THEORY OF

GRAVITATIONAL WAVES

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Topics

- What are gravitational waves?
- Sketch of theory of gravitational waves
- Order-of-magnitude calculations:
  Case study: GW150914
- Outlook

What is a gravitational wave?

A perturbation in spacetime geometry

Modifies Pythagoras' Law:

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

becomes

$$\Delta s^2 = (1 + h_+)\Delta x^2 + h_\times \Delta x \Delta y + (1 - h_+)\Delta y^2 + \Delta z^2$$

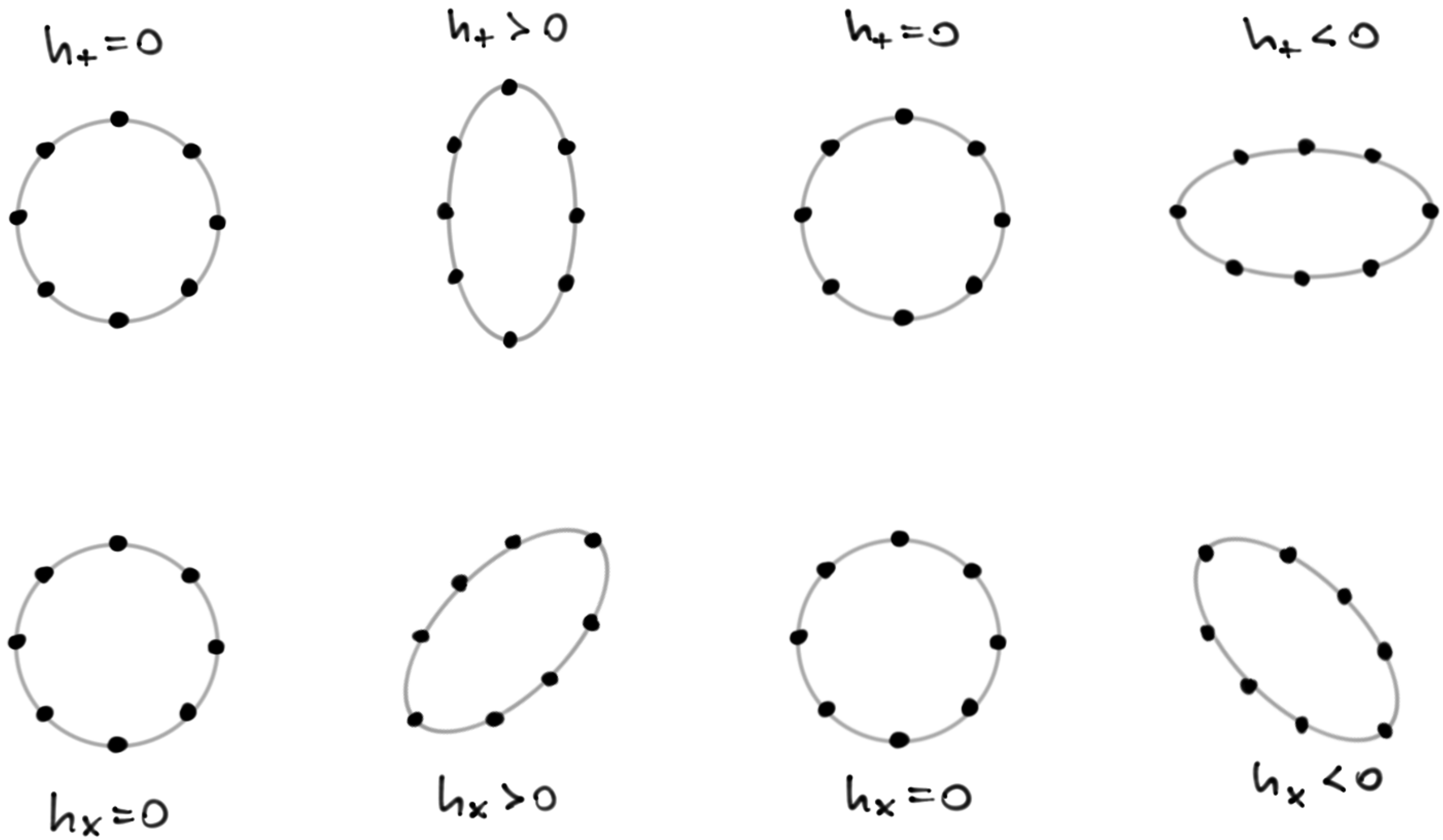
two polarizations, h_+ and $h_\times$

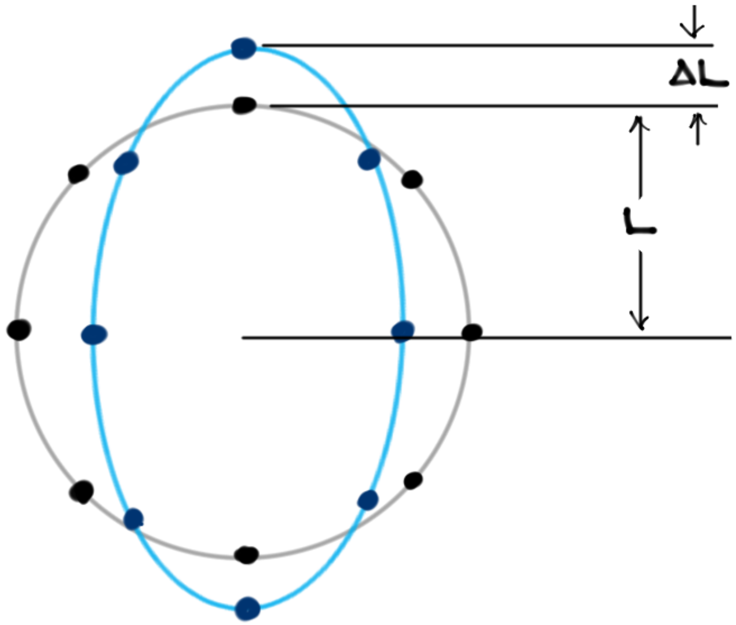
transverse:
no effect with Δz.

(Ignoring many details about gauge etc....)

Effect of a passing gravitational wave...

Consider a hoop of particles on x-y plane :





Strain is $\frac{\Delta L}{L} = h = h_+$ (in case shown)

Why don't we feel gravitational waves everyday ?

Typical strain is

$$h \approx 10^{-21} \sim \frac{\text{hair's width}}{\text{distance to } \alpha \text{ Centauri}}$$

How do we measure such tiny strains?

- **LIGO** Laser Interferometer Gravitational-Wave Observatory

  2 detectors, one in Hanford WA
  one in Livingston LA

  Plus **Virgo** (Piza, Italy)

  Plus **Kagra** (Japan... coming soon)

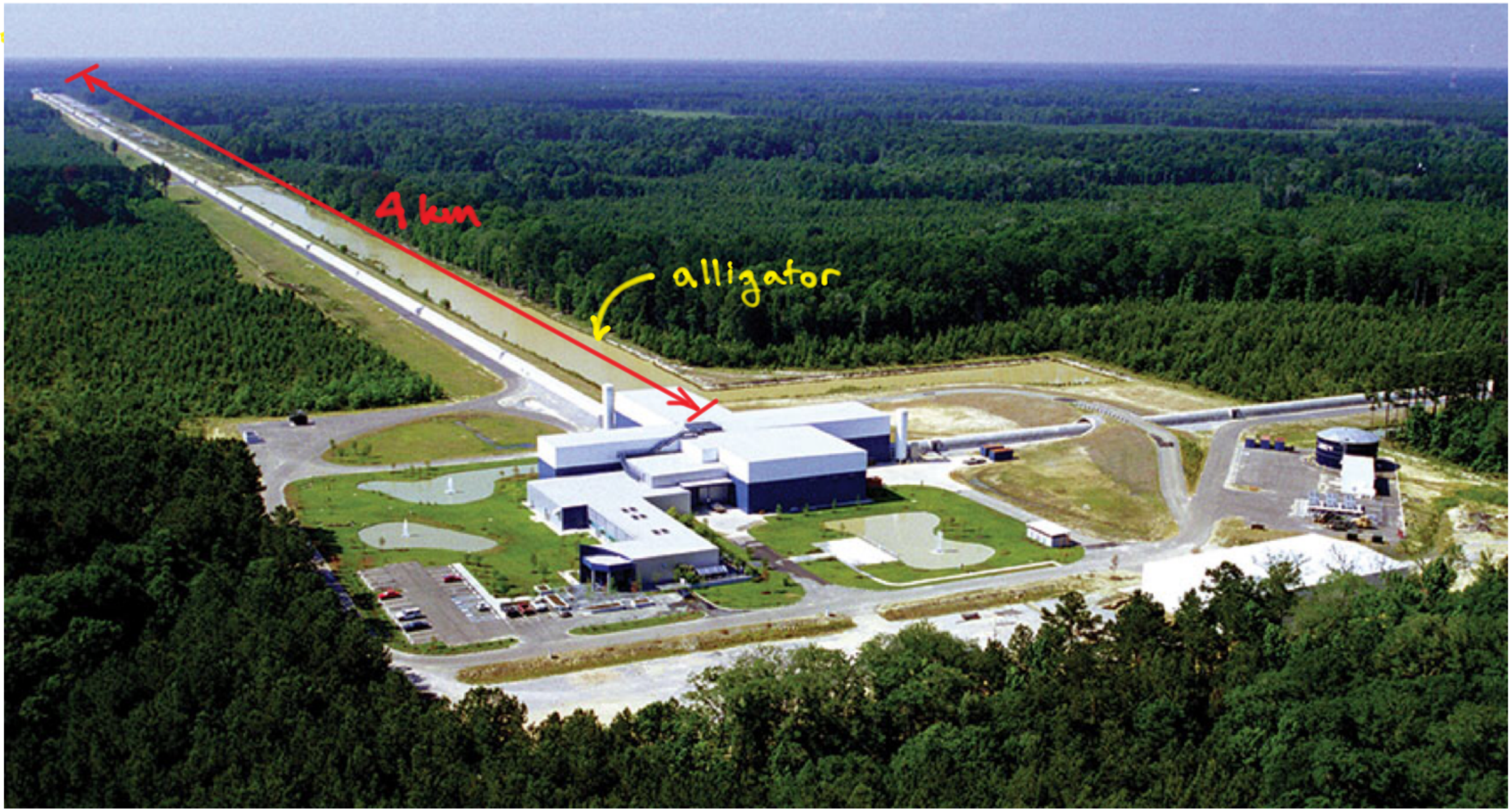
  Plus **LIGO-India** (coming soon)

- Pulsar Timing Arrays (NANOGrav etc....)

- **LISA** Laser Interferometer Space Antenna

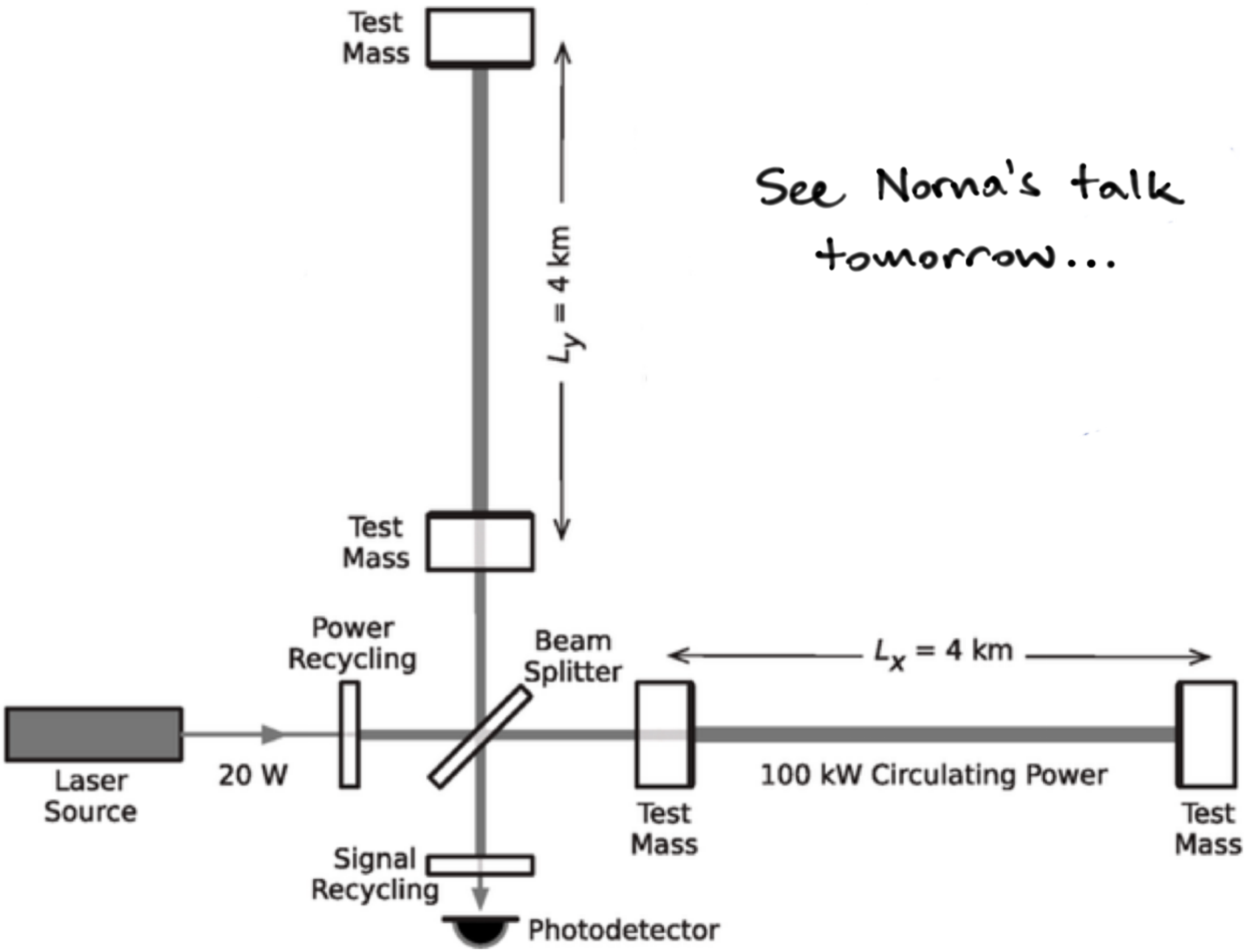
  Heliocentric orbit, coming 2030s.

LIGO Livingston Observatory



Laser Inteferometry

See Norma's talk tomorrow...



General Relativity

$$\mathbf{G} = \frac{8\pi G}{c^4} \mathbf{T}$$

curvature tensor
→ **geometry**

stress-energy tensor
→ **matter**

spacetime tells matter how to move
matter tells spacetime how to curve

– Wheeler

$\mathbf{G}$: 2^{nd} order differential operator on spacetime metric $\mathbf{g}$

Suppose $\quad \mathbf{g} = (\text{flat spacetime}) + \mathbf{h}$ ← perturbation

Then

$$\mathbf{G} = -\frac{1}{2}\Box \mathbf{h} + \begin{pmatrix} \text{terms } 2^{nd} \text{ order} \\ \text{in } \mathbf{h} \end{pmatrix}$$

$$\therefore \quad \Box \mathbf{h} = -\frac{16\pi G}{c^4}\mathbf{T} + O(\mathbf{h}^2)$$

electromagnetism

Poisson equation

gravity

$$\nabla^2 \varphi = -\frac{\rho}{\epsilon_0} \qquad \qquad \nabla^2 \phi = 4\pi G \rho$$

relativity

$$\Box \mathbf{A} = \mu_0 \mathbf{J} \qquad \qquad \Box \mathbf{h} = -\frac{16\pi G}{c^4}\mathbf{T} + O(\mathbf{h}^2)$$

(ignore)

Green's function

$$\mathbf{A} \approx \frac{\mu_0}{4\pi r}\int \mathbf{J}\, d^3x \qquad \qquad \mathbf{h} \approx -\frac{4G}{c^4 r}\int \mathbf{T}\, d^3x$$

Integration by parts

$$\mathbf{A} \approx -\frac{\mu_0}{4\pi r}\int (\nabla\cdot\mathbf{J})\,\mathbf{x}\, d^3x \qquad \qquad \mathbf{h} \approx -\frac{2G}{c^4 r}\int (\nabla\nabla\mathbf{T})\,\mathbf{x}\mathbf{x}\, d^3x$$

conservation law

$$\mathbf{A} \approx \frac{\mu_0}{4\pi r}\int \frac{\partial}{\partial t}(\rho\mathbf{x})\, d^3x \qquad \qquad \mathbf{h} \approx \frac{2G}{c^4 r}\int \frac{\partial^2}{\partial t^2}(\rho\mathbf{x}\mathbf{x})\, d^3x$$

$$\mathbf{A} \approx \frac{\mu_0}{4\pi}\frac{1}{r}\frac{d}{dt}\left[\int \rho\mathbf{x}\, d^3x\right] \qquad \qquad \mathbf{h} \approx \frac{2G}{c^4}\frac{1}{r}\frac{d^2}{dt^2}\left[\int \rho\mathbf{x}\mathbf{x}\, d^3x\right]$$

← Dipole/Quadrupole formula →

Order of magnitude estimates

$$h \sim \frac{2G}{c^4}\,\frac{\ddot{Q}}{r} \qquad \text{where} \qquad Q = \sum_i m_i \|\mathbf{x}_i\|^2$$

(quadrupole moment)

Now $\quad \ddot{Q} \approx 2\sum_i m_i \|\mathbf{v}_i\|^2$

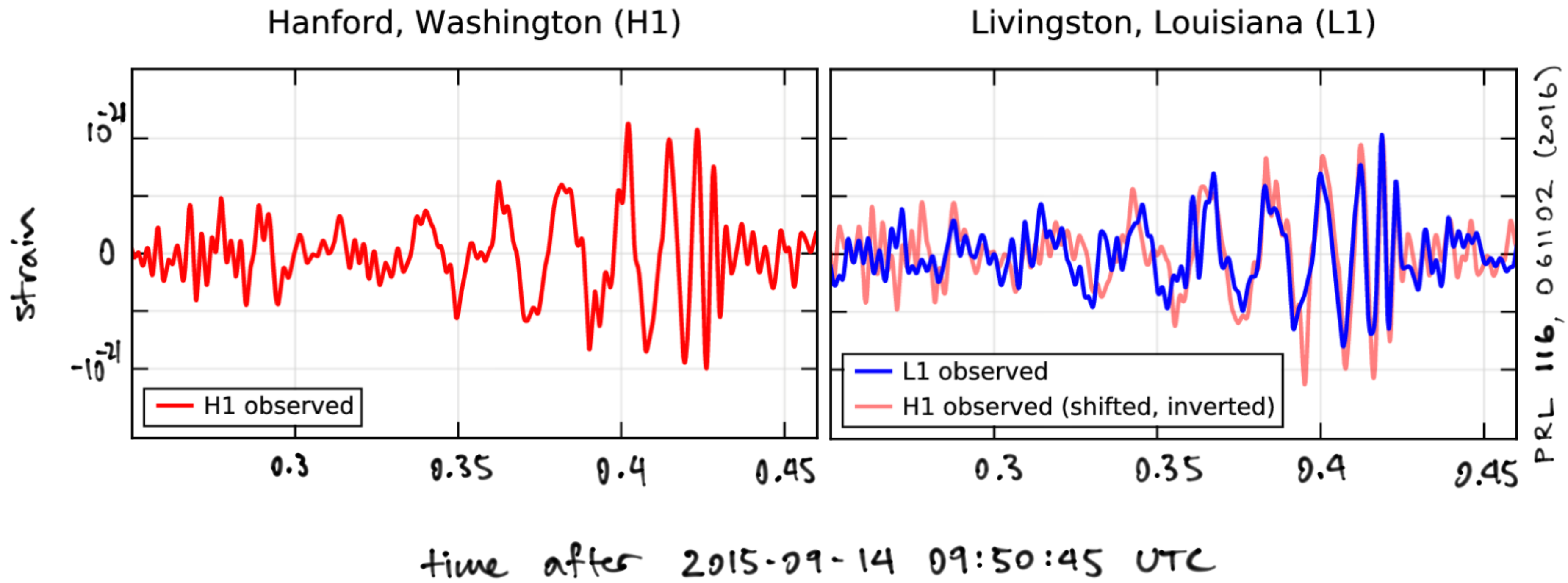
$\sim$ kinetic energy associated with non-symmetrical motion

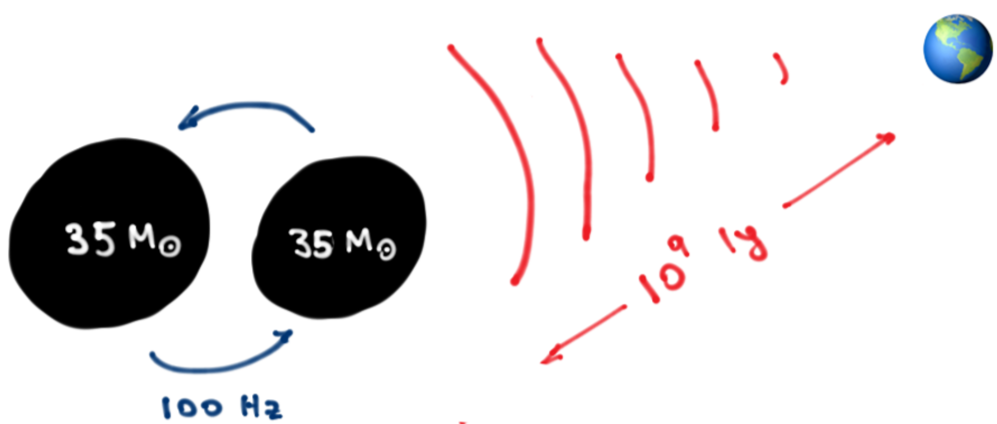
E.g. for a binary system of two equal mass m components moving with relative velocity v (so $v_1 = v_2 = \frac{1}{2}v$)

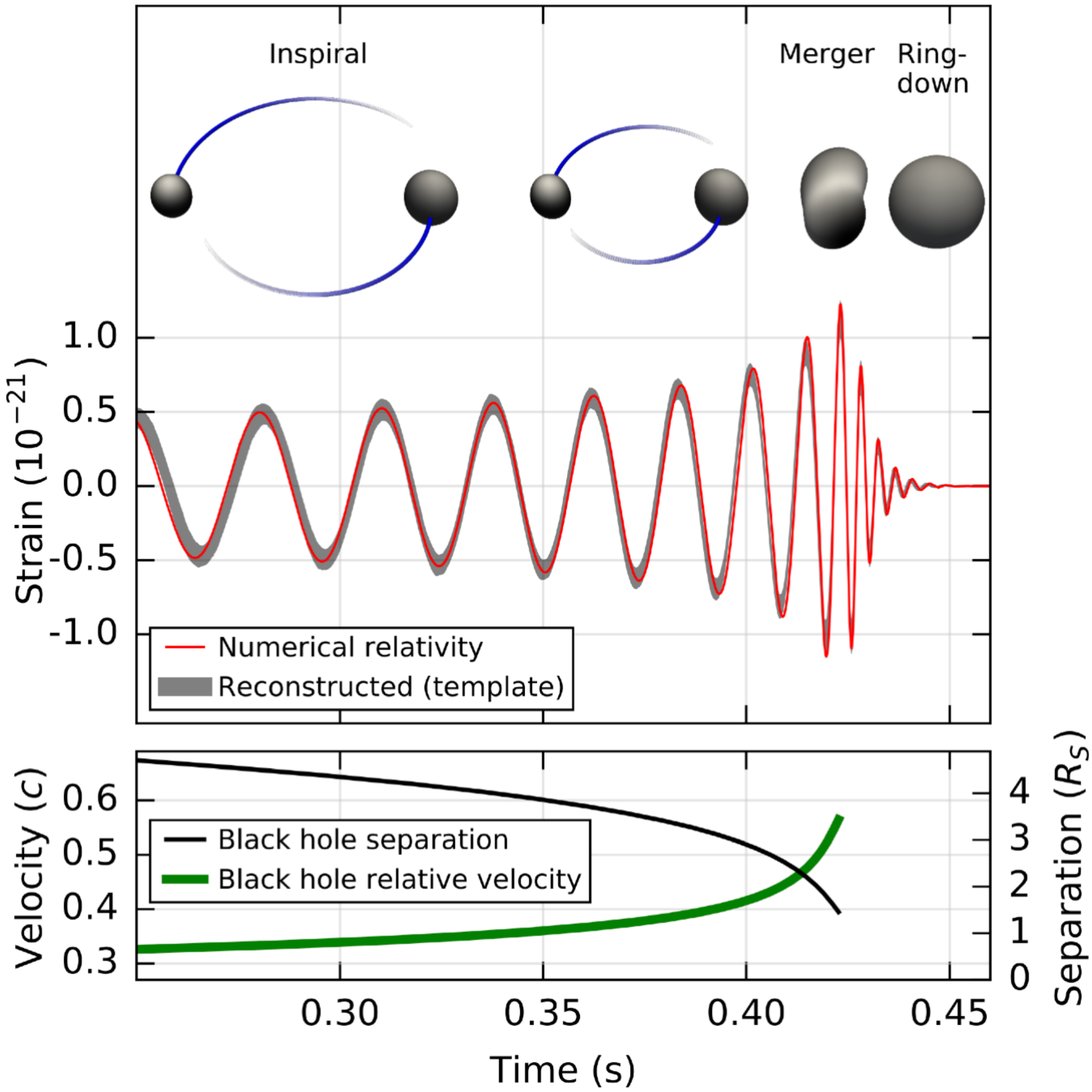
$$\ddot{Q} \approx m v^2$$

so $\quad h \sim \dfrac{2Gm}{c^2 r}\left(\dfrac{v}{c}\right)^2 \qquad$ note: $\dfrac{2Gm}{c^2}$ is Schwarzschild radius of body of mass m.

Case study: GW150914







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Let $m = 35\, M_\odot$ and note $\dfrac{2GM_\odot}{c^2} = 3\,\text{km}$

$$\rightarrow \frac{2Gm}{c^2} \approx 100\ \text{km}$$

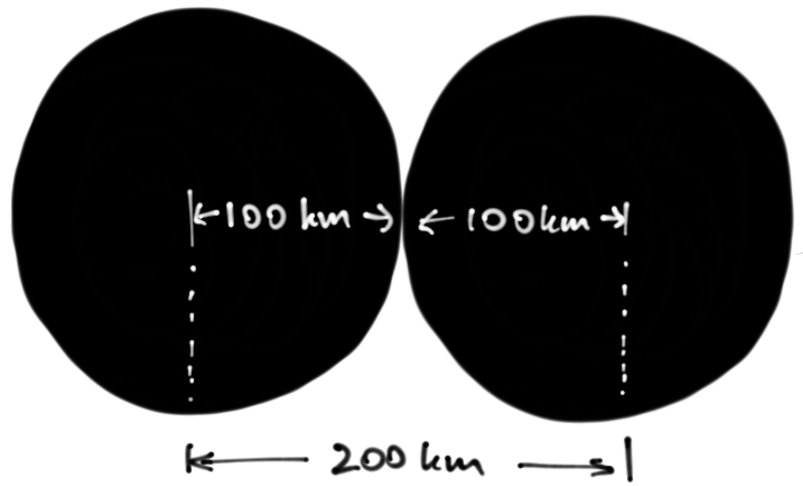
Let $r = \underset{\substack{\uparrow \\ 10^9}}{1\ \text{billion}}\ \underset{\substack{\uparrow \\ 10^{13}\,\text{km}}}{\text{light years}} = 10^{22}\ \text{km}$

$$\therefore \quad h \sim \left(\frac{100\ \text{km}}{10^{22}\ \text{km}}\right)\left(\frac{v}{c}\right)^2$$

$$= 10^{-20}\left(\frac{v}{c}\right)^2$$

(geometric factors omitted)

At 'point of contact' (merger):



$\therefore$ semi-major axis $\quad a \approx 200\ \text{km} = 2 \times \frac{2Gm}{c^2}$

Kepler's 3rd law: $\quad a^3\omega^2 = G(2m)$

$$\rightarrow v^2 = (a\omega)^2 = \frac{2Gm}{a}$$

$$\rightarrow v^2 = \frac{1}{2}c^2$$

$\therefore$ Strain is $\quad h \sim \begin{pmatrix}\text{geometric} \\ \text{factors}\end{pmatrix} \times 10^{-20} \times \frac{1}{2}$

consistent with measured $\quad h_{peak} \approx 10^{-21}$.

Also, the orbital frequency at merger is

$$f_{orb} = \frac{1}{2\pi}\omega = \frac{1}{2\pi}\frac{v}{a} = \frac{1}{2\pi}\frac{\frac{1}{2}c}{200\,km}$$

$$\approx 100 \text{ Hz}$$

and the gravitational wave frequency is

$$f_{GW} = 2 f_{orb} \approx 200 \text{ Hz}$$

quadrupole $\ell = m = 2$ emission

consistent with measured $f_{GW} \approx 150$ Hz at peak.

Energy flux

Recall: $\mathbf{G} = \frac{8\pi G}{c^4}\mathbf{T}$

$$\rightarrow \quad -\frac{1}{2}\Box \mathbf{h} + O(h^2) = \frac{8\pi G}{c^4}\mathbf{T}$$

move this over
and call it $\mathbf{T}_{GW} = \frac{c^4}{8\pi G} O(h^2 \text{ terms})$

Energy density $\sim \frac{c^4}{8\pi G}\left(\frac{1}{2}\frac{\partial h}{\partial t}\right)^2$

recall $h \sim \frac{2G}{c^4}\frac{\ddot{Q}}{r}$

$$\rightarrow \text{Flux} = \frac{dE}{dt\,dA} \sim \frac{c^3}{32\pi G}\left(\frac{\partial h}{\partial t}\right)^2 \sim \frac{1}{4\pi}\frac{G}{c^5}\frac{\dddot{Q}^2}{r^2}$$

$$\text{Luminosity} = \frac{dE}{dt} \sim \frac{1}{4\pi}\frac{G}{c^5}\int r^2 d\Omega \frac{\dddot{Q}^2}{r^2} = \frac{1}{5}\frac{G}{c^5}\dddot{Q}^2$$

from angle average

Now, $\dddot{Q} \approx \ddot{Q}\,\omega \approx m v^2 \omega = m a^2 \omega^3$

but $\quad m = \dfrac{a^3 \omega^2}{2G} \quad$ (Kepler's 3rd law again)

$$\rightarrow \quad \dddot{Q} = \frac{(a\omega)^5}{2G} = \frac{v^5}{2G}$$

$$\therefore \quad \frac{dE}{dt} = \frac{1}{5}\,\frac{G}{c^5}\,\frac{v^{10}}{4G^2} = \frac{1}{20}\,\frac{c^5}{G}\left(\frac{v}{c}\right)^{10}$$

'Planck' luminosity $\simeq 4 \times 10^{52}\ \mathrm{W} = \dfrac{200000\, M_\odot c^2}{s}$

Now $\quad \dfrac{v^2}{c^2} = \dfrac{1}{2}$

$$\rightarrow \quad \frac{dE}{dt} = \frac{1}{20}\,\frac{1}{2^5}\,\frac{c^5}{G} \approx \frac{1}{1000}\,\frac{c^5}{G} \approx \frac{200\, M_\odot c^2}{s}$$

Mass radiated in final orbit ($\sim 0.01\ s$ for $f_{orb} \sim 100\,\mathrm{Hz}$)

$\Delta M \sim 2 M_\odot$ lost to gravitational waves

(actually $\approx 3 M_\odot$ is radiated)

Outlook

GW150914 ...

- Confirmed Einstein's century-old prediction of gravitational waves
- was the first observation of a binary black hole

Have firm detections of 3 black hole mergers to date (GW150914, GW151226, GW170104)

→ gives rate of mergers in the universe (about 1 per 15 minute in entire observable universe)

→ learning about mass spectrum of binary black holes

→ learning about formation of binary black holes

→ provides tests of strong-field gravity.

Next up...

- Detection of other systems

  → binary neutron stars:
  possible progenitors of short gamma-ray bursts;
  laboratory for extreme matter;
  optical counterparts → measurement of Hubble's constant

  → rotating neutron stars

  → supernovae

  → stochastic background of gravitational radiation
  (like cosmic microwave background radiation)

- New detectors

  → 3rd generation ground-based detectors
  → space-based detectors (LISA)
  → pulsar timing arrays