The power of series

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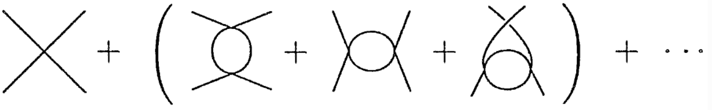
based on **1612.04376**, **1702.04148** with M. Serone and G. Villadoro

Perturbation theory

$$S(\phi) = \int d^d x \, \mathcal{L}(\phi) \, , \qquad Z(\hbar) = \int \mathcal{D}\phi \, Q(\phi) \, e^{iS(\phi)/\hbar}$$

Does perturbative series give exact results?



$$Z(\hbar) = \underbrace{\sum_n c_n \hbar^n}_{\text{pert.}} + \underbrace{\sum_\sigma n_\sigma \, e^{iS_\sigma/\hbar} \sum_n c_{\sigma,n} \, \hbar^n}_{\text{non-pert}}$$

- Non-perturbative contributions $S(\phi_\sigma) = S_\sigma$
- Series are divergent (asymptotic) [Dyson 1952]

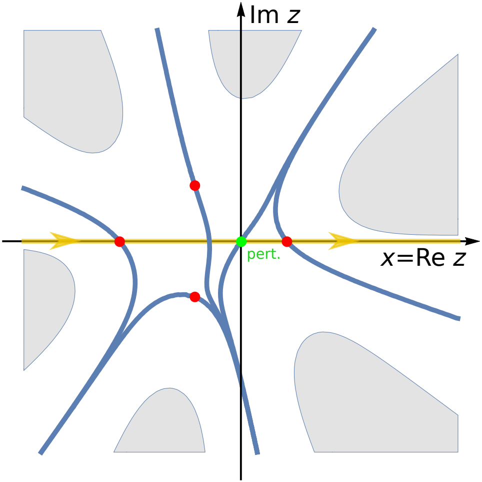
Semiclassical expansion

Picard-Lefschetz theory

$$Z = \int_{\mathbb{R}} dx\, q(x)\, e^{is(x)/\hbar}$$

● ● : saddle points $s'(z_\sigma) = 0$

: steepest descent



Application to path integrals [Witten 2011]

Semiclassical expansion

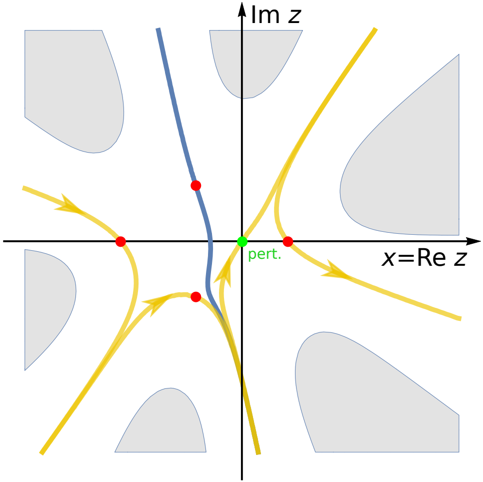
Picard-Lefschetz theory

$$Z = \int_{\mathbb{R}} dx\, q(x)\, e^{is(x)/\hbar}$$

$\color{red}{\bullet}$ $\color{green}{\bullet}$: saddle points $s'(z_\sigma) = 0$

: steepest descent

$$Z = Z_{\text{pert}} + \sum_{\sigma \in \{\color{red}{\bullet}\}} n_\sigma Z_\sigma$$

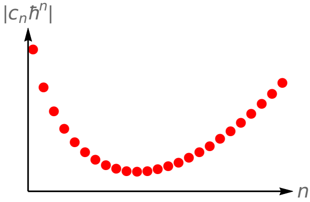


Application to path integrals [Witten 2011]

Divergent Series

Asymptotic series

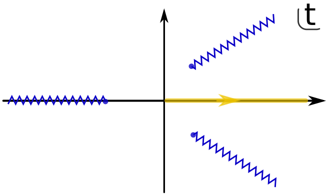
$$\sum_n c_n \hbar^n \,, \qquad c_n \sim a^n n^b \, n!$$



Borel resummation

$$B(t) = \sum_n \frac{c_n}{n!} t^n$$

$$\mathcal{L}B(\hbar) = \int_0^\infty dt\, e^{-t} B(t\,\hbar) \approx \sum_n c_n \hbar^n$$



steepest descents are Borel resummable $\Rightarrow \; Z_\sigma$

Summary

Determine which saddles contribute

$$Z(\hbar) = \sum_n c_n \hbar^n + \sum_\sigma n_\sigma \; e^{iS_\sigma/\hbar} \sum_n c_{\sigma,n}\, \hbar^n$$

Borel summation

When is the perturbative series enough?

euclidean path integral + no other real saddle

$$Z(\hbar) = \int \mathcal{D}\phi\, Q(\phi)\, e^{-S(\phi)/\hbar}$$

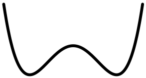
modify Q and S to achieve this

example: SUSY QM

$$Z(\hbar) = \int \mathcal{D}\phi \;\; Q(\phi) \;\; e^{-\;S(\phi)\;/\hbar}$$

$$S(\phi) \to \int dt \left(\frac{1}{2}\dot{\phi} - \frac{1}{4}\phi^2 + \frac{1}{2}\phi^4 + \frac{1}{32} \right)$$

$$Q(\phi) \to \exp \; \left[- \int dt \, \phi \right]$$



Non-perturbative ground state E_0

$$E_0 \sim e^{-1/(3\hbar)}$$

example: SUSY QM

$$Z(\hbar) = \int \mathcal{D}\phi \;\; Q(\phi) \;\; e^{-\; S(\phi) \; / \hbar}$$

$$S(\phi) \to \int dt \left(\frac{1}{2}\dot{\phi} + \frac{\hbar_0}{2}\phi^2 + \frac{1}{2}\phi^4 + \frac{1}{32} \right)$$

$$Q(\phi) \to \exp \left[- \int dt \left(\phi - \left(\frac{1}{2} + \frac{1}{4\hbar_0} \right) \phi^2 \right) \right]$$

Perturbative ground state E_0

$$E_0 \sim e^{-1/(3\hbar)}$$