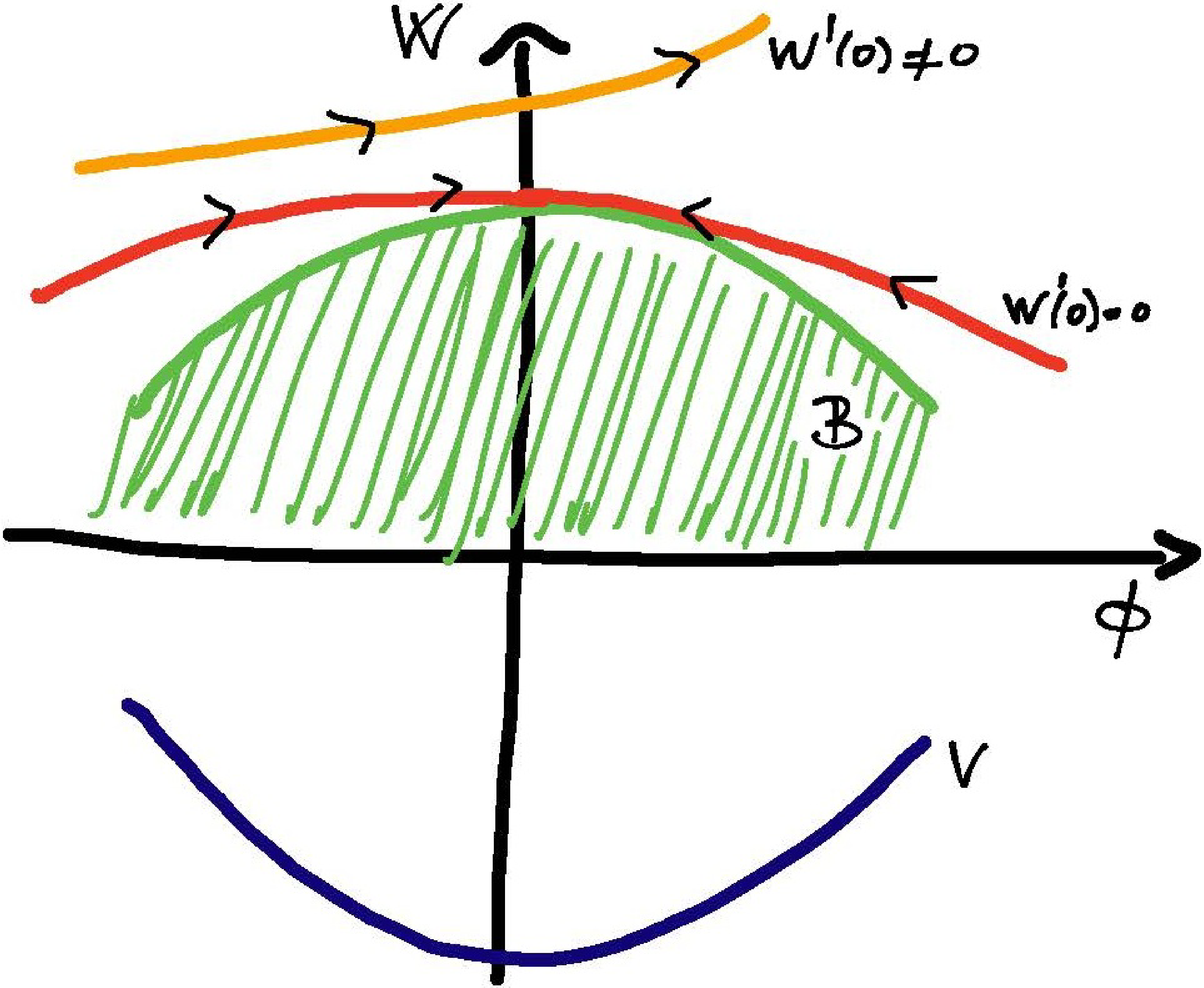
$$\phi(u) = \alpha\ e^{\Delta_-\ u} + \cdots \quad , \quad e^A = e^{-(u-u_0)} + \cdots .$$

• Since $\Delta_- < 0$, small ϕ corresponds to $u \to +\infty$ and $e^A \to 0$.

• This signal we are in the deep interior (IR) of AdS.

• The driving operator has (IR) dimension $\Delta_+ > d$ and a zero vev in the IR.

• Therefore W_- generates locally a flow that arrives at an IR fixed point.


W
W'(0)≠0
W'(0)=0
B
ϕ
V

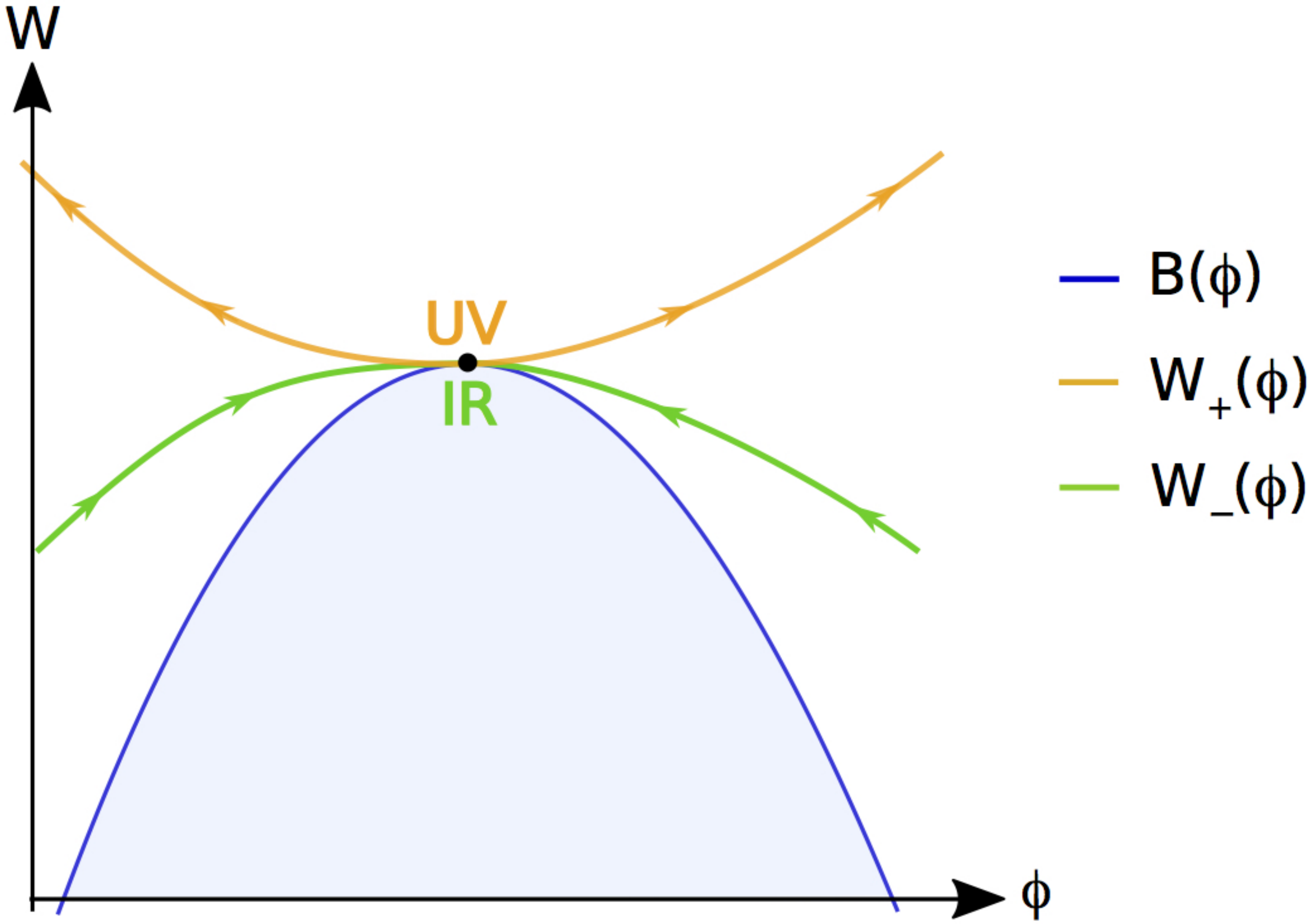
• The W_+ solution is:

$$\phi(u) = \alpha \, e^{\Delta_+ \, u} + \cdots \quad , \quad e^A = e^{-(u-u_0)} + \cdots .$$

• Since $\Delta_+ > 0$ small ϕ corresponds to $u \to -\infty$ and $e^A \to +\infty$.

• This solution described the near-boundary (UV) region of a fixed point.

• This solution is driven by the vev of an operator with (UV) dimension $\Delta_+ > d$ (irrelevant).



♠ A minimum of the potential can be either an IR fixed point or a UV fixed point.

Bounces

- Generic extrema of W : $W'(\phi_B) = 0$ but $V'(\phi_B) \neq 0$.

- From the superpotential equation it is easy to show that

$$W'W'' \simeq V'$$

and therefore:

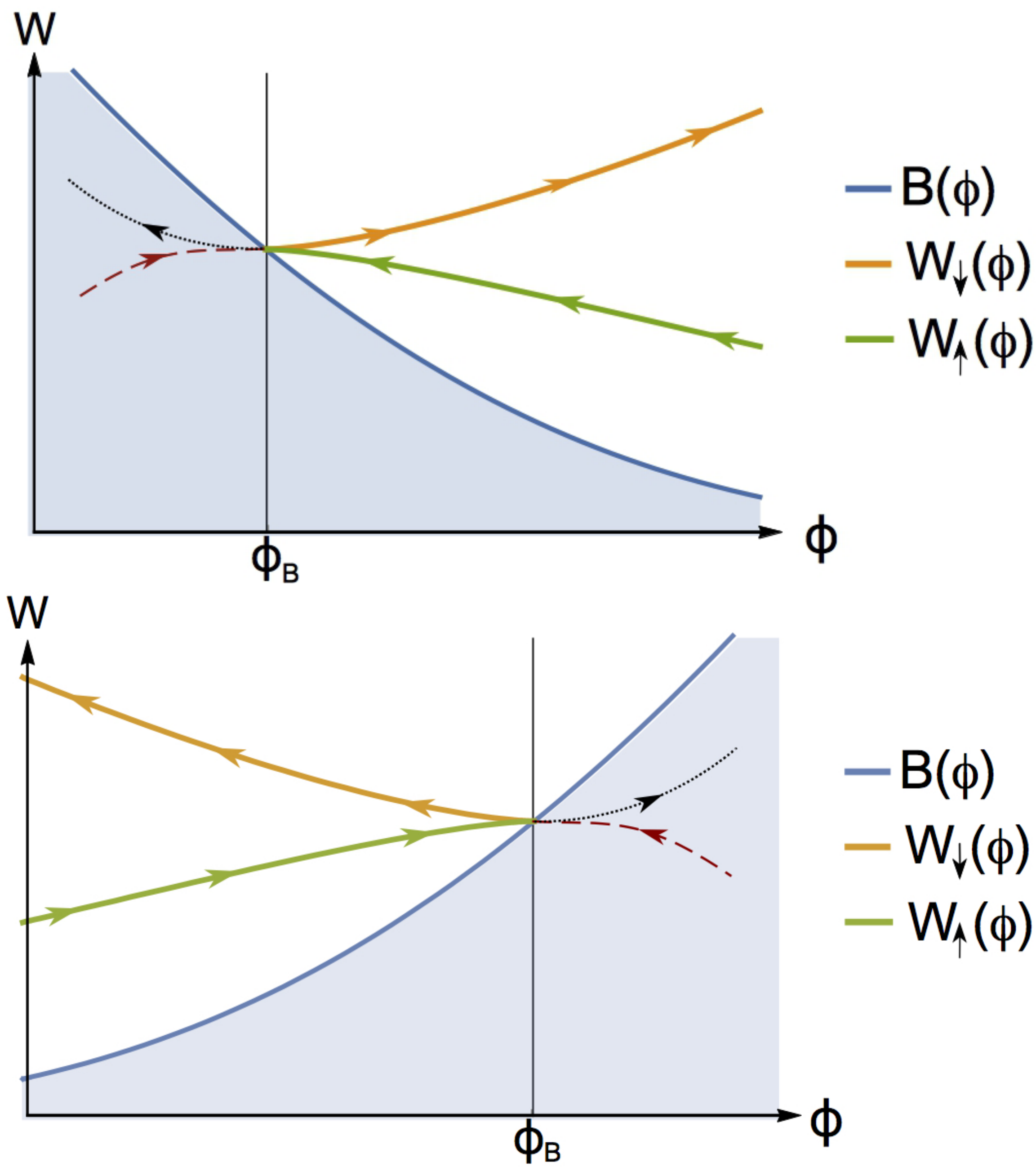
$$W'(\phi_B) = 0 \quad , \quad W''(\phi_B) = \infty$$

- The solution is not analytic but

$$W_\pm(\phi) = B(\phi_B) \pm (\phi - \phi_B)^{\frac{3}{2}} + \cdots$$

- $\pm$ corresponds to the two signs of W'.

- The two branches can be glued together to make a single solution.


W
B(ϕ)
$W_{\downarrow}(\phi)$
$W_{\uparrow}(\phi)$
ϕ_B
ϕ
W
B(ϕ)
$W_{\downarrow}(\phi)$
$W_{\uparrow}(\phi)$
ϕ_B
ϕ

• A flow cannot end at ϕ_B as the resulting geometry is not geodesically complete.

• To obtain a complete geometry we must glue the two solutions.

• Although W is not analytic at ϕ_B, the full solution (geometry+ϕ) is regular at the bounce.

$$\phi(u) = \phi_B + \frac{V'^2}{3}(u-u_B)^2 + \cdots \quad , \quad A(u) = A_B - \sqrt{-\frac{V(\phi_B)}{d(d-1)}}(u-u_B) + \cdots$$

• W as a function of u is both continuous and regular at the bounce.

• W is increasing although W' changes sign!

• The only special thing that happens is that $\dot{\phi} = 0$ at the bounce.

• All bulk curvature invariants are regular at the bounce!

• All fluctuation equations of the bulk fields are regular at the bounce!

• The holographic β-function behaves as

$$\beta = \pm\sqrt{-2d(d-1)\frac{V'(\phi_B)}{V(\phi_B)}(\phi-\phi_B)} + \mathcal{O}(\phi-\phi_B)$$

• The β-function is patch-wise defined. It has a branch cut at the position of the bounce.

• This is non-perturbative behavior.

• Such behavior was conjectured that could lead to limit cycles without violation of the a-theorem.

Curtright+Zachos

Global Regularity

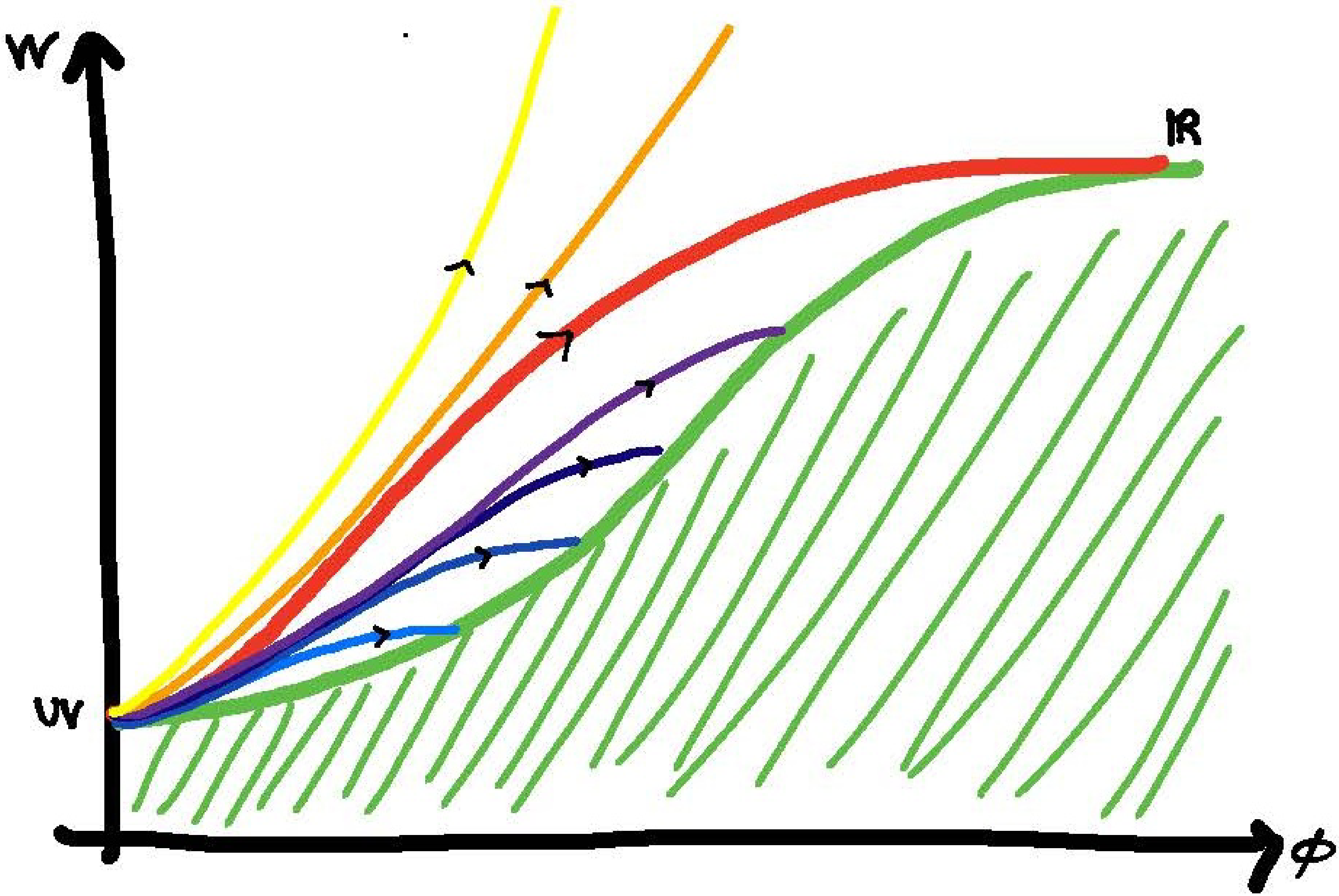
• We have analysed the local behavior of solutions W to the superpotential equation and all its critical points.

• Flows start and end at the extrema of the potential or at $\phi = \pm\infty$.

• For the analytic potentials we assumed, then all regular flows are all solutions for $W(\phi)$, which remain finite along the flow.

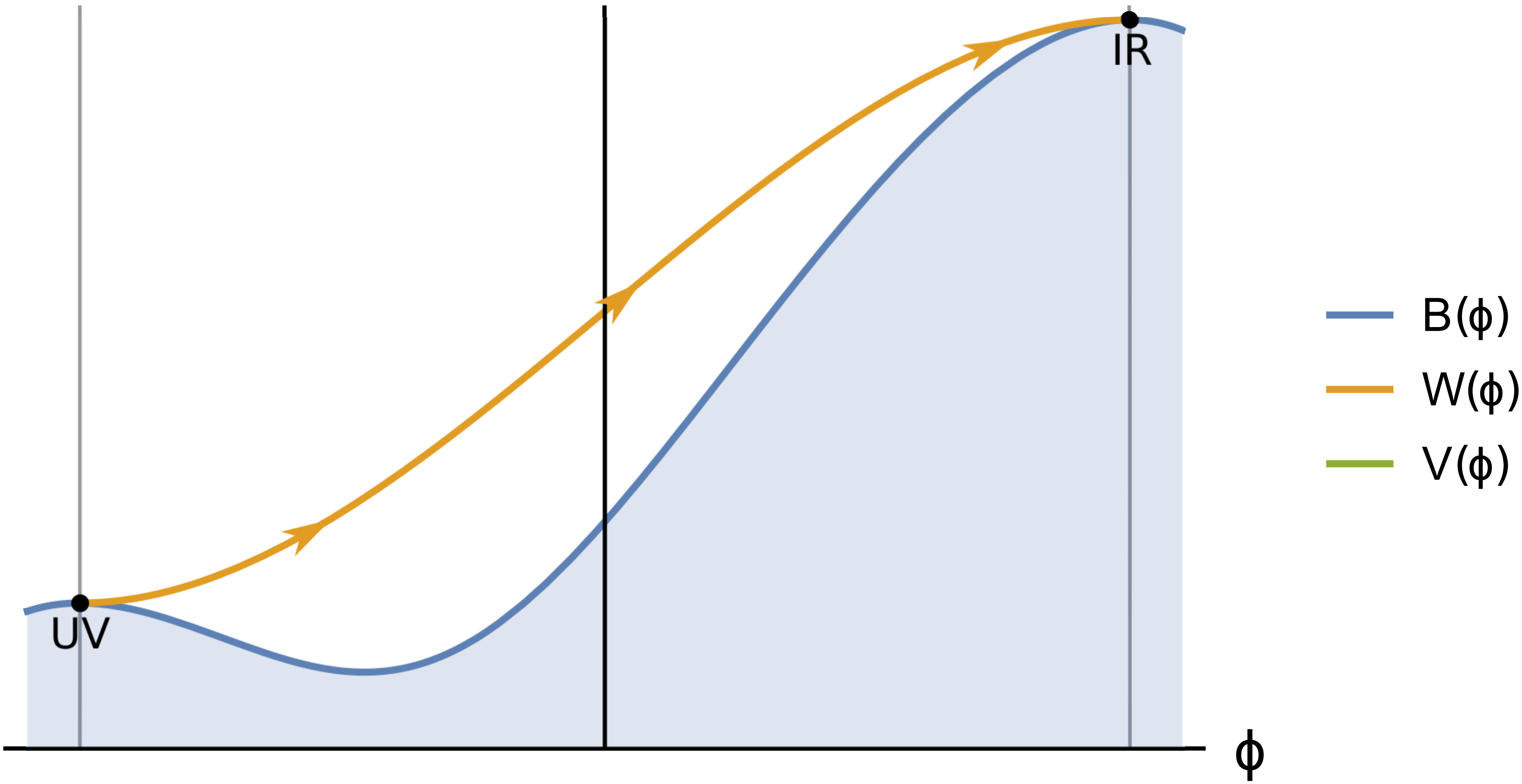
• Regular flows can start and end ONLY at critical points of the potential.

• What these flows are, depends on the details of the potential.


W
IR
UV
φ

Exotica

- Vev flow between two minima of the potential



- Exists only for special potentials

An example was discussed in a cosmological setting.

Libanov+Rubakov+Sibiryakov

- A potential:

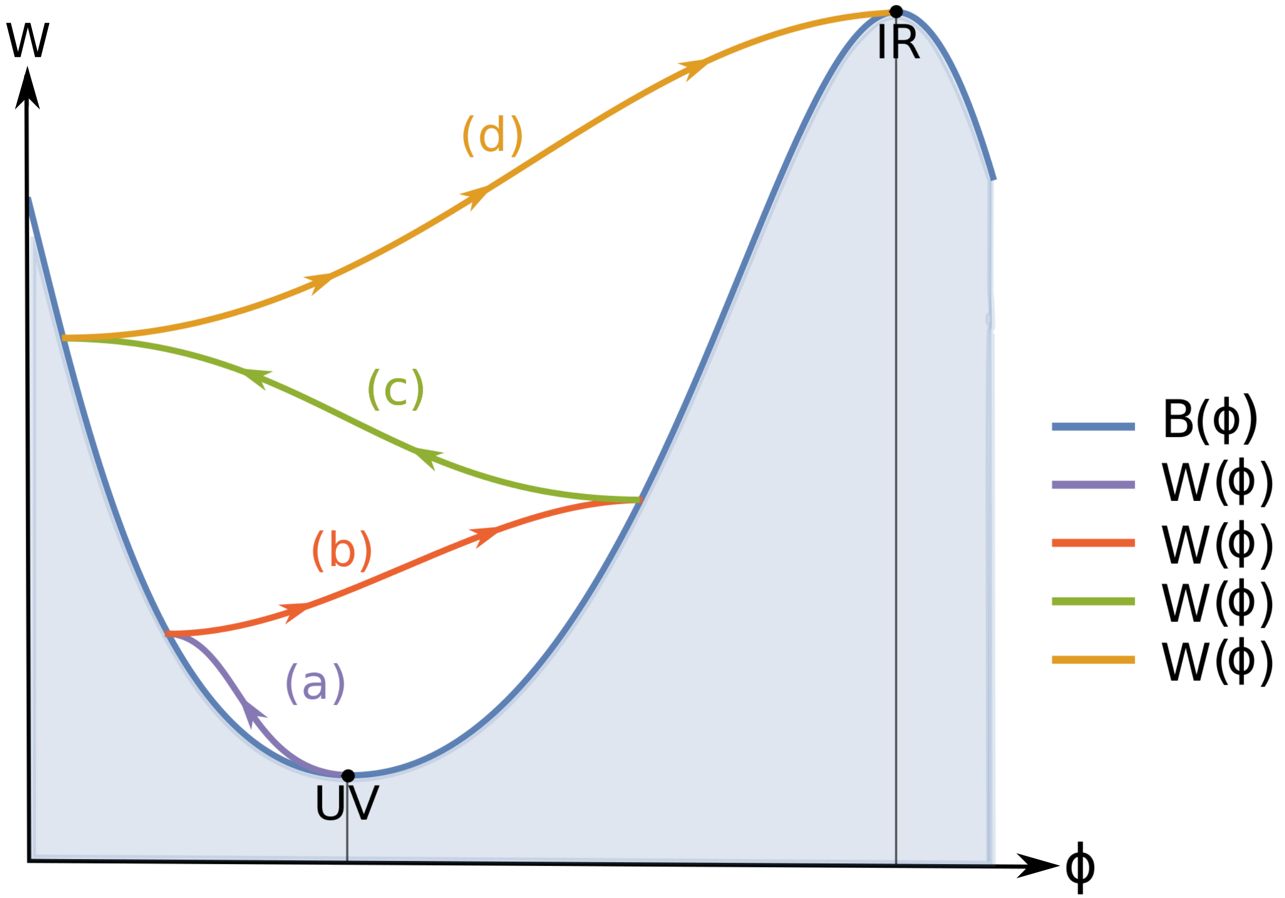
$$V(\phi) = \frac{(kv)^2}{2}\left[1-\left(\frac{\phi}{v}\right)^2\right]^2 - \frac{d}{4(d-1)}\left\{kv^2\left(\frac{\phi}{v}\right)\left[1-\frac{1}{3}\left(\frac{\phi}{v}\right)^2\right]+W_0\right\}^2 .$$

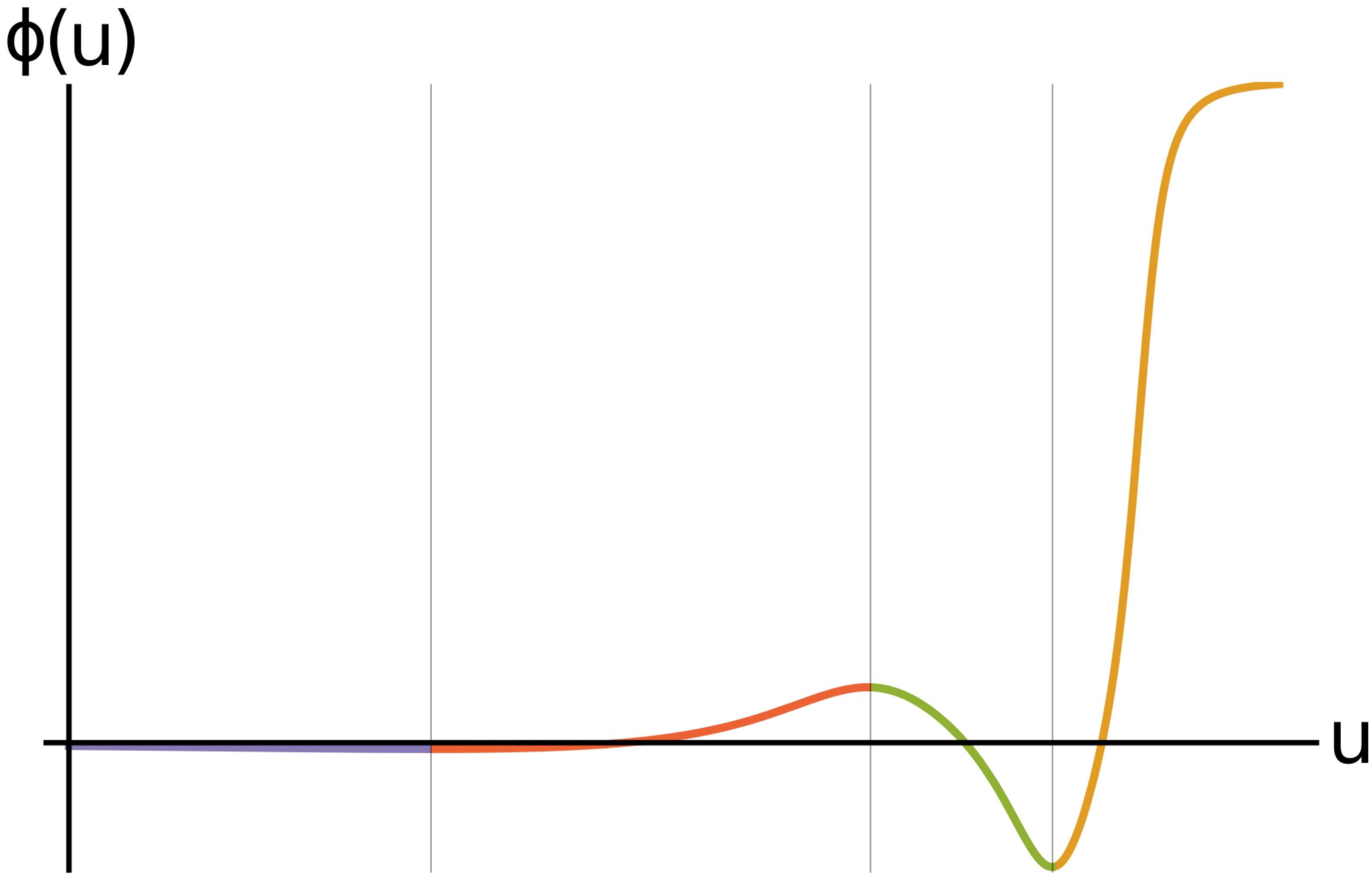
with

$$W(\phi) = kv^2\frac{\phi}{v}\left[1-\frac{1}{3}\frac{\phi^2}{v^2}\right]+W_0$$

$$\phi(u) = v\ \tanh(ku)$$

Regular multibounce flows




ϕ(u)
u


A(u)
u


UV
IR

Curtright, Jin and Zachos gave an example of an RG Flow that is cyclic but respects the strong C-theorem

$$\beta_n(\phi) = (-1)^n\sqrt{1-\phi^2} \quad \to \quad \phi(A) = sin(A)$$

If we define the superpotential branches by $\beta_n = -2(d-1)W_n'/W_n$ we obtain

$$\log W_n = \frac{(2n+1)\pi + 2(-1)^n(\text{arcsin}(\phi) + \phi\sqrt{1-\phi^2})}{8(d-1)}$$

and we can compute the potentials from $V = W'^2/2 - dW^2/4(d-1)$ to obtain $V_n(\phi)$.

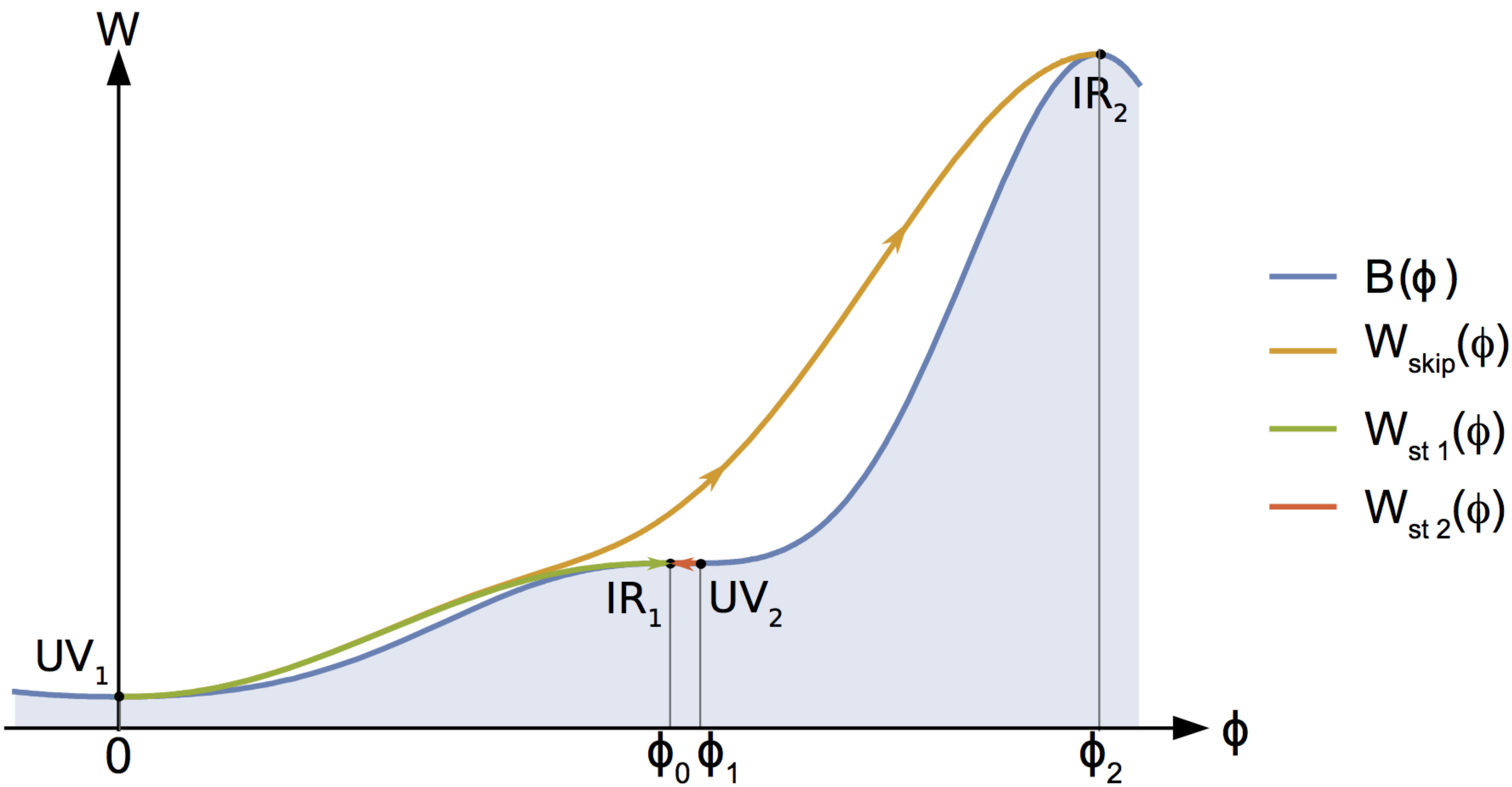
Such piece-wise potentials then satisfy

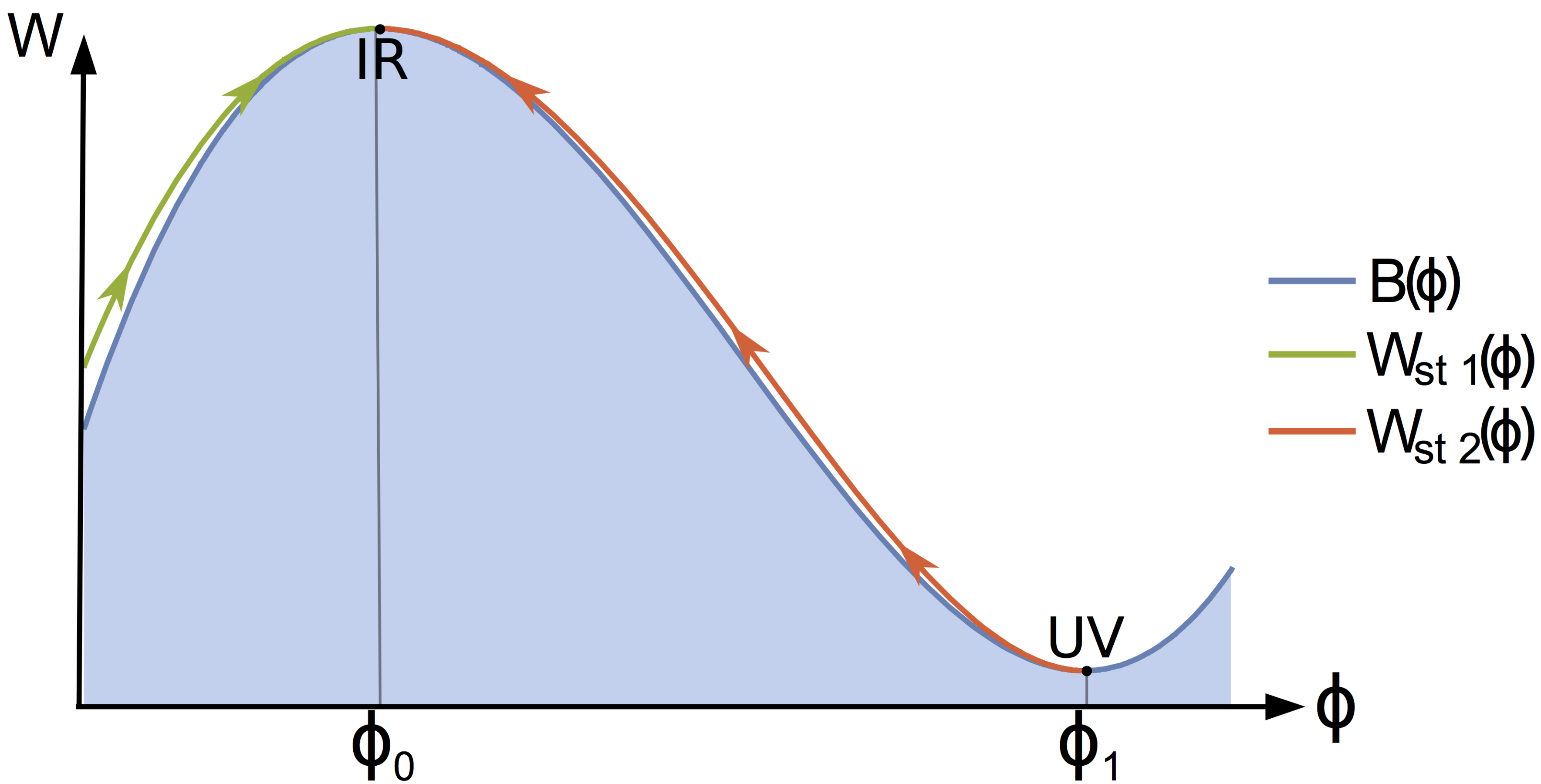
$$V_{n+2}(\phi) = e^{\frac{\pi}{2(d-1)}}\ V_n(\phi)$$

- No such potentials can arise in string theory (I think).

- Holography can provide only "approximate" cycles.

Skipping fixed points




W
IR
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ϕ0
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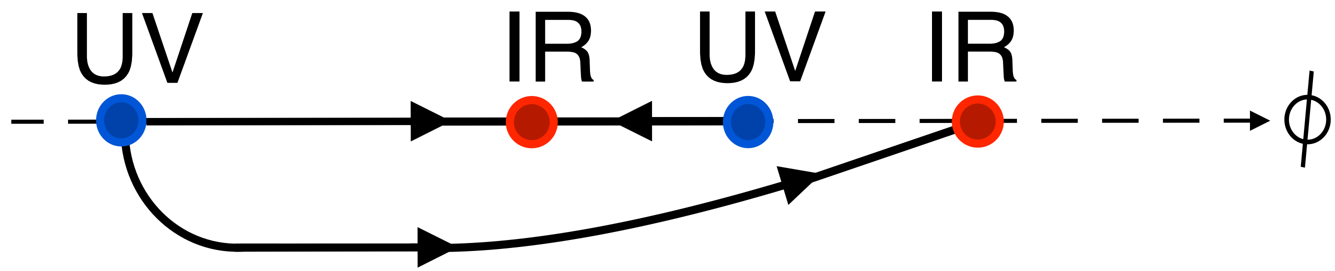
B(ϕ)
Wst 1(ϕ)
Wst 2(ϕ)



• It is not possible in this example to redefine the topology on the line so that the flow looks "normal"

• The two flows $UV_1 \to IR_1$ and $UV_1 \to IR_2$ correspond to the same source but different vev's.

• One can calculate the free-energy difference of these two flows: the one that arrives at the IR fixed point with lowest a, is the dominant one.

Outlook

- Many exotic holographic flows appear for generic potentials

- Do they have fully stable correlators?

- Can they occur in string-derived effective potentials?

- Are they a large-N artifact? Can they occur in strongly-coupled QFTs?

- Can one understand the multiple flows and their dominance from a QFT point of view?

- Are bouncing flows acceptable holographically? Do they have a consistent finite-T behavior? They seem to be intermediate between regular monotonic floes and limit cycles.

- To obtain limit cycles, one needs infinitely multivalued potentials. Do these exist in string theory? Does this exclude holographic limit cycles?

- In gravity the extrema of the potential determine the flows. This is related to Morse theory. On the other hand RG flows are related to bifurcation theory. Does (supergravity) provide a map between the two frameworks? Is this non-trivial?

- Once we allow $V > 0$ cosmology comes in the game, and the behavior of the solutions is richer.

To be continued......

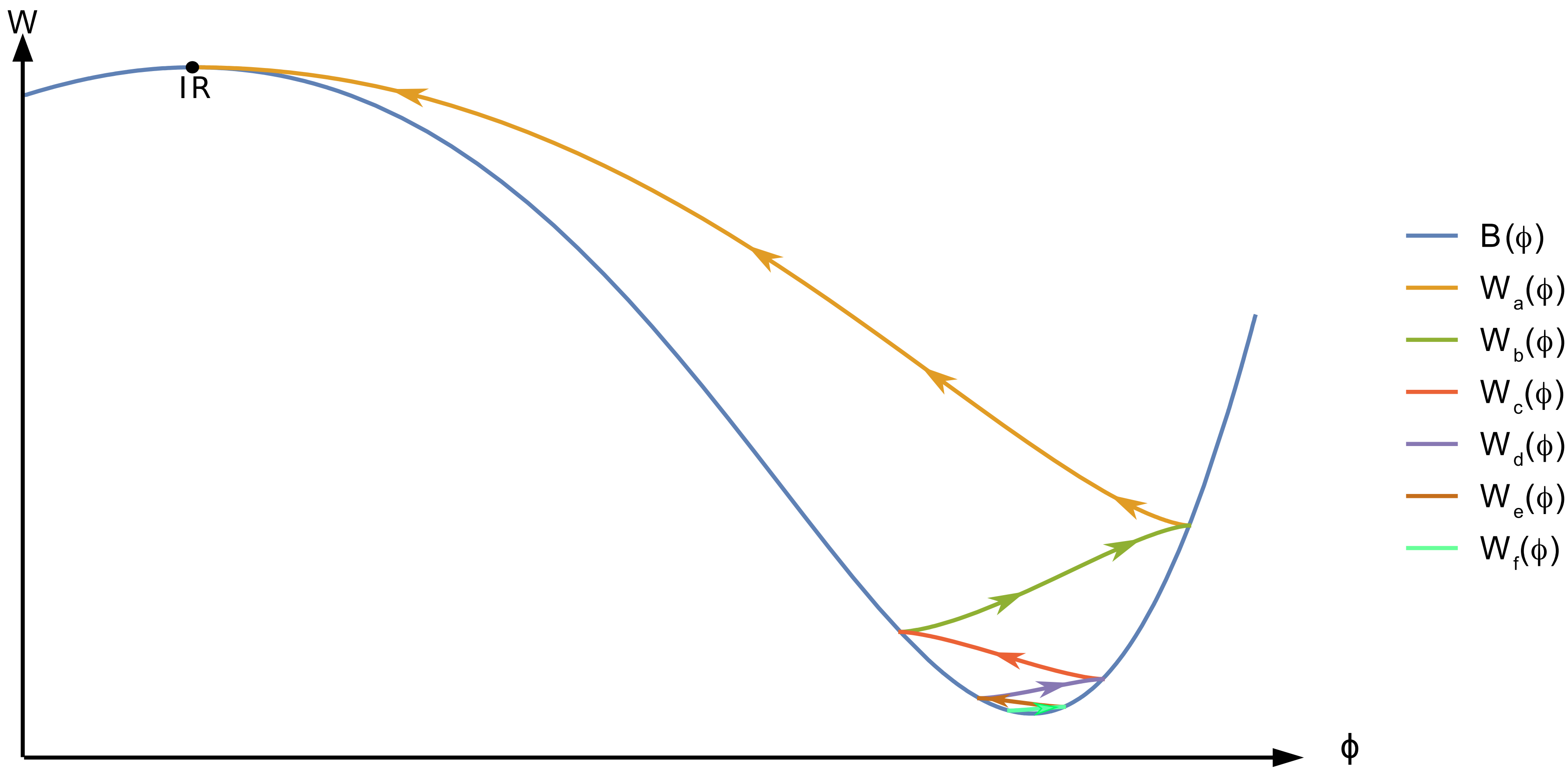
THANK YOU!

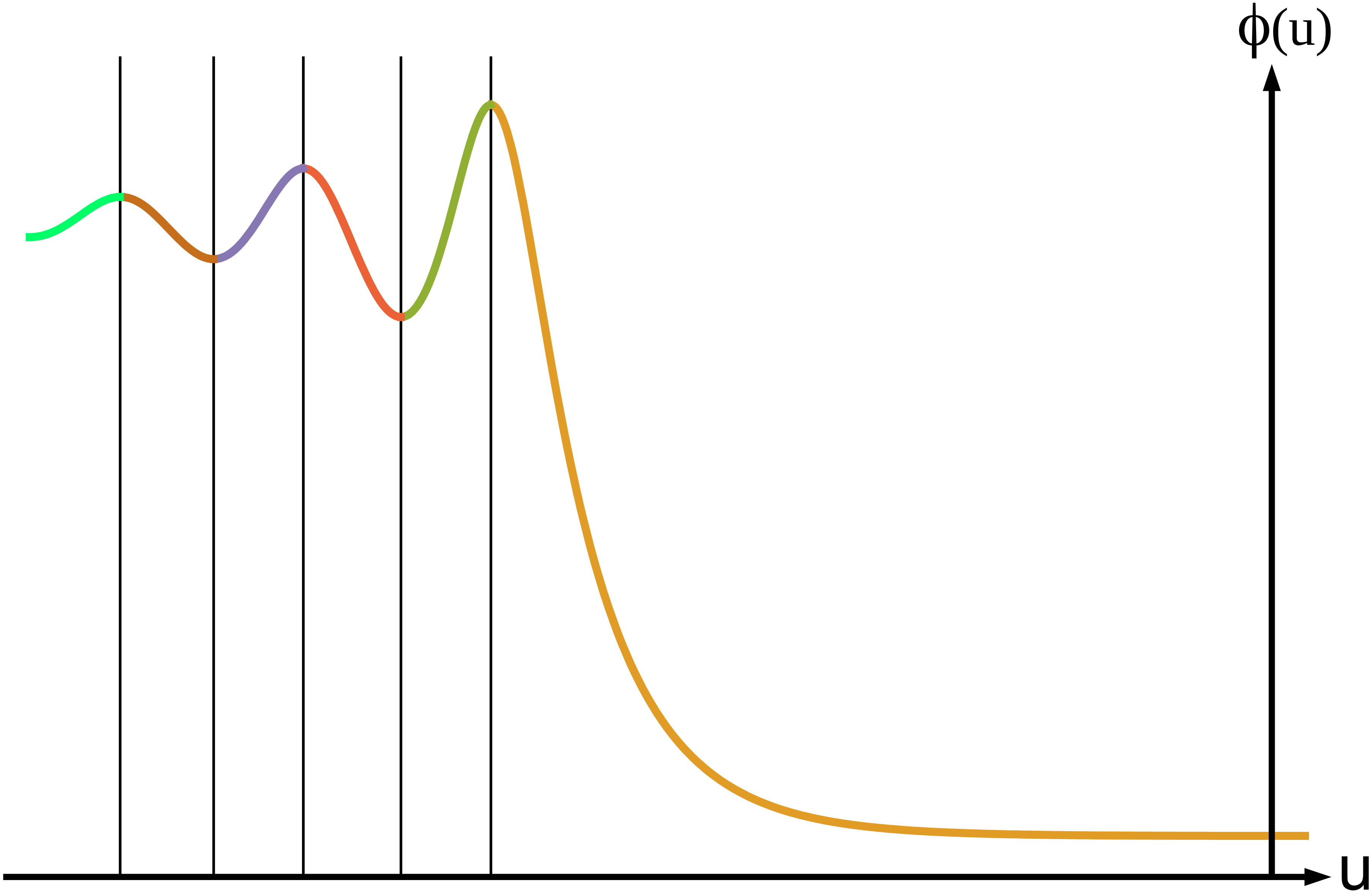
BF violating flows

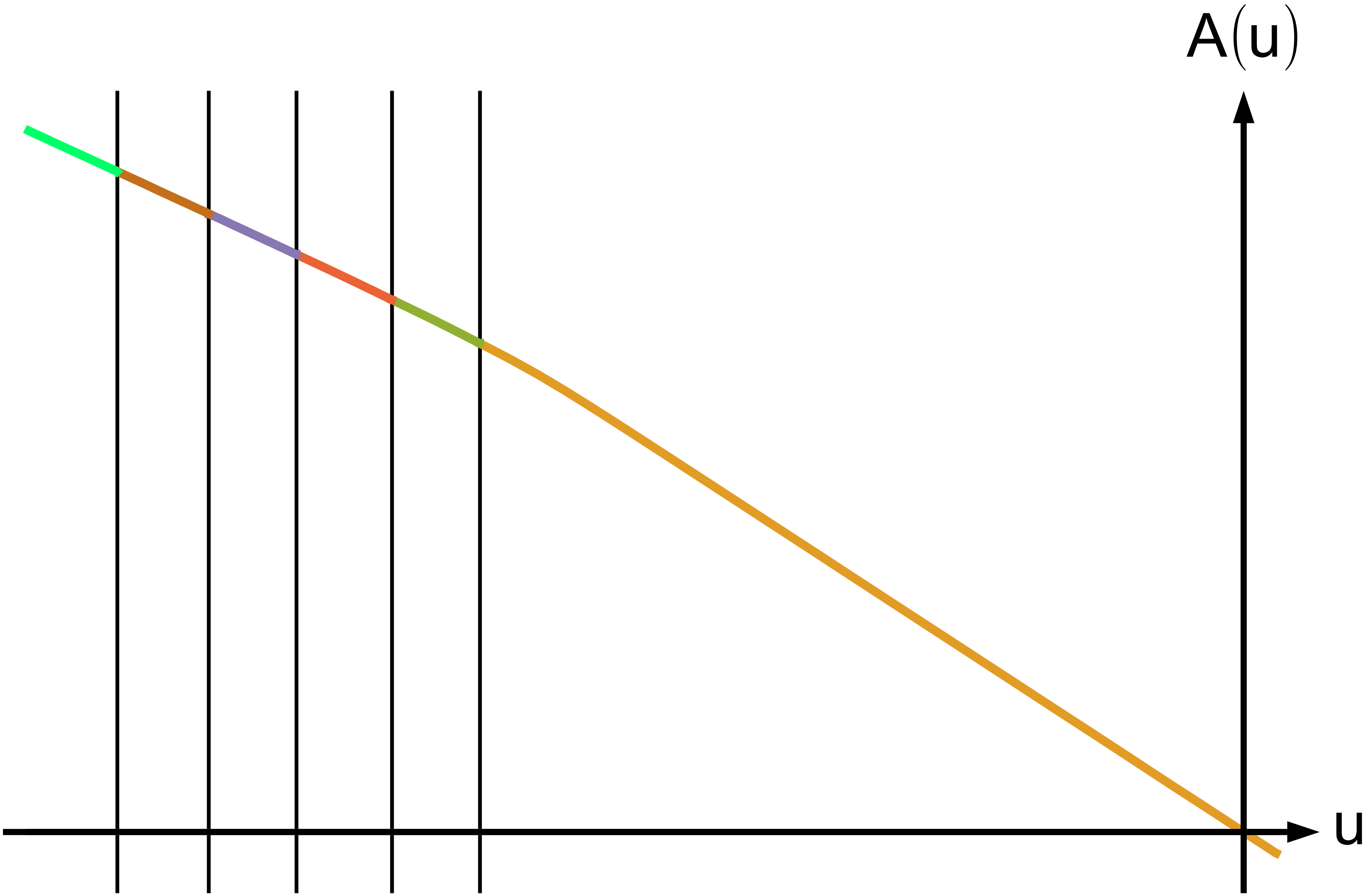
• As mentioned there can be flows out of a BF-violating UV fixed point.

• No β-function description of such flows in the UV.

• Such flows have an infinite-cascade of bounces as one goes towards the UV.




ϕ(u)
u



• Although the flow is regular, it is unstable.

Detailed plan of the presentation

- The critical points of W 26 minutes
- The maxima of V 34 minutes
- The BF bound 35 minutes
- The minima of V 42 minutes
- Bounces 47 minutes
- Global regularity 49 minutes
- Exotica 51 minutes
- Regular Multibounce flows 54 minutes
- Skipping fixed points 56 minutes
- Outlook 58 minutes

- BF-violating flows 60 minutes