Relativistic versions of $E = p^2/2m$ and $p = mv$. (following Okun)

The 2 fundamental equations

- Conservation of energy and momentum are close to the heart of physics.

(They are related to 2 deep symmetries of nature. Discuss.)

All this is looked after in special relativity if we define energy and momentum by the following:

$$E^2 - p^2c^2 = m^2c^4 \quad \cdots\cdots \quad (1)$$

$$\vec{p} = \vec{v}\,\frac{E}{c^2} \quad \cdots\cdots \quad (2)$$

where

E = total energy

$\vec{p}$ = momentum

$\vec{v}$ = velocity

m = ordinary mass as in Newtonian mechanics

Special case 1

Let us call the energy of an object when it is at rest ($v = 0$) E_0.

Consider $E^2 - p^2c^2 = m^2c^4$.

When $\vec{v} = 0$, $\vec{p} = 0$, $E = E_0$; so

$$\underline{E_0 = mc^2}$$

Rest energy is one of the great discoveries of relativity

Special case 2

One could be wondering:
"Why write m for mass here, not m_0?"

To answer, let's consider $v \ll c$.

Then $\vec{p} \approx \vec{v}\dfrac{E_0}{c^2} = m\vec{v}$

and $E = E_0 + T = \sqrt{p^2c^2 + m^2c^4}$

$$= mc^2\left(1 + \frac{p^2c^2}{m^2c^4}\right)^{1/2}$$

$$\approx mc^2\left(1 + \frac{1}{2}\frac{p^2c^2}{m^2c^4} + \cdots\right)$$

$$= mc^2 + \frac{p^2}{2m} + \cdots$$

Thus, in the non-relativistic limit, our two fundamental equations reduce to the Newtonian equations for momentum and KE.

This means that the m in ① is the ordinary Newtonian mass.

If we'd used m_0 in ① our relativistic and non-relativistic notations would not have matched.

Special case 3 (extreme "anti-Newtonian", $m=0$)

If $m=0$, $p = \frac{vE}{c^2} = \frac{v\sqrt{p^2c^2}}{c^2} = \frac{vp}{c}$

$$\Rightarrow \underline{v = c}$$

Such bodies have no rest frame.

Also, $m = 0 \Rightarrow \underline{E = pc}$

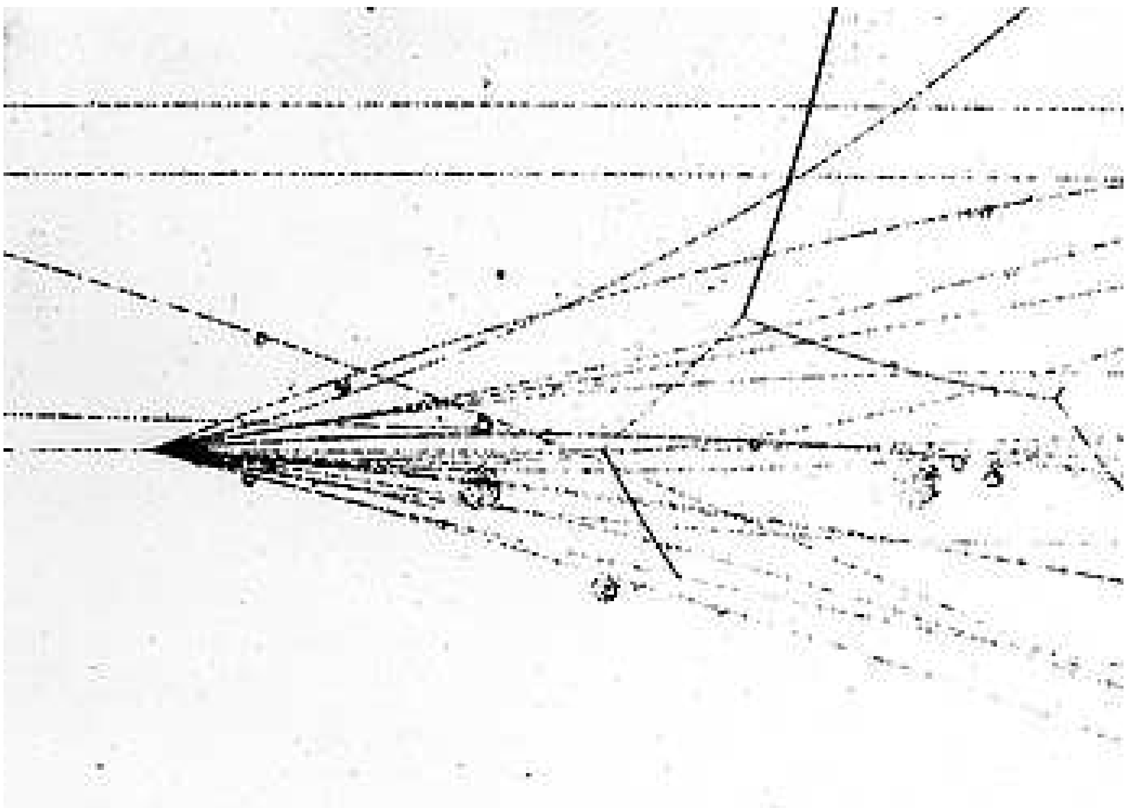
Massless bodies have no rest energy, just KE.

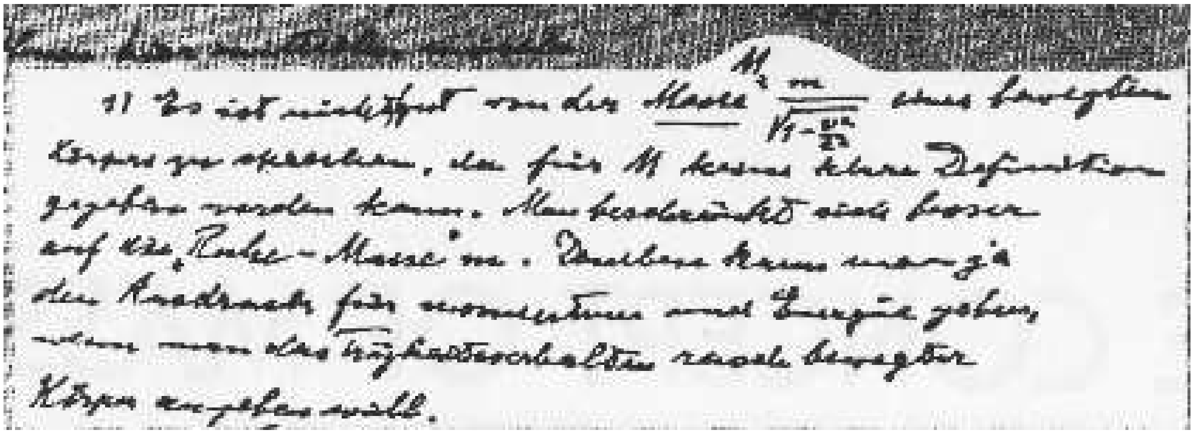
eg photon, graviton...

Summary $E^2 = p^2c^2 + m^2c^4$ and $p = \frac{vE}{c}$ describe the kinematics of a free body for all velocities from 0 to c.

- Also, $E_0 = mc^2$ follows from them.

‘Jet’ of hadrons produced in a pp collision





" Es ist nicht gut von der Masse $M = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}$ eines bewegten Körpers zu sprechen, da für M keine klare Definition gegeben werden kann. Man beschränkt sich besser auf die „Ruhe-Masse" m. Daneben kann man ja den Ausdruck für momentum und Energie geben, wenn man das Trägheitsverhalten rasch bewegter Körper angeben will.

Letter from Albert Einstein to Lincoln Barnett, 19 June 1948. The highlighted passage:

It is not good to introduce the concept of the *mass*

$$M = m/\sqrt{1 - v^2/c^2}$$

of a moving body for which no clear definition can be given. It is better to introduce no other mass concept than the 'rest mass' m**.**
Instead of introducing M **it is better to mention the expression for the momentum and energy of a body in motion.**

Problem Some introduce relativistic kinematics by defining E and p as follows:

$$E = \gamma m c^2 \quad \text{and} \quad p = \gamma m v \qquad (\gamma = \frac{1}{\sqrt{1-v^2/c^2}})$$

Show that these follow from

$$E^2 = p^2c^2 + m^2c^4 \quad \cdots\cdots \; \mathrm{I}$$

$$p = \frac{E}{c^2} v \quad \cdots\cdots \; \mathrm{II}$$

Solution Substituting II into I

$$E^2 = \frac{E^2 v^2}{c^4} c^2 + m^2 c^4$$

$$\Rightarrow E^2\left(1 - \frac{v^2}{c^2}\right) = m^2c^4 \Rightarrow E = \frac{mc^2}{\sqrt{1-v^2/c^2}} = \gamma m c^2$$

Substituting for E from II into I

$$\frac{p^2c^4}{v^2} = p^2c^2 + m^2c^4$$

$$\Rightarrow p^2c^2(c^2 - v^2) = m^2c^4v^2$$

$$\Rightarrow p^2c^4\left(1 - \frac{v^2}{c^2}\right) = m^2c^4v^2$$

$$\Rightarrow p = \frac{mv}{\sqrt{1-v^2/c^2}} = \gamma m v$$

Problem At the Large Electron Positron collider (LEP) at CERN, electrons were accelerated to an energy E of 100 GeV. What is the electron speed?

Solution With $E = 100$ GeV

and $m = 0.511$ MeV/c^2

the mass term in $E^2 = p^2c^2 + m^2c^4$ is miniscule; so we anticipate that v will be very close to c.

From $E = \dfrac{mc^2}{\sqrt{1 - v^2/c^2}}$ we can write

$$E^2 = \frac{m^2c^4}{(1 - v/c)(1 + v/c)} \approx \frac{m^2c^4}{2(1 - v/c)}$$

$$\Rightarrow (1 - v/c) \approx \frac{1}{2}\frac{(mc^2)^2}{E^2}$$

$$= \frac{1}{2}\left(\frac{0.511 \times 10^6\,\text{eV}}{100 \times 10^9\,\text{eV}}\right)^2$$

$$= 0.000000000013$$

$$\Rightarrow \frac{v}{c} = 0.999999999987$$

Gron Tudor Jones
University of Birmingham, UK