Deformation of Dijkgraaf-Vafa Relation
via Spontaneously Broken $\mathcal{N}=2$
Supersymmetry

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Ref.) Hiroshi Itoyama and K. M.,
Partial breaking of $\mathcal{N}=2$ supersymmetry

[Antoniadis-Partouche-Taylor]

[Ittoyama-Fujiwara-Sakaguchi]

$U(N)$ gauge model in which $\mathcal{N}=2$ supersymmetry is broken to $\mathcal{N}=1$ spontaneously has some interesting properties.

The remarkable one is that the model includes $\mathcal{N}=1$, $U(N)$ super Yang-Mills with tree level superpotential as a particular limit;

$e, m, \xi \to \infty$

(a large FI parameters limit)
Motivation

A large FI parameters limit

\[ \mathcal{N} = 1, \text{U(N) super Yang-Mills} + W_{\text{tree}}(\Phi) \]

\[ e, m, \xi \to \infty \]

\[ \text{U(N) gauge model with spontaneously broken } \mathcal{N} = 2 \text{ supersymmetry} \]

\[ W_{\text{eff}}(S) = N \frac{\partial}{\partial S} F_{\text{free}}(S) + \ldots \]

[\text{Dijkgraaf-Vafa}]

\[ S = -\frac{1}{64\pi^2} \text{Tr}_{\text{U(N)}} \mathcal{W}^\alpha \mathcal{W}_\alpha \]
The Lagrangian of U(N) gauge model with spontaneously broken $\mathcal{N}=2$ supersymmetry is

$$
\mathcal{L} = \int d^4 \theta \left[ \frac{-i}{2} \text{Tr} \left( \bar{\Phi} e^{adV} \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi} - \text{h.c.} \right) + \xi V^0 \right] 
+ \int d^2 \theta \left( -\frac{i}{4} \frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi^a \partial \Phi^b} \mathcal{W}^a \mathcal{W}^b + e \Phi^0 + m \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi^0} \right) + \text{h.c.}
$$

We choose the prepotential as

$$
\mathcal{F}(\Phi) = \sum_{k=0}^{n+1} \frac{g_k}{(k+1)!} \text{Tr} \Phi^{k+1}
$$

(degree n+2)

$V^a$: vector superfield

$\Phi^a$: chiral superfield

$a = 0, 1, \ldots, N^2 - 1$

overall U(1)
The model

Spontaneous breaking of $\mathcal{N}=2$ susy

- Vacuum condition: \( \frac{\partial V}{\partial \phi^a} = 0 \) \quad \Rightarrow \quad \left\langle \frac{\partial^2 \mathcal{F}}{\partial \phi^0 \partial \phi^0} \right\rangle = - \left( \frac{e}{m} + i \frac{\xi}{m} \right) 

- The gauge symmetry breaking: \( U(N) \rightarrow \prod_{i=1}^{n} U(N_i) \)

- The Nambu-Goldstone fermion is in the overall $U(1)$ vector part:

\[
\left\langle \delta \left( \frac{\lambda^0 - \psi^0}{\sqrt{2}} \right) \right\rangle \neq 0 \quad \left\langle \delta \left( \frac{\lambda^0 + \psi^0}{\sqrt{2}} \right) \right\rangle = 0
\]

- The mass spectrum

\[
\begin{align*}
\mathcal{N}=1 \text{ massless } & \prod_{i=1}^{n} U(N_i) \text{ vector multiplet} \\
\mathcal{N}=1 \text{ massive } & \prod_{i=1}^{n} U(N_i) \text{ adjoint chiral multiplet} \\
\mathcal{N}=1 \text{ massive vector multiplets corresponding to broken generators}
\end{align*}
\]
Large FI parameters limit

Let us take the limit: \((e, m, \xi) = \Lambda(e', m', \xi'), \quad \Lambda \to \infty.\) (with \(\tilde{g}_k \equiv mg_k\) fixed for \(k \geq 2\))

\[
F(\Phi) = \sum_{k=0}^{n+1} \frac{g_k}{(k+1)!} \text{Tr} \Phi^{k+1} = g_0 \text{Tr} \Phi + \frac{g_1}{2} \text{Tr} \Phi^2 + \mathcal{O}(\Lambda^{-1})
\]

\(\triangleright\) In this limit the model reduces to \(\mathcal{N}=1\) U(N) SYM with \(W_{\text{tree}}(\Phi)\).

\[
\mathcal{L} \quad \longrightarrow \quad \mathcal{L}_{DV} = \text{Im} \left[ \frac{-e' + i\xi'}{m'} \left( 2 \int d^4\vartheta \text{Tr} \bar{\Phi} e^{adV} \Phi + \int d^2\vartheta \text{Tr} \mathcal{W} \mathcal{W} \right) \right] + \int d^2\vartheta W(\Phi) + h.c.,
\]

\(\triangleright\) The overall U(1) part (the Nambu-Goldstone fermion) is decoupled.
Effective superpotential

- We consider the effective superpotential by integrating out massive modes $\Phi$ and $\bar{\Phi}$.

- We treat $\mathcal{W}$ (or $\mathcal{V}$) as the background field.

  The result is represented by

  \[ S = -\frac{1}{64\pi^2} \text{Tr}_{\text{U}(N)} \mathcal{W}^\alpha \mathcal{W}_\alpha \]

- We assume large FI parameters and see the difference between our model and $\mathcal{N}=1$, U(N) SYM, in the leading 1/m order.
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Summary of our result: $\ell$-loop contribution to $W_{\text{eff}}(S)$

$$W_{\text{eff}}^{(\ell)}(S) = N \frac{\partial F^{(\ell)}(S)}{\partial S} \left( \frac{16\pi^2 i \tilde{g}_3}{\tilde{g}_2} \right) \frac{S}{m} + W_{\text{vertex}}^{(\ell)} + \mathcal{O}((1/m)^2)$$

The leading order terms which also exist in DV case  

order 1/m terms (leading difference)

cf.) Dijkgraaf-Vafa relation:  

$$W_{\text{eff}}(S) = N \frac{\partial}{\partial S} F_{\text{free}}(S)$$
We firstly integrate out $\bar{\Phi}$ and consider the perturbation theory with $\Phi$.

- **Propagator**
  \[
  \Delta(p, \pi) = \int_0^\infty ds c^{-s}(p^2 + m' + \frac{1}{2}adW^\alpha_\pi_\alpha - ig_3^I W W)
  \]

- **Vertices**
  1\textsuperscript{st} type.....
  \[m \frac{g_k}{k!} \text{Tr} \Phi^k, \quad \text{for } k = 3 \ldots n + 1\]
  2\textsuperscript{nd} type.....
  \[\frac{-i}{4} \sum_{s=0}^{k-1} \frac{g_k}{k!} \text{Tr}(W \Phi^{s} W \Phi^{k-1-s}), \quad \text{for } k = 4 \ldots n + 1\]

The new terms do contribute to the effective superpotential!
Diagrammatical computation 2

2-loop example

\[
\int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} d^2 \pi_1 d^2 \pi_2 \Delta(p_1, \pi_1) \Delta(p_2, \pi_2) \Delta(-p_1 - p_2, -\pi_1 - \pi_2)
\]

\[
= \int ds_1 ds_2 ds_3 e^{-\sum s_i m'} \left\{ 3N S^2 - ig_3^l (\sum s_i) S^3 \right\}
\]

\[
(S = -\frac{1}{64\pi^2} \text{Tr}_U(N) \mathbb{W}^\alpha \mathbb{W}_\alpha)
\]

\[
\Delta(p, \pi) = \int_0^\infty ds e^{-s(p^2 + m' + \frac{1}{2} tr W^\alpha \pi_\alpha)} e^{s i g_3^l \mathbb{W} \mathbb{W}}
\]

\[
\left( e^{-\frac{1}{2} tr W^\alpha \pi_\alpha} = 1 - \frac{s}{2} tr W^\alpha \pi_\alpha + \frac{s^2}{8} (tr W^\alpha \pi_\alpha)^2 \right)
\]
Alternative method

generalized Konishi anomaly

Let us define the generating functions of the one-point functions:

$$ R(z) = -\frac{1}{64\pi^2} \text{Tr} \left< \mathcal{W}^\alpha \mathcal{W}_\alpha \frac{1}{z - \Phi} \right>, \quad T(z) = \text{Tr} \left< \frac{1}{z - \Phi} \right>.$$  

In terms of these, the effective superpotential is

$$ \frac{\partial W_{\text{eff}}}{\partial g_k} = \frac{m}{k!} \int dz z^k T(z) + \frac{16\pi^2 i}{(k - 1)!} \int dz z^{k-1} R(z), $$

$R(z)$ and $T(z)$ satisfy the following equations:

$$ R(z)^2 = W'(z) R(z) + \frac{1}{4} y(z), $$

$$ 2R(z) T(z) = W'(z) T(z) + \frac{1}{4} c(z) + 16\pi^2 i \mathcal{F}'''(z) R(z) + \frac{1}{4} \tilde{c}(z), $$

which follow from the generalized Konishi anomaly equations.
Conclusion

We have shown that Dijkgraaf-Vafa relation is deformed by spontaneously broken $\mathcal{N}=2$ supersymmetry by two method:

- the diagrammatical computation

\[
W_{\text{eff}}^{(\ell)}(S) = N \frac{\partial F^{(\ell)}(S)}{\partial S} \frac{16\pi^2 i\tilde{g}_3}{g_2} \left( \frac{\partial F^{(\ell)}(S)}{\partial S} \right) \frac{S}{m} + W_{\text{vertex}}^{(\ell)} + \mathcal{O}((1/m)^2)
\]

- the argument based on the generalized Konishi anomaly.
$\mathcal{N}=2$ supersymmetry

1$\text{st}$ susy transformation: $\delta^{(1,\xi)}\mathcal{L}(\xi) = 0$

This is ordinary $\mathcal{N}=1$ susy transformation.

2$\text{nd}$ susy transformation: $\delta^{(2,\xi)}\mathcal{L}(\xi) = 0$

The definition of $\delta^{(2,\xi)}$ is $\delta^{(2,\xi)} = R\delta^{(1,-\xi)}R^{-1}$

Thus, $\delta^{(2,\xi)}\mathcal{L}(\xi) = (R\delta^{(1,-\xi)}R^{-1})(RL(-\xi)R^{-1})$

$= R\left(\delta^{(1,-\xi)}\mathcal{L}(-\xi)\right)R^{-1}$

$= 0$