Counting BPS Solitons and Applications

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The 15th International Conference on Supersymmetry and the Unification of Fundamental Interactions
SUSY 07, Karlsruhe, Germany
July 31, 2007

Work and discussion with M. Eto, T. Fujimori, M. Nitta, K. Ohashi, N. Sakai
hep-th/0703197, to appear NPB
**Introduction**

BPS solitons in supersymmetric gauge theory play essential roles to understand the non-perturbative dynamics and properties.

<table>
<thead>
<tr>
<th>Solitons</th>
<th>Codimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Instanton</td>
<td>4</td>
</tr>
<tr>
<td>Monopole</td>
<td>3</td>
</tr>
<tr>
<td>Voretex</td>
<td>2</td>
</tr>
<tr>
<td>Domain wall</td>
<td>1</td>
</tr>
</tbody>
</table>
It is very important to investigate the structure (topology, singularity, metric, etc.) of the moduli space of these solitons, but I would like to concentrate on calculus of the “volume” of the moduli space of solitons.

Why is the “volume” important?

Application 1

Thermodynamical partition function for the BPS solitons

\[
Z = \frac{1}{(2\pi \hbar)^{2N}} \int_{T^* \mathcal{M}_N} e^{-H/T} \frac{\omega_{\text{can}}^{2N}}{(2N)!} \]

\[
= \left( \frac{T}{2\pi \hbar^2} \right)^N \int_{\mathcal{M}_N} \frac{\omega^N}{N!} \]

\[
= \left( \frac{T}{2\pi \hbar^2} \right)^N \text{Vol}(\mathcal{M}_N) \]
Application 2

Non-perturbative effects of BPS instantons on the prepotential of the $N=2$ supersymmetric gauge theory.

$$Z_{\text{inst}}(\vec{a}; \Lambda; \epsilon) = \sum_{k=0}^{\infty} \Lambda^{2r_k} \int_{\mathcal{M}_{r,k}} \frac{1}{r_k}$$

$$= \sum_{k=0}^{\infty} \Lambda^{2r_k} \text{Vol}(\mathcal{M}_{r,k})$$

$$\Rightarrow \mathcal{F}(\vec{a}; \Lambda) = \lim_{\epsilon \to 0} \epsilon^2 \log Z_{\text{inst}}(\vec{a}; \Lambda; \epsilon)$$

$$= \sum_{k=0}^{\infty} \mathcal{F}_k(\vec{a}) \Lambda^{2r_k}$$

[Prepotential]

[Nekrasov (2002)]
In this talk, I explain how to calculate the volume of the moduli space of BPS solitons, in particular vortices in 2d, or the thermodynamical partition function.

I present a novel and simple method by using a statistical model of gas in 1d.
The Model

We consider the BPS vortices in the supersymmetric gauge theory with 8 supercharges. ($G=U(N_c)$ and $N_f$ flavors)

- $A_\mu$: gauge field
- $\Sigma$: adjoint (hermite) scalar
- $H, \tilde{H}$: hypermultiplets

Bosonic part of the Lagrangian

$$\mathcal{L} = \text{Tr} \left[ -\frac{1}{2g^2} F_{MN} F^{MN} + \mathcal{D}_M H (\mathcal{D}^M H)^\dagger - \frac{g^2}{2} (H H^\dagger - c \mathbf{1}_{N_c})^2 \right]$$
BPS Vortices on $T^2$

From the static energy, we find the BPS equations for the vortices

\[ D_{\bar{z}} H = 0 \]

\[ F_{12} + \frac{g^2}{2} (c N_c - H H^\dagger) = 0 \]

If we consider the vortices on a torus, there exists a bound for the vortex number

\[ k \leq A \]

\[ \frac{1}{N_c} \frac{4\pi}{g^2 c} \]

Number of vortices  Bradlow area  Area of torus  Fayet-Iliopoulos (FI) parameter
D-brane Realization of Vortices

BPS solitons can be realized by using D-brane bound states in superstring theory.

*E.g.* 3-dim model

\[
k \times \text{D0-branes} + N_c \times \text{D2-branes} + N_f \times \text{D6-branes in } \mathbb{R}^{1,2} \times \mathbb{C}^2 / \mathbb{Z}_2 \times \mathbb{R}^3
\]

T-duality in superstring theory maps the vortex configuration to a domain-wall (kinky D-brane) configuration.

\[
\hat{\Sigma}(x^1) = - \frac{1}{2\pi i R} \log \left[ \mathbb{P} \exp \left( i \int_0^{2\pi R} dx^2 A_2(x^1, x^2) \right) \right]
\]

*Vev of adjoint scalar

*Wilson line of the gauge field*
T-duality Mapping of Vortices

\( \frac{1}{R} \rightarrow \infty \)

\( k \) domain walls (\( k \)-times wrapping)

\( \frac{1}{R} = \frac{2}{g^2 c R} \): fixed

\( N_c = N_f = 1 \)}

\( g^2 c \rightarrow \infty \)

\( \frac{1}{R} \rightarrow \infty \)
Gas of hard rods in 1D

\[
\text{Volume of the vortex moduli space) = (Volume of dual domain-wall configuration space) = (Volume of the configuration space of hard rods with length } d \text{ on } S^1) = \frac{L(L - kd)^{k-1}}{k!}
\]
The partition function for the gas of identical hard rods with mass $m$ on $S^1$ with period $L$

$$Z_{\text{rods}} = \left(\frac{mT}{2\pi}\right)^k \frac{L (L - kd)^{k-1}}{k!}$$

where $m=2\pi c$ and $d=2/g^2 c R$ in terms of the gauge theory vortices.

T-duality ($c \rightarrow 2\pi R c$)

The partition function of $k$-vortex system on a torus $T^2$ with area $A$

$$Z_{N_c=N_f=1}^{k,T^2} = \frac{1}{k!} \left( c T \right)^k L (L - kd)^{k-1} (2\pi R)^k$$

$$= \frac{1}{k!} \left( c T \right)^k A \left( A - k \frac{4\pi}{g^2 c} \right)^{k-1}$$

where $A=2\pi RL$
Equation of State

Using the derived partition function, we obtain the van der Waals equation of state

\[ P \left( A - k \frac{4\pi}{g^2 c} \right) = kT \]

which agrees with [Manton-Nasir (1999), Manton-Sutcliffe (2004)] without any knowledge of the vortex moduli space metric!

\[ k \frac{4\pi}{g^2 c} \leq A \]
Other Examples

Similarly, we can calculate

- **$k$ semi-local vortices with $N_c=1$ and general $N_f$**

\[
Z_{k,T^2}^{N_c=1,N_f} = \left( \frac{T}{2\pi} \right)^{kN_f} (2\pi c)^{kN_f} (2\pi R)^{kN_f} \frac{1}{k} \frac{1}{(kN_f - 1)!} L (L - dk)^{kN_f-1}
\]

\[
= (cT)^{kN_f} \frac{1}{k} \frac{1}{(kN_f - 1)!} A \left( A - \frac{4\pi k}{g^2 c} \right)^{kN_f-1}
\]

- **$k$ local non-Abelian vortices with $N_c=N_f=N$**

\[
Z_{k,T^2}^{N_c=N_f=N} = (cT)^{kN} \frac{1}{k!} \left[ \frac{A}{(N-1)!} \left( \frac{4\pi}{g^2 c} \right)^{N-1} \right]^k \left[ 1 - D_N(k - 1) \frac{k}{A} + \mathcal{O} \left( \left( \frac{4\pi}{g^2 cA} \right)^2 \right) \right]
\]

where \[
\frac{D_N}{4\pi/g^2c} = \frac{(2N-2)!!}{(2N-1)!!} = 1, \quad \frac{2}{3}, \quad \frac{8}{15}, \quad \frac{16}{35}, \quad \frac{128}{315}, \ldots
\]
Large Area Limit

For $N_c=N_f=1$, in the limit of $A \to \infty$

$$\text{Vol}(\mathcal{M}_k) = \frac{A(A - kd)^{k-1}}{k!} \simeq \frac{A^k}{k!}$$

This result agrees with Nekrasov-like localization method ($\Omega$-background, Equivariant cohomology, etc.) as follows...
Reduced Matrix Model in $\Omega$-background

Let us consider the following reduced matrix model partition function (from $N=1$ in 4d)

$$Z_k = \int [d\vec{B}] [d\vec{F}] [d\Phi] e^{-S(\vec{B}, \vec{F}, \Phi)}$$

where the action is BRST exact

$$S = \frac{1}{g^2} Q \Xi(\vec{B}, \vec{F}, \Phi)$$

and $\vec{B} = (X, I, H)$, $\vec{F} = (\lambda, \psi, \chi)$

which obey the following BRST transformations

$$QX = \lambda, \quad Q\lambda = [\Phi, X] + \epsilon X$$
$$QI = \psi, \quad Q\psi = \Phi I$$
$$QH = [\Phi, \chi], \quad Q\chi = H$$
$$Q\Phi = 0$$
**Vortex Partition Function**

The partition function reduces to the following residue integral with respect to eigenvalues of $\Phi$.

$$Z_k = \oint \prod_{i=1}^{k} \frac{d\phi_i}{2\pi i \phi_i} \prod_{i<j} \frac{(\phi_i - \phi_j)^2}{(\phi_i - \phi_j)^2 - \epsilon^2}$$

The poles exist at $\phi_i = \epsilon(i - 1)$

$$Z_k = \frac{1}{\epsilon^k k!}$$

In the limit of $\epsilon \to 0$, this gives the "volume" of the moduli space of $k$ vortices in $\mathbb{C}$ with an identification of $A = 1/\epsilon$. 
In general, we can obtain the non-perturbative contribution of $k$-vortices to the twisted superpotential of 2d $N=(2,2)$ supersymmetric gauge theory from the partition function ("volume" of the moduli space of $k$-vortices) [Shadchin 2006]

$$Z_k(\vec{a}, \epsilon) = \sum_{|\vec{k}|=k} \frac{\prod_{l=1}^{N_c} \prod_{f=1}^{N_f} \prod_{i_l=1}^{k_l} (a_l + m_f + \epsilon i_l)}{\prod_{l,n=1}^{N_c} \prod_{i_l=1}^{k_l} (a_l - a_n + \epsilon(k_l - k_m - i_l))} \Lambda^{2N_c k}$$
Conclusion

- We find a novel and simple method to compute the volume (partition function) of the BPS vortex moduli space.
  - It is equivalent to the configuration space of the hard rods system in 1-dimensional circle.
- This derivation does not need the detail structure of the moduli space like metric.
  - This is due to the “localization” property of the supersymmetric gauge theories.
Further Application

• Landscape (counting BPS vacua)
• Counting BPS states in SUGRA
• Kähler potential of Calabi-Yau manifold
• Thermodynamics of vortices in early universe