Crossing Symmetry in Alpha Space

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Motivation

- Understanding QFT landscape starts with
  Conformal Field Theories = fixed points of RG flows
- Describe quantitative + qualitative features of critical points
- RG flows are encoded by "CFT data".
  CFTs provide stepping stone to non-conformal physics.

CFT data

- Local physics of CFTs depends on two sets of c-numbers. These are experimental observables that we want to predict.
- **Spectrum**: dimensions Δ_i of local operators $\mathcal{O}_i$

$$[D, \mathcal{O}_i] = \Delta_i \mathcal{O}_i \, .$$

  In addition need $SO(d)$ or Lorentz reps.
- **OPE coefficients**: dynamical content of CFT. Underlying algebra

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k c_{ijk} \mathcal{O}_k(0) + \ldots$$

- Sufficient to compute all correlation functions of theory:

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0) X \rangle = \sum_k c_{ijk} \langle \mathcal{O}_k(0) X \rangle \, .$$

  Sum converges for x in finite domain.

Bootstrap equations

- OPE gives non-trivial info about 4-pt function $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle$:

$$\sum_k c_{12k}c_{34k}\big(\text{s-channel blocks}\big) = \sum_k c_{23k}c_{14k}\big(\text{t-channel blocks}\big)\,.$$

- Blocks fixed by conformal symmetry; dynamics encoded by c_{ijk}.
- Blocks are known special functions.
  Kinematics easy: in $d = 1$ depend on single conformal invariant

$$z = \frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|}\,.$$

  To be precise:

$$\big(\text{s-channel block}\big) = z^h\,{}_2F_1(h, h; 2h; z) \equiv k_h(z)$$

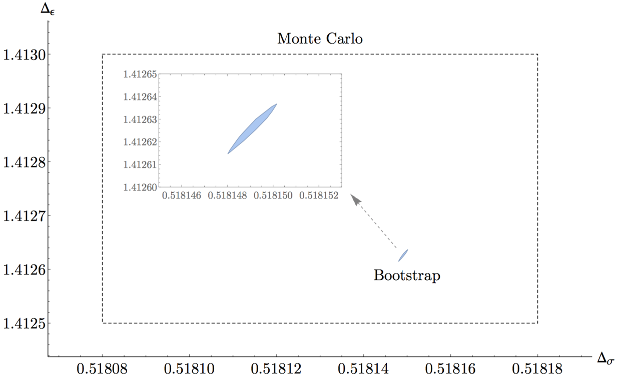
  where $h = \Delta_k/2$. t-channel block similar, with $z \mapsto 1 - z$.

Bootstrap in practice

- Well-defined mathematical problem. Hit equation with linear functional:

$$\langle \sigma\sigma\sigma\sigma \rangle \quad \Rightarrow \quad \sum_k c^2_{\sigma\sigma k} \left(\text{s-channel block} - \text{t-channel block}\right) \equiv 0\,.$$

- Leads to strong constraints on CFT spectrum [Kos et al. 2016]:



Where do we stand?

- This is (second) best thing since sliced bread:
  Proof of principle that $O(1)$ bootstrap eqns are sufficient
  to zone in on interacting CFTs.
  Rigorous, non-perturbative statements about CFT landscape.
  Bottom-up: only symmetry + unitarity.
  Get sharp quantitative predictions.
- Still room for serious improvement.
  Most urgent: study more equations simultaneously, spinning operators (currents, stress tensors). Lot of activity right now.
- Practical downside: will hit a wall at some point...
- Conceptual downside: current method is almost a black box.
  Hard to glean analytic insights from numerical functionals.

Towards analytics

- Different formulations of bootstrap useful in getting analytic results.
  Two main examples:
- Lightcone limit in Minkowski
  $\Rightarrow$ proof of existence of higher-spin towers in CFT spectrum.
- Mellin space:
  $\Rightarrow$ large N, exact critical exponents in ϵ expansion.
- Today: new formulation called alpha space.

Alpha space

- Basic idea: map 4-pt function

$$F_{1234}(z) \ \sim \ \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle$$

  to complex density $\widehat{F}_{1234}(\alpha)$

- CFT data are encoded by position-space correlator in "discrete" fashion:

$$F_{1234}(z) = \sum_j c_{12j}c_{34j}\, k_{h_j}(z)\,.$$

- Will engineer a map $f(z) \to \widehat{f}(\alpha)$ such that CFT data correspond to poles:

$$\text{dimension } h_j \text{ in spectrum } \ \leftrightarrow \ \widehat{F}_{1234}(\alpha) \text{ has pole at } \alpha_j \equiv h_j - \tfrac{1}{2}$$

  So α parametrizes (complexified) "space of conformal dimensions"

Alpha space in practice

- How does this work? Like Fourier, it's an integral transform

$$f(z) \;\mapsto\; \widehat{f}(\alpha) := \int_0^1 \frac{dz}{z^2}\, f(z)\Psi_\alpha(z)$$

for some family of functions $\Psi_\alpha(z)$:

$$\Psi_\alpha(z) = {}_2F_1\left(\begin{matrix} \frac{1}{2}+\alpha,\ \frac{1}{2}-\alpha \\ 1 \end{matrix}\,;\, \frac{z-1}{z}\right).$$

- Rationale: the family $\{\Psi_\alpha\}$ with **imaginary** α forms a complete basis for functions on $[0,1]$ — solve Sturm-Liouville problem for Casimir D.
- Orthogonal w.r.t. natural inner product

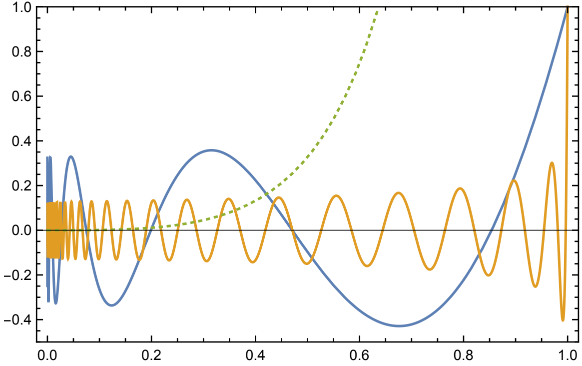
$$\langle f, g\rangle = \int_0^1 \frac{dz}{z^2}\, \overline{f(z)} g(z)\,.$$

Arguing that the Ψ_α are orthogonal...

Orthogonality means that

$$\langle \Psi_\alpha, \Psi_\beta \rangle = \int_0^1 \frac{dz}{z^2}\, \Psi_\alpha(z)\Psi_\beta(z)$$

vanishes for $\alpha \neq \beta$. Proof by picture:



Analytic properties (1)

- Want to prove: poles in α correspond to conformal blocks.
- First check that

$$f(z) \;\mapsto\; \widehat{f}(\alpha) := \int_0^1 \frac{dz}{z^2}\, f(z)\Psi_\alpha(z)$$

  defines meromorphic function of α for "good" functions $f(z)$.
- Rigorous: if $f(z) = O(z^p)$, then $\widehat{f}(\alpha)$ analytic in strip

$$|\mathrm{Re}(\alpha)| < p - 1/2\,.$$

- Also $f_p(z) = z^p$ has well-defined, meromorphic alpha-space partner $\widehat{f_p}(\alpha)$. (By analytic continuation if $\mathrm{Re}(p) < 1/2$.)
- CFT correlators are sums $\sum_n c_n\, z^{h_n}(\ldots)$. Proceed by subtraction.

Analytic properties (2)

- Sequel: prove that poles in α correspond to conformal blocks.
- Method: reconstruct $f(z)$ from alpha space partner $\widehat{f}(\alpha)$

$$f(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\alpha \, \widehat{f}(\alpha) \Psi_\alpha(z)$$

together with connection formula:

$$\Psi_\alpha(z) = Q(\alpha) k_{1/2+\alpha}(z) \; + \; (\alpha \to -\alpha) \, .$$

- Cauchy: single pole at $\alpha = \alpha_j$ gives contribution

$$f(z) = \left[-Q(\alpha_j) \widehat{f}(\alpha) \big|_{\alpha=\alpha_j} \right] k_{1/2+\alpha_j}(z) + \sum \text{other poles}$$

Conclusion: match with CB expansion if

$$h_j = \frac{1}{2} + \alpha_j \, , \quad c_{12j} c_{34j} = -Q(\alpha_j) \, \text{Res} \, \widehat{f}(\alpha) \big|_{\alpha=\alpha_j} \, .$$

- Contour prescription.

Crossing (1)

- So far: (almost) generic functions $f(z)$.
  But CFT correlators obey crossing identity. Let $F_\sigma(z) \propto \langle \sigma\sigma\sigma\sigma \rangle$. Then

$$F_\sigma(z) = \left(\frac{z}{1-z}\right)^{2h_\sigma} F_\sigma(1-z)\,.$$

- Expand both sides in terms of basis functions $\Psi_\alpha(z)$:

$$\int d\alpha\, F_\sigma(\alpha)\Psi_\alpha(z) = \int d\beta\, F_\sigma(\beta)\left(\frac{z}{1-z}\right)^{2h_\sigma} \Psi_\beta(1-z)\,.$$

- By slight massaging, can recast this as integral equation:

$$\boxed{F_\sigma(\alpha) = \int_{-i\infty}^{i\infty} d\beta\; K(\alpha,\beta|h_\sigma)\, F_\sigma(\beta)}$$

  for some definite kernel $K(\alpha,\beta|h_\sigma)$.

Crossing (2)

$$F_\sigma(\alpha) = \int_{-i\infty}^{i\infty} d\beta\; K(\alpha, \beta | h_\sigma)\, F_\sigma(\beta)$$

- Turns bootstrap into complex analysis problem:
  find meromorphic function $F_\sigma(\alpha)$ that obeys above integral equation.
- No more z-dependence: completely "integrated out" position space.
- Positivity (if applicable) = residues of $F_\sigma(\alpha)$ are sign-definite.
- Defined on **imaginary axis** whereas physical poles live on $\mathbb{R}$.
- Contour prescription: spurious poles coming from
  (a) $F_\sigma(\alpha)$ for operators with $h < 1/2$, and
  (b) poles in $K(\alpha, \beta | h_\sigma)$ that depend on h_σ.

Crossing kernel

$$F_\sigma(\alpha) = \int_{-i\infty}^{i\infty} d\beta \; K(\alpha,\beta|h_\sigma)\, F_\sigma(\beta)$$

- In practice, need to compute integral kernel $K(\alpha,\beta|h_\sigma)$:

$$K(\alpha,\beta|h) = \int_0^1 \frac{dz}{z^2}\left(\frac{z}{1-z}\right)^{2h} \Psi_\alpha(z)\Psi_\beta(1-z)\,.$$

- Meromorphic, two-variable complex function; obeys duality relation

$$K(\alpha,\beta|h) = K(\beta,\alpha|1-h)\,.$$

- Do integral $\Rightarrow$ sum of two ${}_4F_3(1)$ hypergeometrics.
- Known to mathematicians...
  $K(\alpha,\beta|h)$ is an example of *Wilson function* $W(\alpha,\beta|c_1,c_2,c_3,c_4)$, studied in [Groenevelt 2003] + follow-ups.

Special functionology

- Lesson: $K(\alpha, \beta | h_\sigma)$ is [proportional to] a Wilson function. So what?
- Interpretation: integral operator

$$f(\alpha) \;\mapsto\; \int_{-i\infty}^{i\infty} d\beta \, W(\alpha, \beta | c_1, c_2, c_3, c_4) f(\beta)$$

  as unitary map between two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_{\text{dual}}$, which depend on $c_1, \ldots, c_4$. For us $\mathcal{H} = \mathcal{H}_{\text{dual}}$.
- Constructive proof; basis of $\mathcal{H}$ consists of *Wilson polynomials*. For us: find that

$$F_\sigma(\alpha) = \sum_n c_n \, \Gamma(2h_\sigma \pm \alpha) \times (\text{Wilson poly})$$

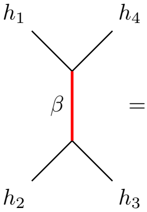
  is solution to crossing.
- Integer-spaced spectrum. Not unitary for generic choice of coefs $\{c_n\}$.

General case

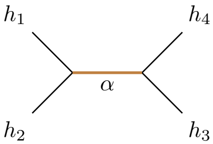
- General $1d$ CFT correlator $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle$ has 2 different decompositions, in s- and t-channel.
- Same logic, expand in 2 different alpha space basis functions $\Psi^s_\alpha(z)$ and $\Psi^t_\alpha(z) \ \Rightarrow\ $ 2 different densities $F^{s,t}(\alpha)$.
- Instead of single integral equation, get 2×2 system:

$$F^s(\alpha) = \int_{-i\infty}^{i\infty} d\beta\, K(\alpha, \beta | h_1, h_2, h_3, h_4) F^t(\beta)$$

- Should be related to 6j symbol of $SL(2, \mathbb{R})$:



$$= \int d\alpha\, K(\alpha, \beta | h_1, h_2, h_3, h_4)$$



Not worked out; partial results in [Groenevelt 2005].

Where next?

a Think hard about Wilson kernel, algebraic/integrability properties etc.

b Revisit known results, e.g. [chiral half of] minimal models. Doable in some cases; Virasoro blocks not known in general.
Recycle DOZZ/Ponsot-Teschner solution to Liouville?

c Study crossing equation as honest integral equation:

$$F(\alpha) = (\text{source term}) + \int_{-i\infty}^{i\infty} d\beta \; K(\alpha, \beta | h_\sigma) \, F(\beta) \, .$$

Source term comes from (i) low-dimension operators and (ii) finite set of poles from $K(\alpha, \beta | h_\sigma)$.
Classic topic in mathematics (Hilbert, Fredholm, ...). Formal solution:

$$F = (\mathsf{K} - \mathrm{id})^{-1} \cdot (\text{source term}) \; + \; \mathrm{Ker}\,(\mathsf{K} - \mathrm{id}) \, .$$

Need to make sense of this. Approximations of $K(\alpha, \beta)$ useful?

Other settings/generalizations

- Obvious: d-dimensional scalar 4-pt function. WIP by other groups.
- More modest: scalar 2-pt functions on non-trivial backgrounds; planar boundaries & real projective space $\mathbb{RP}^d$.

Other settings/generalizations

- Boundary case is classic bootstrap problem:

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\rangle = \sum_{\text{bulk DOF}} c_k \text{ (bulk blocks)} \equiv \sum_{\text{bdy DOF}} e_j \text{ (boundary blocks)}$$

- Need to solve 2 different Sturm-Liouville problems. Get two different bases:

$$\Psi_\alpha^{\text{bulk}}(\xi) \quad \text{and} \quad \Psi_\nu^{\text{bdy}}(\xi)$$

with $\alpha, \nu \in i\mathbb{R}$. Formal properties completely similar to $1d$ case.

- Boundary bootstrap becomes integral equation:

$$\int_{i\mathbb{R}} d\alpha\, F_{\text{bulk}}(\alpha)\Psi_\alpha^{\text{bulk}}(\xi) \equiv \int_{i\mathbb{R}} d\nu\, F_{\text{bulk}}(\nu)\Psi_\nu^{\text{bdy}}(\xi)\,.$$

- Find 2×2 system of integral equations with kernels $E_{\text{bulk}\to\text{bdy}}(\nu,\alpha)$, $E_{\text{bdy}\to\text{bulk}}(\alpha,\nu)$. Both are sums of ${}_4F_3(1)$ hypergeometrics.

A universal point of view

- Similar integral equation exists for case of $\mathbb{RP}^d$.
  Kernel is again sum of ${}_4F_3(1)$'s. What's the story?
- In fact: the 2 BCFT kernels and crosscap one are special cases of the $1d$ kernel

$$K(\alpha,\beta|h_1,h_2,h_3,h_4)$$

  with weights h_i set to appropriate values. Proof: inspect some integrals.
- Formal properties are therefore the same.
- Trivial to prove in alpha space — *in hindsight* evident in position space too.

Summary/discussion

- Developed new framework to study CFT correlators.
  Crossing becomes [set of] integral equations.
- Not out of the woods yet. Interesting solutions solve a very non-trivial integral equation. Some ideas how to proceed, but appears to be hard work (with potentially huge insights to be gained).
- Plenty of generalizations . . .
- Limits of “heavy” operators?
- See how this fits into Isachenkov-Schomerus ideas about integrability.