Counting BPS Solitons and Applications

Kazutoshi Ohta[1a]

Department of Physics, Graduate School of Science, Tohoku University, Sendai 980-8578, Japan

Abstract. We propose a novel and simple method of computing the volume of the moduli space of BPS solitons in supersymmetric gauge theory. We use a D-brane realization of vortices and T-duality relation to domain walls. We there use a special limit where domain walls reduce to gas of hard (soft) one-dimensional rods for the Abelian (non-Abelian) cases. In the simpler cases of the Abelian-Higgs model on a torus, our results agree with exact results which are geometrically derived by an explicit integration over the moduli space of the vortices. On the other side of the limit, we can compute the volume of the moduli space in the combinatorial way, where the problem on the random (plane) partition appears as well as the four dimensional instanton calculus. A part of this talk is based on collaboration with M. Eto, T. Fujimori, M. Nitta, K. Ohashi and N. Sakai [hep-th/0703197].

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1 Introduction

BPS solitons play essential roles to understand the non-perturbative dynamics and properties in supersymmetric gauge theory. It is an important task to evaluate the non-perturbative effects from various kinds of BPS solitons, like instantons, monopoles, vortices and domain-walls, which are classified by their codimensions. To carry out this, we need to investigate the whole detail structure (topology, metric, singularity, etc.) of the moduli space of the solitons, but I would like to concentrate on calculations of the "volume" of their moduli space in this talk. In general, the moduli space of the solitons is non-compact and the volume of the moduli space diverges. So we need a regularization in the calculation of the volume. The volume in double-quotes means it is evaluated with a suitable regularization.

There are important and interesting applications of the "volume" of the moduli space of the BPS solitons. One of them appears in a thermodynamical partition function of diluted BPS soliton gas. If we assume the BPS solitons do not interact with each other and behave as free particles, integration over phase space, which is a cotangent bundle over the moduli space, reduces to integration over the moduli space with suitable metric. Then the thermodynamical partition function is proportional to the "volume" of the moduli space

$$Z = \left(\frac{T}{2\pi\hbar^2}\right)^N \mathrm{Vol}(\mathcal{M}_N),$$

where T is temperature of the system and $\mathrm{Vol}(\mathcal{M}_N)$ stands for the "volume" of the moduli space of N BPS solitons.

Another application is proposed by Nekrasov [1]. He has shown that the prepotential of $\mathcal{N}=2$ supersymmetric gauge theory, which includes all non-perturbative instanton corrections, can be obtained from a statistical partition function summing over Young tableaux. The partition function measures a regularized "volume" of the moduli space of k instantons with gauge group $U(r)$ and the prepotential is given by a leading term of free energy in an asymptotic expansion with respect to a regularization parameter, which corresponds to the so-called Ω-background.

In this talk, we propose a novel and simple derivation of the "volume" of the moduli space of the BPS solitons, in particular BPS vortices, by evaluating or counting a pictorial configuration space of solitons. The configuration space is realized by a D-brane configuration. First, we consider the configuration space of vortices on a compact torus. In this case, effective size of vortices affects the entire volume of the moduli space, since the vortices can not overlap with each over due to their size. We naively expect that the vortices behave like finite size disks on the two-dimensional torus, but the D-brane picture shows us that a configuration space of domain-walls, which is T-dual to the vortices, is equivalent to a configuration of finite size hard rods on a one-dimensional circle in a special limit of parameters [2]. This limit is an approximation, but surprisingly the result is exact because of the localization.

Secondary, we also consider a large area limit of the base space where the vortices exist. In this limit,

[a] *Email:* kohta_at_tuhep.phys.tohoku.ac.jp

the effective size of the vortices can be ignored. So the configuration space of the vortices or T-dual domain-walls reduces to a configuration of rectangular kinks. We can identify the kink configurations with three-dimensional (3d) Young diagrams (plane partitions). So the evaluation of the volume of the moduli space reduces to a counting of the skew Young tableaux associated with the kink configurations. This is also an approximation, but we expect that this gives an exact answer. We can compare with a non-perturbative F-term of two-dimensional supersymmetric gauge theory in evidence [3].

2 Thermodynamics of vortices on T^2

We consider BPS vortices in supersymmetric $SU(N_c)$ gauge theory with 8 supercharges, which has N_f flavor hypermultiplets. The theory possesses gauge fields A_μ, adjoint scalar fields Σ in the vector multiplet and the hypermultiplets H which are expressed by an $N_c \times N_f$ matrix. After Bogomol'nyi completion of energy density, we find BPS equation for the vortices

$$\mathcal{D}_{\bar{z}} H = 0, \quad F_{z\bar{z}} + \frac{g^2}{2}(c\mathbf{1}_{N_c} - HH^\dagger) = 0,$$

where g is a gauge coupling constant and c is a Fayet-Iliopoulos parameter. If we consider solutions of the above BPS equations on a compact two-dimensional surface, like a torus T^2, there exists a bound between the number of vortices k and the area of the torus A

$$k\frac{1}{N_c}\frac{4\pi}{g^2 c} \leq A.$$

This means that the vortex has a minimal effective area $\frac{1}{N_c}\frac{4\pi}{g^2c}$, which excludes the other vortices. This unit area is called the Bradlow area.

Now we realize these BPS vortices as D-brane bound states in Type IIA superstring theory. For example, if we consider three-dimensional model, it is realized by a bound state of k D0-branes, N_c D2-branes and N_f D6-branes in $\mathbb{R}^{1,2}\times\mathbb{C}^2/\mathbb{Z}_2\times\mathbb{R}^3$, where the orbifold preserves a half of supercharges and removes extra moduli (flat directions). As discussed in [4], the T-duality maps a Wilson line of the gauge field which express the vortex configuration to a domain-wall (kinky D-brane) configuration. If we take a T-duality along one cycle (with radius R) of the torus, the k vortices are mapped to the kinky D-branes which wrap k-times on a dual cycle. It is convenient to see the kinky D-brane configuration in covering space of the dual cycle. In the covering space, the kinky D-branes is expressed as kink configurations interpolating between the flavor D5-branes. In general, the kink configuration is smooth functions, but if we take a special limit of $g^2c \to \infty$ and $1/R \to \infty$ with $d \equiv 2/g^2cR$ fixed, then the shape of the kinks become sharp and is represented by piece-wise linear functions. For example, if we consider the simplest case, namely k-vortices with $N_c = N_f = 1$ (ANO vortices), the above process is depicted in Fig.1.

Once we take the limit and get the piece-wise linear configuration, we can identify the domain-wall kink configuration with a system of k hard rods with length d on S^1 with radius L[1]. Therefore, if we want to calculate the volume of the configuration (moduli) space of the vortices, we need to calculate the volume of the T-dual domain-wall configuration space, which furthermore reduces to the volume of the configuration space of hard rods on S^1. This reduced problem is easy to calculate. The answer to the k ANO vortices is proportional to $\frac{2\pi L(2\pi L-kd)^{k-1}}{k!}$. The statistical partition function of gas of k identical hard rods with mass m on S^1 with radius L

$$Z_{\text{rods}} = \left(\frac{mT}{2\pi}\right)^k \frac{2\pi L(2\pi L - kd)^{k-1}}{k!}.$$

Note that these parameters are expressed in terms of the dual picture. So, in order to obtain the thermodynamical partition function of the original vortex system, we replace m with $(2\pi)^2Rc$ and d with $2/g^2cR$. Then we finally obtain the partition function of the vortices

$$Z^{N_c=N_f=1}_{k,T^2} = \frac{1}{k!}(cT)^k A\left(A - k\frac{4\pi}{g^2c}\right)^{k-1}, \qquad (1)$$

where $A = (2\pi)^2RL$ is the area of the torus. From the partition function (1), we can derive the van der Waals equation of state of the vortex gas

$$P\left(A - k\frac{4\pi}{g^2c}\right) = kT, \qquad (2)$$

where the Bradlow area $\frac{4\pi}{g^2c}$ appears and pressure diverges at $A = k\frac{4\pi}{g^2c}$. This agrees with the arguments over the BPS equations.

In the above derivation, we have used the special limit of the parameters where the kink configuration approximates to the piece-wise linear function. Thus we can evaluate easily the volume of the moduli space. However, in spite of the approximate calculation, we find the results (1) and (2) are exact as compared with [5]. This means that a kind of localization works in the calculation of the volume of the BPS vortex moduli space as like as in the instanton calculus, that is, the volume does not depends on the detailed structure of the moduli space and is determined by fixed point structure of isometries. This localization reduces the problem to the simple statistical or combinatorial one. We will see another evidence in the next section.

Our argument can be extended to the general N_c and N_f case. The problem of the volume calculation of the moduli space also reduces to the one-dimensional statistical system. For $N_f > 1$, there appear short length hard rods which correspond to domain-walls connecting the Higgs vacua within a period of the covering space. In the limit of $1/R \to \infty$, these shot rods

[1] L is a radius of another cycle of the torus against the T-dual direction.

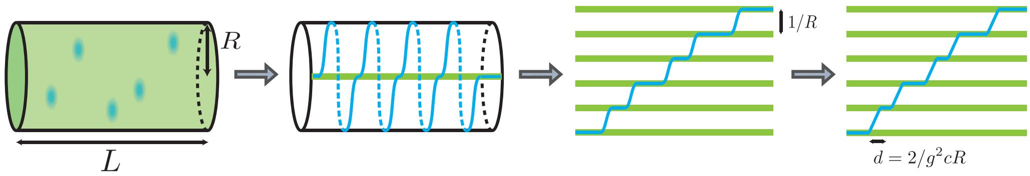

Fig. 1. Vortices as a bound state of D-branes. T-duality maps it to a kinky D-brane configuration. If we consider the configuration in a covering space, it is equivalent to a system of hard rods in one-dimension.

are regarded as particles bound on the long rods corresponding to the kinky D-branes wrapping around the dual circle. In addition, the hard rods now can be overlapped with each other, so the rods effectively become "soft" for the general non-Abelian case. Thus the configuration space of the non-Abelian vortices is equivalent to the rods with the particles in one-dimension. It is however difficult to integrate over the whole configuration space of the reduced system of the rods. We can perform it for the some special cases. For example, in the case of k local non-Abelian vortices with $N_c = N_f = N$, we obtain an expansion of the partition function in terms of $1/A$

$$Z_{k,T^2}^{N_c=N_f=N} = (cT)^{kN}\frac{1}{k!}\left[\frac{A}{(N-1)!}\left(\frac{4\pi}{g^2c}\right)^{N-1}\right]^k \times\left[1 - D_N(k-1)\frac{k}{A} + \mathcal{O}\left(\left(\frac{4\pi}{g^2cA}\right)^2\right)\right],$$

where $\frac{D_N}{4\pi/g^2c} = \frac{(2N-2)!!}{(2N-1)!!}$. The partition function for the k semi-local vortices with $N_c = 1$ and general N_f is give by

$$Z_{k,T^2}^{N_c=1,N_f} = (cT)^{kN_f}\frac{1}{k}\frac{1}{(kN_f-1)!}A\left(A - \frac{4\pi k}{g^2c}\right)^{kN_f-1}.$$

We can also derive the equation of state for these vortex gases.

3 Large area limit

So far we have been treating the vortex system on the compact 2-dimensional surface, but here we take a large area limit of the base space, namely let us consider the vortices on $\mathbb{C}$. If we assume k vortices behave as point particles, the moduli space is a symmetric product space $\mathcal{M}_k \simeq \mathbb{C}^k/S_k$, where S_k is the symmetric group of order k. The volume of the moduli space, of course, diverges since $\mathbb{C}$ is non-compact, and is proportional to $(\mathrm{Vol}(\mathbb{C}))^k$. As in a spirit of [1], we expect that it makes a sense to pick up a coefficient of the divergent volume, that is, a regularized "volume" has important information to investigate the structure of the vortex moduli space and we can apply it to various physical problems.

Before explaining how to evaluate the regularized "volume", we notice that there is one-to-one correspondence between the Higgs vacua and the Young diagrams. We are considering the supersymmetric $SU(N_c)$ Yang-Mills theory (SQCD) with 8 supercharges. Assuming $N_c < N_f$ and N_f matter hypermultiplets have non-degenerate masses which are ordered as $m_1 < m_2 < \cdots < m_{N_f}$. The eigenvalues of the vev for the adjoint scalar in the Higgs phase are given by choosing N_c masses from m_i's. So the number of the Higgs vacua is ${}_{N_f}C_{N_c} = \frac{N_f!}{N_c!\tilde{N}_c!}$, where $\tilde{N}_c = N_f - N_c$. In the language of the brane configuration, the Higgs vacua correspond to bound states of D1 and D5-branes, whose positions are related to the adjoint scalar vev and hypermultiplet masses, respectively. There works a kind of exclusion principle and only one D1 can bind to D5 due to the orbifolding by $\mathbb{Z}_2$. Identifying the D1 brane positions with right-up edges of the $-45°$ rotated Young diagram, that is, using the Maya diagram (free fermion Fock space) / Young diagram correspondence, the Higgs vacuum corresponds to a Young diagram within $N_c \times \tilde{N}_c$ boxes. (See Fig.2.) Thus the Higgs vacuum can be labeled by the Young diagram λ.

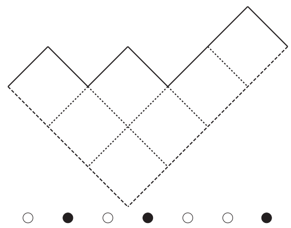

Fig. 2. D-brane bound states associated with the Higgs vacua relate to Young diagrams via the Maya / Young diagram correspondence. White and black circles represents D5-branes and D1-D5 bound states, respectively.

The domain-wall in this supersymmetric system interconnects two different Higgs vacua. If we choose two different vacua (Young diagram) as λ_1 and λ_2 and determine orientation of a "time" direction which is transverse to the domain-wall, the BPS condition says that the Young diagram λ_1 should be include inside the λ_2, namely λ_2 must be constructed by just adding some boxes to λ_1. We represent this inclusion relation by $\lambda_1 \prec \lambda_2$. If we arrange k domain-walls by a "time" series, the $k+1$ vacua $(\lambda_1, \lambda_2, \ldots, \lambda_{k+1})$ should satisfy

$$\lambda_1 \prec \lambda_2 \prec \cdots \prec \lambda_{k+1}.$$

This means that k BPS domain-walls consists a plane partition (3d Young diagram). This correspondence, however, is abstract since the domain-walls have thickness and transition between vacua is smooth. If we take a limit of $g^2c \to \infty$, the transition becomes steep

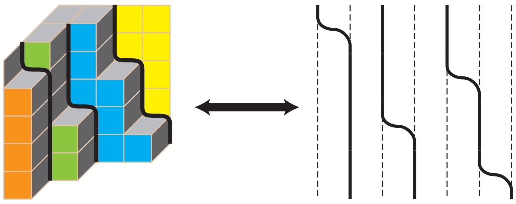

Fig. 3. There is one-to-one correspondence between 3d Young diagrams and kink flows (kinky D-branes).

since the kink profile of the domain-wall gets rectangular (similar to the limit of $d \to 0$ in the previous section). Thus the correspondence between the BPS domain-walls and 3d Young diagram is exact in the above limit. We will consider this situation in the following.

Heights of the 3d Young diagram relate to the positions of the domain-walls, or we can equivalently express the positions as numerical numbers inside the boxes of a skew Young diagram λ_{k+1}/λ_1, which is removal of boxes of λ_1 from ones of λ_{k+1}. The number of domain-walls is given by $k = |\lambda_{k+1}| - |\lambda_1|$, where $|\lambda|$ stands for the number of the boxes in the Young diagram λ. We call the skew Young diagram with the numerical numbers inside the boxes the skew Young tableau. In order to calculate the volume of the moduli (configuration) space of the domain-walls, we need to count the number of all possibilities of the Young tableau. It is similar to evaluation of a dimension of a symmetric group associated with the skew Young diagram, but it diverges in our case since the transverse direction to the domain-walls is continuous and non-compact. To regularize the counting of the domain-wall configuration space, we consider a finite interval, discretize the transverse direction, and label the positions by integers from one to sufficiently large N. Therefore, we find the volume of the domain-wall moduli space $\mathcal{M}_k^{\rm DW}$ is given by

$$\mathrm{Vol}(\mathcal{M}_k^{\rm DW}) = \lim_{N\to\infty, q\to 1} \frac{1}{N^k} s_{\lambda_{k+1}/\lambda_1}(1, q, q^2, \ldots, q^{N-1}),$$

where $s_{\lambda/\lambda'}(x_1, x_2, \ldots, x_N)$ is the skew Schur function. If we simply choose $\lambda_1 = \emptyset$ and $\lambda_{k+1} = \lambda$, then

$$\mathrm{Vol}(\mathcal{M}_k^{\rm DW}) = \frac{d_\lambda}{k!} = \prod_{i<j} \frac{\mu_i - \mu_j - i + j}{-i + j},$$

where d_λ is the dimension of the symmetric group and μ_i, which satisfies $\mu_1 \geq \mu_2 \geq \cdots$ and $\sum_{i=1}^{\infty} \mu_i = k$, is the number of the boxes in the i-th row of the Young diagram λ. This volume is related to the volume of the moduli space of the large N $U(N)$ flat connections on the two-dimensional disk [6]. It also corresponds to just a "half" of the moduli space volume of non-commutative $U(1)$ instantons. This fact is reminiscent of the observation by Hanany and Tong [7].

To apply the above domain-wall result to the vortex, we have to consider the multiple domain-wall configuration in the dual covering space as similar to the previous section. For example, the k ANO vortices in the $N_c = N_f = 1$ theory are equivalent to k domain-walls of the $N_c = 1$ and $N_f \to \infty$ by T-duality. Indeed, if we count the number of configurations of k discrete positions in N, it gives

$$\mathrm{Vol}(\mathcal{M}_{k,\mathbb{C}}^{\rm ANO}) = \lim_{N\to\infty} \frac{1}{N^k} \frac{N!}{k!(N-k)!} = \frac{1}{k!}.$$

This agrees with the large area limit of (1) by identifying the divergent area A^k with N^k. Similarly we can count the dual domain-wall configurations to the vortices for the particular cases in the combinatorial way. It precisely gives a coefficient of the large area divergent part.

4 Conclusion and Discussions

In this talk, I give a simple schematic and combinatorial calculation of the volume of the vortex and domain-wall moduli space. These BPS objects have codimension 2 and 1. The problem of the moduli space volume calculation reduces to the evaluation of the volume of the one-dimensional hard rod configuration space, or the counting of the number of 3d Young diagrams (plane partitions). These calculations also related to the counting of the BPS D-brane configurations in the realization of supersymmetric gauge theory. We expect that these schematic and D-brane picture can be applied to other solitons like instantons and monopoles.

On the other hand, the calculation of the moduli space volume has another application to the derivation of the effective prepotential or superpotential (F-terms) in supersymmetric gauge theory. For two-dimensional supersymmetric Yang-Mills theory, the non-perturbative effective superpotential is calculated in [3] by using equivariant cohomology or topological matrix model. This is an effective theory in the Coulomb phase, but similar combinatorial objects (Gamma functions) appears in the partition function, even though our calculation is done in the Higgs phase. We also hope that our calculation sheds light on the relation between effective theories in the Coulomb and Higgs phase, and makes clear non-perturbative dynamics in supersymmetric gauge theories.

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