Lattice Formulation of Two Dimensional Topological Field Theory

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1. Introduction

Lattice construction of SUSY gauge theory is difficult.

\[ P = \{ Q, \bar{Q} \} \]

\[ \{ Q_\alpha, \bar{Q}_\beta \} = 2 P_\mu (\gamma^\mu)_{\alpha\beta} \]

SUSY breaking \[\rightarrow\] Fine-tuning problem

Difficult

\{ * \text{taking continuum limit} \}

\{ * \text{numerical study} \}
Candidate to solve fine-tuning problem

A lattice model of Extended SUSY preserving a partial SUSY $Q$

$Q$ : does not include the translation (BRST charge of TFT (topological field theory))
Candidate Models

**CKKU models** (Cohen-Kaplan-Katz-Unsal)
2-d $N=(4,4)$, 3-d $N=4$, 4-d $N=4$ etc. super Yang-Mills theories

**Sugino models**
B635 (2006) 218-224)

**Catterall models**

( Relationship between them: T.T (JHEP 07 (2007) 010) )
Do they really recover the target continuum theory?
Perturbative studies

Lattice

continuum limit $a \to 0$

All right!

Target continuum theory
Non-perturbative studies

No sufficient study

I have done it with the study of Topological Field Theory
Non-perturbative study

For 2-d \( N=(4,4) \) CKKU models

\[ \mathcal{O} \mathcal{O} = 0, \quad \mathcal{O} \neq Q(\text{something}) \]

BRST-cohomology

Imply

2-d \( N=(4,4) \) CKKU

Topological field theory

Must be realized

Target continuum theory

Topological field theory

Non-perturbative quantity
2.1 The target continuum theory (2-d N=(4,4))


\[
S = \frac{1}{g_2^2} \int d^2x Q \Xi(\mathcal{B}, \mathcal{F}, \Phi),
\]

\[
\Xi(\mathcal{B}, \mathcal{F}, \Phi) = \text{Tr} \left[ \frac{1}{4} \eta[\Phi, \bar{\Phi}] + \bar{\chi} \cdot (\bar{H} - i\bar{E}) 
+ \left\{ -i\lambda_{\mu} D^{\mu} \bar{\Phi} + \xi_{s_0}[s_0, \bar{\Phi}] + \xi_{s_3}[s_3, \bar{\Phi}] \right\} \right],
\]

\( A_{\mu} \) : gauge field

(Set of Fields) \( = \mathcal{A} \cup \{\Phi\} \)
**BRST transformation**

\[ QA_{\mu} = \lambda_{\mu}, \quad Q\lambda_{\mu} = iD_{\mu}\Phi, \]
\[ QS_{0} = \frac{1}{2}(\lambda + \lambda^\dagger) \equiv (\xi_{s0}), \quad Q\xi_{s0} = [\Phi, s_0], \]
\[ QS_{3} = \xi_{s3}, \quad Q\xi_{s3} = [\Phi, s_3], \]
\[ QH^C = [\Phi, \chi^C], \quad Q\chi^C = H^C, \]
\[ QH^{C\dagger} = [\Phi, \chi^{C\dagger}], \quad Q\chi^{C\dagger} = H^{C\dagger}, \]
\[ QH^R = [\Phi, \chi^R], \quad Q\chi^R = H^R, \]
\[ Q\bar{\Phi} = \eta, \quad Q\eta = [\Phi, \bar{\Phi}], \]
\[ Q\Phi = 0. \]

**Questions**

**(I)** Is BRST transformation homogeneous?

**(II)** Does \( Q \) change the gauge transformation laws?
Answer for (I) and (II)

(I) \( Q \) is not homogeneous

\[ Q \lambda_\mu = i \partial_\mu \Phi - [\Phi, A_\mu] : \text{not homogeneous of } A \]

(II) \( Q \) change the gauge transformation law

\[ A_\mu \rightarrow g A_\mu g^{-1} + g \partial_\mu g^{-1} \]

\[ \lambda_\mu \rightarrow g \lambda_\mu g^{-1} \]
2.2 BRST cohomology in the continuum theory


\[
\begin{align*}
\mathcal{W}_0 &= \text{Tr} \, \Phi^2 \\
\mathcal{W}_1 &= \text{Tr} \, \Phi \lambda \\
\mathcal{W}_2 &= \text{Tr} \, \Phi \, F + \lambda \wedge \lambda
\end{align*}
\]

Integration of \( \mathcal{W}_k \) over k-homology cycle

\[
\mathcal{O}_k \equiv \int_{\gamma_k} \mathcal{W}_k
\]

satisfying descent relation

\[
Q \mathcal{W}_k = d \mathcal{W}_{k-1}
\]

\( \mathcal{O}_k \) are **BRST cohomology** composed by \( \lambda, A, \Phi \)
Due to (II)

$\mathcal{O}_k$ can be BRST cohomology

$\mathcal{O}_k \quad \rightarrow \quad$ formally BRST exact

$\int_{\gamma_1} \mathcal{W}_1 = \int_{\gamma_1} Q \text{Tr} A \Phi$

$\int_{\gamma_2} \mathcal{W}_2 = \int_{\gamma_2} Q \text{Tr} \lambda \wedge A$

not BRST exact!

not gauge invariant

BRST exact $\quad \rightarrow \quad Q$ (gauge invariant quantity)

$Q$ change the gauge transformation law (II)
3.1 Two dimensional $N=(4,4)$ CKKU action

(K.Ohta, T.T (2007))

**BRST exact form**

$$S = \frac{1}{g^2} \sum_n Q \Xi(\vec{B}_n, \vec{F}_n, \Phi_n),$$

$$\Xi(\vec{B}_n, \vec{F}_n, \Phi_n) = \text{Tr} \left[ \frac{1}{4} \eta_n[\Phi_n, \Phi_n] + \bar{\chi}_n \cdot (\vec{H}_n - i \vec{E}_n) + \frac{1}{2} \left\{ \lambda_n (X_n^\dagger \Phi_n - \Phi_n^+ X_n^\dagger) + \bar{\lambda}_n^\dagger (X_{n-1} \Phi_n - \Phi_{n-1} X_{n-1}) + \bar{\lambda}_n (Y_n^\dagger \Phi_n - \Phi_n^+ Y_n^\dagger) + \bar{\lambda}_n^\dagger (Y_{n-j} \Phi_n - \Phi_{n-j} Y_{n-j}) \right\} \right],$$

**Set of Fields**

$$= \vec{A}_n \cup \Phi_n$$
BRST transformation on the lattice

\[ QX_n = \lambda_n, \quad Q\lambda_n = \Phi_n X_n - X_n \Phi_n, \]
\[ QY_n = \tilde{\lambda}_n, \quad Q\tilde{\lambda}_n = \Phi_n Y_n - Y_n \Phi_n, \]
\[ QH^R_n = [\Phi_n, \chi^R_n], \quad Q\chi^R_n = H^R_n, \]
\[ QH^C_n = \Phi_n \chi^C_n - \chi^C_n \Phi_n, \quad Q\chi^C_n = H^C_n, \]
\[ Q\Phi_n = \eta_n, \quad Q\eta_n = [\Phi_n, \Phi_n], \]
\[ Q\Phi_n = 0. \]

(I) Homogeneous transformation of

In continuum theory,

\[ QA^\mu = \lambda^\mu, \quad Q\lambda^\mu = iD^\mu \Phi, \]
\[ Qs_0 = 1/2(\lambda + \chi^\dagger) \equiv (\xi_{s0}), \quad Q\xi_{s0} = [\Phi, s_0], \]
\[ Qs_3 = \xi_{s3}, \quad Q\xi_{s3} = [\Phi, s_3], \]
\[ QH^C = [\Phi, \chi^C], \quad Q\chi^C = H^C, \]
\[ QH^C\dagger = [\Phi, \chi^C\dagger], \quad Q\chi^C\dagger = H^C\dagger, \]
\[ QH^R = [\Phi, \chi^R], \quad Q\chi^R = H^R, \]
\[ Q\Phi = \eta, \quad Q\eta = [\Phi, \Phi], \]
\[ Q\Phi = 0. \]

(I) Not Homogeneous transformation of

\[ A_n = (\vec{B}_n, \vec{F}_n) \]
Homogeneous property

\( Q \longrightarrow \text{tangent vector} \)

with

\[
\tilde{Q} = \sum_n \left[ \lambda_n \frac{\partial}{\partial \lambda_n} + X_n^\dagger \frac{\partial}{\partial \lambda_n} + Y_n \frac{\partial}{\partial \lambda_n} + Y_n^\dagger \frac{\partial}{\partial \lambda_n} + \Phi_n \frac{\partial}{\partial \phi_n} + \chi_n \frac{\partial}{\partial H_n} \right]
\]

Number operator counts the number of fields in

\[
\{Q, \tilde{Q}\} = \hat{N}_A \frac{\partial}{\partial \hat{F}_n} + \eta_{\phi_n} \frac{\partial}{\partial \hat{F}_n}
\]

\[
\hat{A}_n = (\hat{B}_n, \hat{F}_n)
\]

\( Q \) closed term including the field of

\[
h_{nA} = n_A^{-1} \hat{N}_A \cdot h_{nA} = n_A^{-1} \{Q, \tilde{Q}\} h_{nA} = n_A^{-1} Q \cdot (\tilde{Q} h_{nA})
\]
\textbf{(II) Gauge symmetry under } \mathcal{Q} \text{ on the lattice}

\textbf{* (II) Gauge transformation laws do not change under BRST transformation}

\[
\begin{align*}
A_\mu & \rightarrow g A_\mu g^{-1} + g \partial_\mu g^{-1} \\
\lambda_\mu & \rightarrow g \lambda_\mu g^{-1}
\end{align*}
\]

in continuum theory

\[ \mathcal{Q} \text{ BRST} \]
3.2 BRST cohomology on the lattice theory
(K. Ohta, T.T (2007))

BRST cohomology cannot be realized!

Only the polynomial of $\phi$
can be BRST cohomology
Essence of the proof of the result

Closed terms $h_{nA}$ including the fields in $\tilde{A}_n$ must be BRST exact.

(I) Homogeneous property of $Q$

Only polynomial of $\phi$ can be BRST cohomology

(II) $h_{nA}$: gauge invariant $\xrightarrow{\tilde{Q}} h_{nA}$: gauge invariant
BRST cohomology must be composed only by $N=(4,4)$ CKKU model without mass term would not recover the target theory non-perturbatively.

$N=(4,4)$ CKKU model

Topological field theory

Target theory

Topological field theory

BRST cohomology are composed by $\lambda, A, \phi$
5. Summary

The topological property (like as BRST cohomology) could be used as a non-perturbative criteria to judge whether supersymmetric lattice theories (which preserve BRST charge) have the desired continuum limit or not.
We apply the criteria to $N = (4,4)$ CKKU model without mass term

The target continuum limit would not be realized

*Implication by an explicit form.*

**Perturbative study**

*did not show it*

*It can be a powerful criteria.*
Comment on the No-go result

(I) and (II) plays the crucial role.

These relate with the gauge transformation law on the lattice.

Gauge parameters are defined on each sites as the independent parameters.

\[ A_\mu \rightarrow g A_\mu g^{-1} + \frac{Q}{2} \left[ \Phi, A_\mu \right] \]

Contribute to the realization of BRST cohomology in the continuum theory.
BRST cohomology

\[ \mathcal{O}_k \equiv \int_{\gamma_k} \mathcal{W}_k \]

Topological quantity defined by the inner product of homology and the cohomology

The realization is difficult due to the independence of gauge parameters (Singular gauge transformation)

Admissibility condition etc. would be needed