Instantons in Deformed Super Yang-Mills Theories

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Abstract. We study the instanton effective action in deformed four-dimensional $\mathcal{N}=2$ and $\mathcal{N}=4$ super Yang-Mills (SYM) theories. These deformed gauge theories are defined on the D-brane world-volume in the presence of constant, self-dual Ramond-Ramond (R-R) 3-form field strength background $\mathcal{F}$ which is scaled as $(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}=$ fixed in the zero-slope limit $\alpha'\to 0$. The instanton effective action is obtained by solving equations of motion of the deformed $\mathcal{N}=2$ SYM action. We show that this effective action correctly reproduces the string theory result derived from D3/D(−1)-branes system in the lowest order of the backgrounds and gauge coupling constant. We comment on the generalization of $\mathcal{N}=2$ results to $\mathcal{N}=4$ case.

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1 Introduction

Deformation of supersymmetric gauge theories is useful for the study of non-perturbative effects. For example, the Ω-background and noncommutative deformations of $\mathcal{N}=2$ SYM theory is useful to evaluate integration over the instanton moduli space [1]. Similar deformations can be provided by introducing supergravity backgrounds. Especially, constant R-R backgrounds play also an interesting role in the instanton calculus.

In fact, it was shown that in D3/D(−1)-branes system with self-dual graviphoton and vector backgrounds, the deformed effective action of the D(−1)-branes is nothing but the instanton effective action of $\mathcal{N}=2$ super Yang-Mills theory in the Ω-background at the lowest order in the background [2]. Due to the graviphoton and vector backgrounds, the effective action on the D-brane world-volume is deformed. We investigated the vacuum structure and supersymmetry of the deformed $\mathcal{N}=2$ and $\mathcal{N}=4$ SYM actions defined on the D3-brane world-volume [3]. The instanton effective action of these deformed SYM theories should reproduce the result of D3/D(−1)-branes system. In [4], we derived the instanton equations in these deformed $\mathcal{N}=2,\mathcal{N}=4$ SYM theories and solved it. The solutions were obtained through the ADHM construction [5]. These solutions are expressed by the ADHM moduli parameters and we found that the instanton effective action for the deformed $\mathcal{N}=2$ SYM theory agrees with the result from the D3/D(−1)-branes system obtained in [2] to first order in the background and gauge coupling constant. In the following, we see this equivalence in $\mathcal{N}=2$ case. We also comment that it is possible to generalize this result to $\mathcal{N}=4$ SYM theory. A detailed results will be explained in [4].

2 Deformed gauge theories in constant, self-dual R-R 3-form background

To derive the explicit form of the deformed actions on the D3-brane world-volume (including fermions), we calculate open string disk amplitudes with the insertions of the R-R vertex operators and take the zero-slope limit $\alpha'\to 0$. The appropriate boundary condition for the vertex operators is imposed. We consider the ordinary NSR formalism of type IIB superstring theory and the R-R background is treated in a perturbative way. The background $\mathcal{F}^{\alpha\beta AB}$ is scaled as $(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{\alpha\beta AB}\equiv C^{\alpha\beta AB}=$ fixed. Here α,β are space-time spinor and A,B are internal indices. In addition to this zero-slope scaling, the self-duality condition is imposed on the space-time index of the background. Due to the background, non-trivial interactions are induced on the D3-brane world-volume and the supersymmetry which is present in the vanishing background case is generically broken.

Let us start from the $\mathcal{N}=4$ case in the following subsection. After that, we derive the deformed $\mathcal{N}=2$ SYM action by orbifold projecting the deformed $\mathcal{N}=4$ action.

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2.1 $\mathcal{N}=4$ deformed action

We consider the deformed four-dimensional $\mathcal{N}=4$ $U(N)$ SYM theory in the presence of constant R-R 3-form background $\mathcal{F}^{\alpha\beta AB}$ with the self-dual constraint in the four-dimensional space-time part. The deformed action is realized on the N Euclidean D3-branes world-volume located in the flat $\mathbf{R}^{10}$.

We focus on the leading order corrections of the background, namely, the deformed action is obtained by calculating tree-level open string amplitudes (disk amplitudes) with the insertion of one R-R closed string vertex operator.

The vertex operators corresponding to open string fields are $V_A^{(n)}, V_\varphi^{(n)}, V_\Lambda^{(n)}, V_{\overline{\Lambda}}^{(n)}$ which represent a gauge field A_μ, adjoint scalar fields φ_a $(a=1,\cdots 6)$, gauginos $\Lambda^{\alpha A}, \overline{\Lambda}_{\dot\alpha A}$, $(A=1,\cdots 4)$ respectively. Here the notation (n) $(n=0,-1/2,-1)$ stands for the picture number. The vertex operators $V_{H_{AA}}^{(0)}$, $V_{H_{A\varphi}}^{(0)}$, $V_{H_{\varphi\varphi}}^{(0)}$ for the auxiliary fields [2] are also introduced. The vertex operator $V_{\mathcal{F}}^{(-1/2,-1/2)}$ corresponds to the constant R-R background.

The following selection rules are useful to find the non-zero amplitudes:

1. Internal charge cancellation condition among open and closed string vertex operators
2. α' power counting
3. Lorentz and internal index structure

The conditions 1 and 3 are necessary for non-vanishing disk amplitudes while the condition 2 becomes important when we take the zero-slope limit $\alpha'\to 0$ and move to the field theory description. Regarding the condition 2, it is crucial that the background R-R field is scaled as $(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}=\text{fixed}=C$ in the zero-slope limit. Considering all the conditions 1, 2 and 3, we find that there are only three non-zero amplitudes which include one R-R vertex operator:

$$\begin{aligned}&\langle\!\langle V_A^{(0)}(p_1)V_\varphi^{(-1)}(p_2)V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\!\rangle,\\ &\langle\!\langle V_{H_{AA}}^{(0)}(p_1)V_\varphi^{(-1)}(p_2)V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\!\rangle,\\ &\langle\!\langle V_\Lambda^{(-1/2)}(p_1)V_\Lambda^{(-1/2)}(p_2)V_{\mathcal{F}}^{(-1/2,-1/2)}\rangle\!\rangle.\end{aligned}\tag{1}$$

After calculating all the amplitudes, taking into account the symmetric factor, taking the zero-slope limit and integrating out all the auxiliary fields, we find the deformed Lagrangian $\mathcal{L}_{\mathcal{N}=4}=\mathcal{L}^{(0)}_{\mathcal{N}=4}+\mathcal{L}^{(1)}_{\mathcal{N}=4}+\mathcal{L}^{(2)}_{\mathcal{N}=4}+\cdots$ where $\mathcal{L}^{(0)}_{\mathcal{N}=4}$ is an ordinary $\mathcal{N}=4$ $U(N)$ SYM action and $\mathcal{L}^{(n)}_{\mathcal{N}=4}$ is a correction in n-th power of the background. The first and second order corrections are

$$\begin{aligned}\mathcal{L}^{(1)}_{\mathcal{N}=4}&=\frac{1}{\kappa g^2}\mathrm{Tr}\left[iF_{\mu\nu}\varphi_a\right]C^{\mu\nu a}\\ &\quad-\frac{1}{\kappa g^2}\mathrm{Tr}\left[\varepsilon_{ABCD}\Lambda_\alpha{}^A\Lambda_\beta{}^B\right]C^{(\alpha\beta)[CD]},\end{aligned}\tag{2}$$

$$\mathcal{L}^{(2)}_{\mathcal{N}=4}=\frac{1}{2}\frac{1}{\kappa g^2}\mathrm{Tr}\left[\varphi_a\varphi_b\right]C_{\mu\nu}{}^aC^{\mu\nu b}\tag{3}$$

where $\mathrm{Tr}(T^mT^n)=\kappa\delta^{mn}$ for $U(N)$ generators T^m and g is a gauge coupling constant. Here we have defined the deformation parameter by

$$\begin{aligned}C^{\mu\nu a}&\equiv-2\pi(2\pi\alpha')^{\frac{1}{2}}(\sigma^{\mu\nu})_{\alpha\beta}\left(\overline{\Sigma}^a\right)_{AB}\mathcal{F}^{(\alpha\beta)[AB]},\\ C^{(\alpha\beta)[AB]}&\equiv-2\pi(2\pi\alpha')^{\frac{1}{2}}\mathcal{F}^{(\alpha\beta)[AB]}.\end{aligned}\tag{4}$$

$\overline{\Sigma}^a$ is a six-dimensional sigma matrix. Let us check the consistency between our deformed action and the known D-brane effective action. In general, the D-brane effective action is expressed as

$$S=S_{\mathrm{DBI}}+S_{\mathrm{CS}}\tag{5}$$

where S_{DBI} is the Dirac-Born-Infeld action and S_{CS} is a Chern-Simons term. The R-R background appears in the Chern-Simons term in the D-brane effective action

$$S_{\mathrm{CS}}=\frac{\mu_3}{\kappa}\mathrm{STr}\int_{\mathcal{M}_4}\sum_n P[e^{i\lambda\mathrm{i}_\varphi^2}\lambda^{\frac{1}{2}}\mathcal{A}^{(n)}]e^{\lambda F}.\tag{6}$$

The symbol P denotes the pull-back of ten-dimensional fields and i_φ is the interior product by φ_a. μ_3 is a R-R charge of the D3-brane, $\lambda=2\pi\alpha'$ and STr is a symmetric trace of the $U(N)$ gauge group. Finally, $\mathcal{A}^{(n)}$ is an n-form R-R potential.

It is not difficult to check that the linear term in C in the bosonic part of the deformed action (2) precisely agrees with the eq. (6). Since our deformation is due to the R-R background, it would be expected that there is a fuzzy sphere configuration of the vacuum. Indeed, we showed that there is a fuzzy S^2 solution of the vacuum in the deformed $\mathcal{N}=4$ SYM theory. For more detail, see [3].

2.2 $\mathcal{N}=2$ deformed action

The deformed $\mathcal{N}=2$ SYM action can be derived by orbifold projecting the deformed $\mathcal{N}=4$ action obtained in the previous subsection. The $\mathcal{N}=2$ $U(N)$ SYM theory is realized on the N (fractional) D3-branes located in the orbifold fixed point. Directions perpendicular to the D3-branes are orbifolded as $\mathbf{C}\times\mathbf{C}^2/\mathbf{Z}_2$. The vertex operators are appropriately projected and the surviving open string degrees of freedom are A_μ, φ, $\bar\varphi$ corresponding to gauge field, adjoint scalars, and $\Lambda^{\alpha I}$, $\overline{\Lambda}_{\dot\alpha I}$ $(I=1,2)$ corresponding to gauginos. The R-R background is now decomposed into the form

$$C^{\alpha\beta AB}=-\frac{i}{2\sqrt{2}}\begin{pmatrix}C^{\alpha\beta}&0\\0&-\bar C^{\alpha\beta}\end{pmatrix}\tag{7}$$

by the orbifold projection. Here $C^{\alpha\beta}$ is a graviphoton and $\bar C^{\alpha\beta}$ is a vector background. Both of these satisfy the self-duality condition. By this projection, the $\mathcal{N}=2$ deformed Lagrangian $\mathcal{L}_{\mathcal{N}=2}=\mathcal{L}^{(0)}_{\mathcal{N}=2}+\mathcal{L}^{(1)}_{\mathcal{N}=2}+\mathcal{L}^{(2)}_{\mathcal{N}=2}+\cdots$ is obtained from the deformed $\mathcal{N}=4$ SYM Lagrangian. Here $\mathcal{L}^{(0)}_{\mathcal{N}=2}$ is an ordinary $\mathcal{N}=2$ SYM

Lagrangian and

$$\mathcal{L}^{(1)}_{\mathcal{N}=2}+\mathcal{L}^{(2)}_{\mathcal{N}=2}=\frac{1}{\kappa g^2_{\rm YM}}{\rm Tr}\Big[i(\tilde{C}^{\mu\nu}\bar{\varphi}+\bar{C}^{\mu\nu}\varphi)F_{\mu\nu} -\frac{1}{\sqrt{2}}\bar{C}^{\mu\nu}\Lambda^I\sigma_{\mu\nu}\Lambda_I+\frac{1}{2}(\tilde{C}^{\mu\nu}\bar{\varphi}+\bar{C}^{\mu\nu}\varphi)^2\Big]. \quad (8)$$

The supersymmetry of this deformed action is generically broken due to the background. However, depending on the rank of the background, there are unbroken deformed supersymmetries in this model. The number of remaining supersymmetries is summarized in table 1. For more detail, see the reference [3].

Table 1. The number of unbroken supersymmetries in the deformed $\mathcal{N}=2$ SYM action [3]. $R[M]$ is the rank of the matrix M.

		$R[C^{(\alpha\beta)}]$		
		0	1	2
$R[\bar{C}^{(\alpha\beta)}]$	0	$\mathcal{N}=(1,1)$	$\mathcal{N}=(1,0)$	$\mathcal{N}=(1,0)$
	1	$\mathcal{N}=(\frac{1}{2},1)$	$\mathcal{N}=(\frac{1}{2},0)$	$\mathcal{N}=(\frac{1}{2},0)$
	2	$\mathcal{N}=(0,1)$	$\mathcal{N}=(0,0)$	$\mathcal{N}=(0,0)$

3 Instanton calculus in deformed super Yang-Mills theories

Let us focus on the instanton effective action in the deformed $\mathcal{N}=2$ SYM theory. To derive the instanton effective action, we need to find the instanton solution which is expressed by the ADHM moduli parameters [6]. After writing the action to the perfect square form, the gauge field strength part and the last term in (8) become

$$\mathcal{L}_{\rm gauge}=\frac{1}{\kappa}{\rm Tr}\left[-\frac{1}{2}(F^{(-)}_{\mu\nu})^2 -\frac{1}{2}\left(F^{(+)}_{\mu\nu}-ig(C^{\mu\nu}\bar{\varphi}+\bar{C}^{\mu\nu}\varphi)\right)^2\right], \quad (9)$$

where $F^{(\pm)}_{\mu\nu}=\frac{1}{2}(F_{\mu\nu}\pm\tilde{F}_{\mu\nu})$, $\tilde{F}_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}$. The (anti)instanton equation is derived from $\mathcal{L}_{\rm gauge}$ as

$$F^{(-)}=0 \text{ (instanton)},$$
$$F^{(+)}_{\mu\nu}-ig(C^{\mu\nu}\bar{\varphi}+\bar{C}^{\mu\nu}\varphi)=0 \text{ (anti-instanton)}. \quad (10)$$

Since we are interested in the low-energy effective theory of the $\mathcal{N}=2$ pure SYM theory, we introduce the VEV of the adjoint fields $\varphi,\bar{\varphi}$. However, it is known that when the adjoint scalar fields have a VEV, the super instanton solution is expanded in the gauge coupling constant g and the solution is determined in a perturbative way. Let us focus on the self-dual condition. In this case, the gauge coupling expansion of the solution is

$$A_\mu=g^{-1}A^{(0)}_\mu+gA^{(1)}_\mu+\cdots,$$
$$\Lambda^I=g^{-\frac{1}{2}}\Lambda^{(0)I}+g^{\frac{3}{2}}\Lambda^{(1)I}+\cdots,$$
$$\bar{\Lambda}_I=g^{\frac{1}{2}}\bar{\Lambda}^{(0)}_I+g^{\frac{5}{2}}\bar{\Lambda}^{(1)}_I+\cdots,$$
$$\varphi=g^0\varphi^{(0)}+g^2\varphi^{(1)}+\cdots,$$
$$\bar{\varphi}=g^0\bar{\varphi}^{(0)}+g^2\bar{\varphi}^{(1)}+\cdots. \quad (11)$$

The equation of motion up to leading order in g is now

$$F^{(0)(-)}_{\mu\nu}=0,$$
$$\nabla^2\bar{\varphi}^{(0)}+iF^{(0)}_{\mu\nu}\bar{C}^{\mu\nu}=0,$$
$$\nabla^2\varphi^{(0)}+i\sqrt{2}\Lambda^{(0)I}\Lambda^{(0)}_I+iF^{(0)}_{\mu\nu}C^{\mu\nu}=0,$$
$$(\sigma^\mu)_{\alpha\dot{\beta}}\nabla_\mu\overline{\Lambda}^{(0)\dot{\beta}}_I+\sqrt{2}[\bar{\varphi}^{(0)},\Lambda^{(0)}_{I\alpha}]+\sqrt{2}\Lambda^{(0)\beta}{}_I\bar{C}_{(\beta\alpha)}=0,$$
$$(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\nabla_\mu\Lambda^{(0)I}_\beta=0,$$
$$\nabla_\mu(F^{(0)\mu\nu}+\tilde{F}^{(0)\mu\nu})=0. \quad (12)$$

Here ∇_μ is a gauge covariant derivative defined by zeroth order gauge field $A^{(0)}_\mu$. $F^{(0)}_{\mu\nu}$ is a field strength of $A^{(0)}_\mu$. It is easy to see that the self-duality condition $F^{(0)(-)}_{\mu\nu}=0$ is consistent with the equation of motion. The solution to this equation can be expressed by the ADHM moduli parameters $(a'_\mu,\mathcal{M}',\mu,\bar{\mu},\chi,\bar{\chi},w,\bar{w})$ [4]. First of all, the gauge field equation is not deformed by the background. Thus the gauge field part is an ordinary k instanton background. For the same reason, the solution for Λ is not deformed. On the other hand, $\overline{\Lambda}$ and adjoint scalar equations in this background is deformed by the non-zero background[1]. The result is

$$A^{(0)}_\mu=-i\overline{U}\partial_\mu U,$$
$$\Lambda^{(0)I}_\alpha=\Lambda_\alpha(\mathcal{M}^I)=\overline{U}(\mathcal{M}^I f\bar{b}_\alpha-b_\alpha f\overline{\mathcal{M}}^I)U,$$
$$\varphi^{(0)}=i\frac{\sqrt{2}}{4}\epsilon_{IJ}\overline{U}\mathcal{M}^I f\overline{\mathcal{M}}^J U+\overline{U}\begin{pmatrix}\phi & 0\\ 0 & \chi\mathbf{1}_2+\mathbf{1}_k C\end{pmatrix}U,$$
$$\bar{\varphi}^{(0)}=\overline{U}\begin{pmatrix}\bar{\phi} & 0\\ 0 & \bar{\chi}\mathbf{1}_2+\mathbf{1}_k\bar{C}\end{pmatrix}U, \quad (13)$$

where $C_\alpha{}^\beta=(\sigma^{\mu\nu})_\alpha{}^\beta C^{\mu\nu}$, $\mathcal{M}=(\mu,\mathcal{M}')^T$ and $\phi=\langle\varphi^{(0)}\rangle$, $\bar{\phi}=\langle\bar{\varphi}^{(0)}\rangle$ are VEVs of the adjoint scalars. $U_{\lambda u}$ is an $(N+2k)\times N$ matrix which satisfies the conditions

$$\overline{\Delta}_i{}^{\dot{\alpha}\lambda}U_{\lambda u}=0,\quad \overline{U}_u{}^\lambda U_{\lambda u}=\delta_{uv},$$
$$\Delta_{\lambda i\dot{\alpha}}f_{ij}\overline{\Delta}_j{}^{\dot{\alpha}\rho}=\delta_\lambda{}^\rho-U_{\lambda u}\overline{U}^{u\rho},$$
$$\Delta_{\lambda j\dot{\alpha}}(x)\equiv\Delta_{(u+i\alpha)j\dot{\alpha}}=\begin{pmatrix}w_{uj\dot{\alpha}}\\ \delta_{ij}x_{\alpha\dot{\alpha}}+(a'_{\alpha\dot{\alpha}})_{ij}\end{pmatrix} \quad (14)$$

Here we have introduced the indices $\lambda=1,2,\cdots,N+2k$, $i,j=1,2,\cdots,k$, $u,v=1,2,\cdots,N$. f is an x-dependent $k\times k$ matrix. $\chi,\bar{\chi}$ should satisfy the constraint

$$\mathbf{L}\chi=-i\frac{\sqrt{2}}{4}\epsilon_{IJ}\overline{\mathcal{M}}^I\mathcal{M}^J+\bar{w}^{\dot{\alpha}}\phi w_{\dot{\alpha}}+C^{\mu\nu}[a'_\mu,a'_\nu],$$
$$\mathbf{L}\bar{\chi}=\bar{w}^{\dot{\alpha}}\bar{\phi}w_{\dot{\alpha}}+\bar{C}^{\mu\nu}[a'_\mu,a'_\nu], \quad (15)$$

[1] We need not to solve the equation of motion for $\overline{\Lambda}$ because $\overline{\Lambda}$ contributes to the classical potential through the sub-leading order in g.

where the operator $\mathbf{L}$ is defined by

$$\mathbf{L}* = \frac{1}{2}\{\bar{w}^{\dot{\alpha}} w_{\dot{\alpha}}, *\} + [a'_\mu, [a'^\mu, *]]. \tag{16}$$

The classical action is now expanded as

$$S = \frac{8\pi^2 k}{g^2} + ik\theta + g^0 S^{(0)}_{\text{eff}} + \mathcal{O}(g^2), \tag{17}$$

where we have introduced the θ angle. The leading order action is

$$\begin{aligned} S^{(0)}_{\text{eff}} = \frac{1}{\kappa}\int d^4x \ \text{Tr}\Big[&-\nabla_\mu\varphi^{(0)}\nabla^\mu\bar{\varphi}^{(0)} \\ &-\frac{i}{\sqrt{2}}\Lambda^{(0)I}[\bar{\varphi}, \Lambda^{(0)}_I] + i\bar{\varphi}^{(0)}F^{(0)}_{\mu\nu}C^{\mu\nu} \\ &+i\varphi^{(0)}F^{(0)}_{\mu\nu}\bar{C}^{\mu\nu} + i\frac{\sqrt{2}}{2}\Lambda^{(0)I}_\alpha\Lambda^{(0)}_{\beta I}\bar{C}^{(\alpha\beta)}\Big]. \end{aligned} \tag{18}$$

After plugging the solution (13) into the eq. (18), the instanton effective action is evaluated as

$$\begin{aligned} S^{(0)}_{\text{eff}} = 4\pi^2\text{tr}_k\Big[&-i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mu}^I\bar{\phi}\mu^J + \frac{1}{2}\bar{w}^{\dot{\alpha}}(\bar{\phi}\phi + \phi\bar{\phi})w_{\dot{\alpha}} \\ &-\bar{\chi}\mathbf{L}^{-1}\chi - i\frac{\sqrt{2}}{8}\bar{C}^{(\alpha\beta)}\epsilon_{IJ}\mathcal{M}'^I_\alpha\mathcal{M}'^J_\beta\Big] \\ &+(\bar{C}C\text{-term}). \end{aligned} \tag{19}$$

It is possible to rewrite the result in [2] in the above form just by imposing an ADHM constraint, dropping the Lagrange multiplier term in [2] and solving $\chi, \bar{\chi}$ in terms of other moduli. We can see that our result precisely reproduces the known result which was obtained in D3/D(−1)-brane system in the presence of self-dual R-R background up to the $\mathcal{O}(C\bar{C})$ terms. Thus our deformed action reproduces the result which is consistent with string theory calculations.

Let us next consider the anti-self-dual configuration $F^{(+)} = 0$. The solution is expanded as

$$\begin{aligned} A_\mu &= g^{-1}A^{(0)}_\mu + gA^{(1)}_\mu + \cdots, \\ \Lambda^I &= g^{\frac{1}{2}}\Lambda^{(0)I} + g^{\frac{5}{2}}\Lambda^{(1)I} + \cdots, \\ \bar{\Lambda}_I &= g^{-\frac{1}{2}}\bar{\Lambda}^{(0)}_I + g^{\frac{3}{2}}\bar{\Lambda}^{(1)}_I + \cdots, \\ \varphi &= g^0\varphi^{(0)} + g^2\varphi^{(1)} + \cdots, \\ \bar{\varphi} &= g^0\bar{\varphi}^{(0)} + g^2\bar{\varphi}^{(1)} + \cdots. \end{aligned} \tag{20}$$

The leading order equation is then

$$\begin{aligned} &F^{(0)(+)}_{\mu\nu} = 0, \\ &\nabla^2\bar{\varphi}^{(0)} - i\sqrt{2}\overline{\Lambda}^{(0)}_I\overline{\Lambda}^{(0)I} = 0, \\ &\nabla^2\varphi^{(0)} = 0, \\ &(\sigma^\mu)_{\alpha\dot{\beta}}\nabla_\mu\overline{\Lambda}^{(0)\dot{\beta}}_I = 0, \\ &(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\nabla_\mu\Lambda(0)I_\beta - \sqrt{2}[\varphi^{(0)}, \overline{\Lambda}^{(0)I\dot{\alpha}}] = 0, \\ &\nabla_\mu(F^{(0)\mu\nu} + \tilde{F}^{(0)\mu\nu}) = 0. \end{aligned} \tag{21}$$

Thus the equation of motion does not receive corrections from $C, \bar{C}$. Moreover, $S^{(0)}_{\text{eff}}$ in (17) for the anti-self-dual gauge coupling expansion is

$$\begin{aligned} S^{(0)}_{\text{eff}} = \frac{1}{\kappa}\int d^4x \ \Big[&-\nabla_\mu\varphi^{(0)}\nabla^\mu\bar{\varphi}^{(0)} + \frac{i}{\sqrt{2}}\overline{\Lambda}^{(0)}_I[\varphi^{(0)}, \overline{\Lambda}^{(0)I}] \\ &+iF^{(0)}_{\mu\nu}\bar{\varphi}^{(0)}C^{\mu\nu} + iF^{(0)}_{\mu\nu}\varphi^{(0)}\bar{C}^{\mu\nu}\Big]. \end{aligned} \tag{22}$$

Due to the self-duality condition on the background $C, \bar{C}$, there are no background corrections in the instanton effective action. Therefore we conclude that the anti-self-dual sector is not deformed by the self-dual background.

The super instanton solution in the $\mathcal{N} = 4$ deformed SYM theory is also obtained in [4].

The deformed instanton equation is derived in a straightforward way and the solution to the equation is expressed by the ADHM moduli parameters as in the case of $\mathcal{N} = 2$ deformed SYM theory. By using this solution, we can calculate the instanton effective action in principle.

However, there is an efficient way to derive the deformed $\mathcal{N} = 4$ instanton effective action. Since it is known that the $\mathcal{N} = 2$ deformed SYM action is obtained from the $\mathcal{N} = 4$ theory by the orbifold projection, the $\mathcal{N} = 4$ instanton effective action is derived from the the consistency of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ instanton effective actions by the same orbifold projection. The result can be found in [4].

These deformed gauge theories and the deformed instanton effective actions have very similar structure to the Ω-background deformation [2]. It would be expected that these deformation effects play a role similar to the Ω-background in the instanton calculus. For more detail, see the reference [4] and future works.

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References

1. N. A. Nekrasov, Adv. Theor. Math. Phys. **7** (2004) 831, hep-th/0206161.
2. M. Billó, M. Frau, F. Fucito and A. Lerda, JHEP **0611** (2006) 012, hep-th/0606013.
3. K. Ito, H. Nakajima and S. Sasaki, JHEP **0707** (2007) 068, arXiv:0705.3532 [hep-th].
4. K. Ito, H. Nakajima and S. Sasaki, to appear in arXiv.
5. M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Yu. I. Manin, Phys. Lett. A **65** (1978) 185.
6. N. Dorey, T. J. Hollowood, V. V. Khoze and M. P. Mattis, Phys. Rept. **371** (2002) 231, hep-th/0206063.