Supersymmetric three dimensional conformal sigma models

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Abstract. We construct supersymmetric conformal sigma models in three dimensions[1]. Nonlinear sigma models in three dimensions are nonrenormalizable in perturbation theory. We use the Wilsonian renormalization group equation method, which is one of the nonperturbative methods, to find the fixed points. Existence of fixed points is extremely important in this approach to show the renormalizability. Conformal sigma models are defined as the fixed point theories of the Wilsonian renormalization group equation. The Wilsonian renormalization group equation with anomalous dimension coincides with the modified Ricci flow equation. The conformal sigma models are characterized by one parameter which corresponds to the anomalous dimension of the scalar fields. Any Einstein-Kähler manifold corresponds to a conformal field theory when the anomalous dimension is $\gamma = -1/2$. Furthermore, we investigate the properties of target spaces in detail for two dimensional case, and find the target space of the fixed point theory becomes compact or noncompact depending on the value of the anomalous dimension.

1 Introduction

Nonlinear sigma models with $\mathcal{N} = 2$ supersymmetry in two or three dimensions are defined by the so-called Kähler potential $K(\phi, \bar{\phi})$, which is a function of the chiral and anti-chiral superfields, ϕ^i and $\bar{\phi}^{\bar{j}}$. The bosonic fields $\varphi^i(x)$ play the role of the coordinates of the target manifold $\mathcal{M}$. The metric, characterizing the target manifold $\mathcal{M}$, is obtained by the second derivative of this Kähler potential

$$g_{i\bar{j}} = \frac{\partial^2 K(\varphi, \bar{\varphi})}{\partial \varphi^i \partial \bar{\varphi}^{\bar{j}}} \equiv K_{,i\bar{j}}.$$

The manifold defined by a Kähler potential is called the Kähler manifold. This metric is an arbitrary function of the scalar fields. The Lagrangian of nonlinear sigma model with $\mathcal{N} = 2$ supersymmetry reads

$$\mathcal{L} = g_{i\bar{j}} \partial_\mu \varphi^i \partial^\mu \bar{\varphi}^{\bar{j}} + i g_{i\bar{j}} \bar{\psi}^{\bar{j}} (D\!\!\!/\psi)^i + \frac{1}{4} R_{i\bar{j}k\bar{l}} \psi^i \psi^k \bar{\psi}^{\bar{j}} \bar{\psi}^{\bar{l}},$$

$$(1)$$

where the covariant derivative for the fermion fields is given by

$$(D_\mu \psi)^i = \partial_\mu \psi^i + \partial_\mu \varphi^j \, \Gamma^i{}_{jk} \psi^k.$$

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The first term of the Lagrangian (1) shows infinite number of the derivative interactions. In two dimensions, the scalar fields are dimensionless and the Lagrangian are perturbatively renormalizable. However, in three dimensions, all these interaction terms are perturbatively nonrenormalizable, since the scalar fields have the canonical dimension $d_\varphi = 1/2$.

We will investigate these two and three dimensional NLσMs using the nonperturbative renormalization group method.

2 Two dimensional case

Any NLσM is renormalizable within perturbation theories in two dimensions. It is convenient to use the Wilsonian renormalization group (WRG) equation for the nonperturbative study of field theories with infinitely many coupling constants. In the paper[2], we derived the β function for 2-dimensional $\mathcal{N} = 2$ supersymmetric NLσM using the WRG equation, obtaining

$$\beta(g_{i\bar{j}}) = \frac{1}{2\pi} R_{i\bar{j}} + \gamma \left[\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}} g_{i\bar{j},\bar{k}} + 2 g_{i\bar{j}} \right]. \quad (2)$$

The WRG equation describes the variation of the Wilsonian effective action when the cutoff scale is changed [3]. The first term, proportional to the Ricci tensor of the target space, comes from the one-loop diagrams,

whereas the second term, proportional to the anomalous dimension γ of fields, comes from the rescaling of fields needed to properly normalize the kinetic term. The presence of the anomalous dimension reflects the nontrivial continuum limit of the fields.

When the anomalous dimension of the field vanishes, scale invariance is realized for NLσMs on Ricci-flat Kähler (Calabi-Yau) manifolds [4]. Calabi-Yau metrics have been explicitly constructed for some non-compact manifolds [5], in the case that the number of isometries is sufficient to reduce the Einstein equation to an ordinary differential equation.

However, when the anomalous dimension of the fields does not vanish, the condition of scale invariance is quite different. In the paper[6], we study novel conformal field theories with anomalous dimensions by solving for the condition at the fixed point: $\beta = 0$. We assume $\mathbf{U}(N)$ symmetry to reduce a set of partial differential equations to an ordinary differential equation. The conformal theories obtained have one free parameter corresponding to the anomalous dimension of the scalar fields. The geometry of the target manifolds depends strongly on the sign of the anomalous dimensions.

In particularly, the solution of the $\beta(g_{i\bar{j}}) = 0$ is very simple in the case that the target manifold is of one complex dimension. The properties of the target manifold of the solutions depend strongly on the sign of the parameter $a = -4\pi\gamma$, here γ is the anomalous dimension of the scalar fields.

When $a > 0$, the anomalous dimension is negative. Because the line element is given by in polar coordinates, with $x = re^{i\phi}$,

$$ds^2 = \frac{1}{1 + ar^2}\Big((dr)^2 + r^2(d\phi)^2\Big), \qquad (3)$$

the volume and the distance from the origin ($r = 0$) to infinity ($r = \infty$) are divergent, while the length of the circumference at infinity is finite. Therefore, the shape of the target manifold is that of a semi-infinite cigar. The volume integral of the scalar curvature is also finite, giving the Euler number is equal to that of a disc. The theory is known as Witten's Eucleaden black hole solution[7].
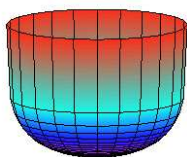

Fig. 1. The target manifold for $a = +1$ embedded in 3-dimensional flat Euclidean spaces. It takes the form of a semi-infinite cigar with radius $\sqrt{\frac{1}{a}}$.

Figure 1 shows the manifold embedded in 3-dimensional flat Euclidean spaces. The distance between any two points is measured along the shortest path on the surface in the Euclidean space.

When $a < 0$, the anomalous dimension is positive. In this case, the metric and scalar curvature read

$$g_{i\bar{j}} = \frac{1}{1 - |a|x}, R = \frac{-|a|}{1 - |a|x}. \qquad (4)$$

This metric is ill-defined at the boundary $|x| \sim \frac{1}{\sqrt{-a}}$. This is not merely a coordinate singularity, because the scalar curvature is divergent at the boundary. Although the volume integral is divergent, the distance to the boundary is finite. Now, let us embed this manifold in a flat space. The manifold is embedded as a space-like surface in a flat Minkowski space. Figure 2 shows the manifold embedded in a 3-dimensional flat Minkowski space.
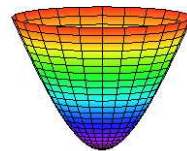

Fig. 2. The target manifold for $a = -1$, embedded in a flat Minkowski space. The vertical axis has negative signature. In the asymptotic region $\rho \to \infty$, the surface approaches the lightcone.

3 Renormalizability of three dimensional sigma models

Renormalization group (RG) equation for the metric of the target manifold $\mathcal{M}$ in three dimensional sigma models has been derived in [2,8]

$$-\frac{d}{dt}g_{i\bar{j}} = \frac{1}{2\pi^2}R_{i\bar{j}} + \gamma\Big(2g_{i\bar{j}} + \varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}}g_{i\bar{j},\bar{k}}\Big)$$
$$+ \frac{1}{2}\Big(\varphi^k g_{i\bar{j},k} + \varphi^{*\bar{k}}g_{i\bar{j},\bar{k}}\Big).$$

This RG equation, derived by using the so-called Kähler normal coordinate[9], can be written in a covariant form

$$-\frac{d}{dt}g_{i\bar{j}} = \frac{1}{2\pi^2}R_{i\bar{j}} - g_{i\bar{j}} + \nabla_i\xi_{\bar{j}} + \nabla_{\bar{j}}\xi_i \qquad (5)$$

if we define a vector field

$$\xi^i = \left(\frac{1}{2} + \gamma\right)\varphi^i \qquad (6)$$

in the Kähler normal coordinate. In other coordinate system, we have to choose a vector field corresponding to the scale transformation of the target manifold.

Let us consider the theories whose target spaces are Einstein-Kähler manifolds. The Einstein-Kähler manifolds satisfy the condition

$$R_{i\bar{j}} = \frac{h}{a^2} g_{i\bar{j}}, \tag{7}$$

where a is the radius of the manifold, which is related to the coupling constant λ by $\lambda = \frac{1}{a}$.

Using the Einstein Kähler condition (7), we can obtain the anomalous dimension and β function respectively, because only λ depends on t:

$$\gamma = -\frac{h\lambda^2}{4\pi^2}. \tag{8}$$

$$\beta(\lambda) \equiv -\frac{d\lambda}{dt} = -\frac{h}{4\pi^2}\lambda^3 + \frac{1}{2}\lambda. \tag{9}$$

We have an IR fixed point at $\lambda = 0$, and we also have a UV fixed point at $\lambda^2 = \frac{2\pi^2}{h} \equiv \lambda_c^2$ for positive h. *Therefore, if the constant h is positive, it is possible to take the continuum limit by choosing the cutoff dependence of the bare coupling constant as*

$$\lambda(\Lambda) \overset{\Lambda\to\infty}{\longrightarrow} \lambda_c - \frac{M}{\Lambda}, \tag{10}$$

where M is a finite mass scale. With this fine tuning, $\mathcal{N} = 2$ supersymmetric nonlinear σ models are renormalizable, at least in our approximation, if the target spaces are Einstein-Kähler manifolds with positive curvature.

It should be emphasized that although the RG equation obtained in the perturbation theory has the similar form with the RG equation obtained in the Wilson's renormalization method, it is valid only in the vicinity of the free field theory, whereas the Wilsonian RG equation can be used to study even nontrivial conformal field theories located far away from the free field theory.

When the constant h is positive, the target manifold is a compact Einstein-Kähler manifold [10]. In this case, the anomalous dimensions at the fixed points are given by

$\gamma_{IR} = 0$ at the IR fixed point (Gaussian fixed point),

$\gamma_{UV} = -\frac{1}{2}$ at the UV fixed point.

At the UV fixed point, the scaling dimension of the scalar fields (x_φ) is equal to the canonical plus anomalous dimensions:

$$x_\varphi \equiv d_\varphi + \gamma_\varphi = 0. \tag{11}$$

Thus the scalar fields and the chiral superfields are dimensionless in the UV conformal theory, as in the case of two dimensional field theories. Above the fixed point, the scalar fields have non-vanishing mass, and the symmetry is restored.
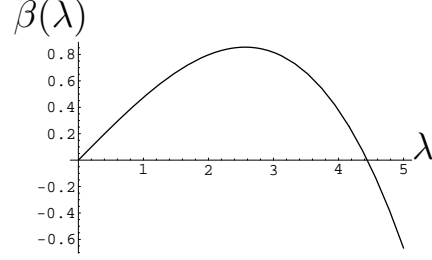

Fig. 3. The beta function for Einstein-Kähler theories with positive scalar curvature.

4 Conformal sigma models

The RG equation (5), called the modified Ricci flow in mathematical literature[11], describes the deformation of the target manifold of the effective theory.

The fixed point, invariant under the change of the mass scale, is obtained by solving an equation

$$\frac{1}{2\pi^2} R_{i\bar{j}} - g_{i\bar{j}} + \nabla_i \xi_{\bar{j}} + \nabla_{\bar{j}} \xi_i = 0. \tag{12}$$

The metric $g_{i\bar{j}}$ satisfying this equation defines a conformal field theory, and such solution is called the Kähler-Ricci soliton [12].

From now, we put the parameter c as

$$c \equiv \frac{1}{2} + \gamma, \tag{13}$$

which corresponds to the conformal dimension of the scalar fields at fixed point.

Although it is difficult to solve eq.(12) explicitly for $\gamma \neq -\frac{1}{2}$, it can be solved for two-dimensional target space $\mathcal{M}$ by using a graphical method. We use real variables to describe the target manifold $\mathcal{M}$, and choose a special gauge where the line element of $\mathcal{M}$ takes the following form

$$ds^2 = dr^2 + e^2(r)d\phi^2. \tag{14}$$

Since our target spaces are complex manifolds, we have assumed rotational symmetry in the ϕ direction corresponding to the $U(1)$ symmetry, and normalize the range of ϕ to $0 \leq \phi < 2\pi$. Then $e(r)$ denotes the radius of a circle for a fixed value of r.

The fixed point of the RG equation written in terms of real coordinates corresponds to the solution of

$$a^2 R_{ij} - g_{ij} + \nabla_i \xi_j + \nabla_j \xi_i = 0, \tag{15}$$

where

$$a^2 = \frac{1}{2\pi^2}.$$

Now, we have to find the vector field $\xi^i = (\xi^r, \xi^\phi)$. The vector field ξ^i, representing an infinitesimal scale transformation of the target space, has to be proportional to $(cr, 0)$ at least around the origin $r = 0$, that is renorlization condition to normalize the kinetic term in the RG equation. Since we assume the rotational symmetry ($U(1)$), it is natural to assume $\xi_\phi = 0$. Then the

vector field in this coordinate system is fixed by the consistency of the coupled differential equation (15). We obtain the RG equation in this gauge

$$-a^2 e'' - e + 2cee' = 0. \tag{16}$$

When $c = 0$, namely for $\gamma = -1/2$, the solution of this equation is easily obtained

$$e(r) = a \sin \frac{r}{a},$$

which defines the line element of the round S^2 with radius a.

On the other hand, when $c \neq 0$, it is convenient to rewrite the second order differential equation to a set of the first order differential equations

$$e' = p \tag{17}$$

$$p' = -\frac{1}{a^2} e(1 - 2cp)$$

with the boundary condition
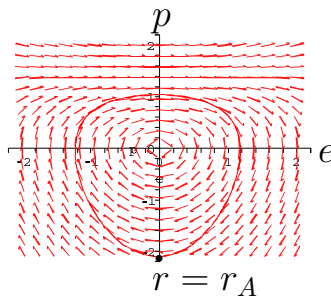
$$e(0) = 0, \quad p(0) = 1 \tag{18}$$

$$r = r_A$$

Fig. 4. Flow of the first order differential equations (17) for $0 < 2c < 1$ in the "phase space" $(e(r), p(r))$. The solid line represents the solution specified by the boundary condition.
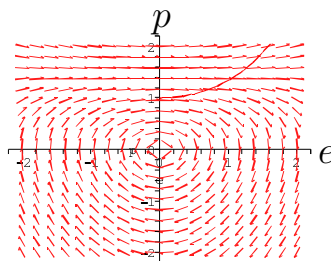

Fig. 5. Flow of the first order differential equations (17) for $2c \geq 1$ in the phase space. The solid line represents the solution specified by the boundary condition.

The vector field of the flow (17) is shown in Fig.4. When $0 \leq 2c < 1$, this equation defines a compact manifold since the trajectory starting from the initial point (18) comes back to $e = 0$ at a finite $r = r_A$

implying the circumference of the circle at that r vanishes. We may call it the "deformed shpere". Similarly, we have compact target space for $c < 0$.

On the other hand, Fig.5 shows the radius $e(r)$ becomes larger and larger when r goes to infinity, then the solution corresponds to a noncompact manifold for $2c \geq 1$. To see the asymptotic behavior of $e(r)$ for large r, we will be able to neglect the second term in (16)

$$\frac{de}{dr} = 1 + \frac{c}{a^2} e^2,$$

which can be integrated to obtain $e(r)$

$$e(r) = \frac{a}{\sqrt{c}} \tan\left(\frac{\sqrt{c}}{a} r\right).$$

Since $e(r)$ defines the radius of the circle when the geodesic distance from the origin r is fixed, this asymptotic behavior implies that the manifold for $2c > 1$ has an increasingly large radius for large r.

Acknowledgment

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